University Edition.

A

MANUAL

OF

SPHERICAL AND PRACTICAL ASTRONOMY:

EMBRACING

THE GENERAL PROBLEMS OF SPHERICAL ASTRONOMY, THE SPECIAL APPLICATIONS TO NAUTICAL ASTRONOMY, AND THE THEORY AND USE OF FIXED AND PORTABLE ASTRONOMICAL INSTRUMENTS.

WITH AN APPENDIX ON THE METHOD OF LEAST SQUARES.

BY

WILLIAM CHAUVENET,

PROFESSOR OF MATHEMATICS AND ASTRONOMY IN WASHINGTON UNIVERSITY, SAINT LOUIS

VOL. I.

CONTENTS.

CHAPTER XII.

CHAPTER XIII.

SPHERICAL ASTRONOMY.

CHAPTER I.

THE CELESTIAL SPHERE—SPHERICAL AND RECTANGULAR CO-ORDINATES.

1. **From** whatever point of space an observer be supposed to view the heavenly bodies, they will appear to him as if situated upon the surface of a sphere of which his eye is the centre. If, without changing his position, he directs his eye successively to the several bodies, he may learn their relative directions, but cannot determine either their distances from himself or from each other.

The position of an observer on the surface of the earth is, however, constantly changing, in consequence, 1st, of the diurnal motion, or the rotation of the earth on its axis; 2d, of the annual motion, or the motion of the earth in its orbit around the sun.

The changes produced by the diurnal motion, in the apparent relative positions or directions of the heavenly bodies, are different for observers on different parts of the earth's surface, and can be subjected to computation only by introducing the elements of the observer's position, such as his latitude and longitude.

But the changes resulting from the annual motion of the earth, as well as from the proper motions of the celestial bodies themselves, may be separately considered, and the directions of all the known celestial bodies, as they would be seen from the centre of the earth at any given time, may be computed

according to the laws which have been found to govern the motions of these bodies, from data furnished by long series of observations. The complete investigation of these changes and their laws belongs to *Physical Astronomy*, and requires the consideration of the distances and magnitudes as well as of the directions of the bodies composing the system.

Spherical Astronomy treats specially of the *directions* of the heavenly bodies; and in this branch, therefore, these bodies are at any given instant regarded as situated upon the surface of a sphere of an indefinite radius described about an assumed centre. It embraces, therefore, not only the problems which arise from the diurnal motion, but also such as arise from the annual motion so far as this affects the apparent positions of the heavenly bodies upon the celestial sphere, or their directions from the assumed centre.

SPHERICAL CO-ORDINATES.

2. The direction of a point may be expressed by the angles which a line drawn to it from the centre of the sphere, or point of observation, makes with certain fixed lines of reference. But, since such angles are directly measured by arcs on the surface of the sphere, the simplest method is to assign the position in which the point appears when projected upon the surface of the sphere. For this purpose, a great circle of the sphere, supposed to be given in position, is assumed as a *primitive circle* of reference, and all points of the surface are referred to this circle by a system of *secondaries* or great circles perpendicular to the primitive and, consequently, passing through its poles. The position of a point on the surface will then be expressed by two *spherical co-ordinates:* namely, 1st, the distance of the point from the primitive circle, measured on a secondary; 2d, the distance intercepted on the primitive between this secondary and some given point of the primitive assumed as the *origin* of co-ordinates.

We shall have different systems of co-ordinates, according to the circle adopted as a primitive circle and the point assumed as the origin.

3. *First system of co-ordinates.—Altitude and azimuth.*—In this system, the primitive circle is the *horizon*, which is that great circle of the sphere whose plane touches the surface of the

earth at the observer.* The plane of the horizon may be conceived as that which sensibly coincides with the surface of a fluid at rest.

The *vertical line* is a straight line perpendicular to the plane of the horizon at the observer. It coincides with the direction of the plumb line, or the simple pendulum at rest. The two points in which this line, infinitely produced, meets the sphere, are the *zenith* and *nadir*, the first above, the second below the horizon.

The zenith and nadir are the poles of the horizon.

Secondaries to the horizon are *vertical circles*. They all pass through the zenith and nadir, and their planes, which are called *vertical planes*, intersect in the vertical line.

Small circles parallel to the horizon are called *almucantars*, or *parallels of altitude*.

The *celestial meridian* is that vertical circle whose plane passes through the axis of the earth and, consequently, coincides with the plane of the terrestrial meridian. The intersection of this plane with the plane of the horizon is the *meridian line*, and the points in which this line meets the sphere are the *north* and *south points* of the horizon, being respectively north and south of the plane of the equator.

The *prime vertical* is the vertical circle which is perpendicular to the meridian. The line in which its plane intersects the plane of the horizon is the *east and west line*, and the points in which this line meets the sphere are the *east* and *west points* of the horizon.

The north and south points of the horizon are the poles of the prime vertical, and the east and west points are the poles of the meridian.

* In this definition of the horizon we consider the plane tangent to the earth's surface as sensibly coinciding with a parallel plane passed through the centre; that is, we consider the radius of the celestial sphere to be infinite, and the radius of the earth to be relatively zero. In general, any number of parallel planes at *finite* distances must be regarded as marking out upon the *infinite* sphere the same great circle. Indeed, since in the celestial sphere we consider only *direction*, abstracted from distance, all lines or planes having the same direction—that is, all parallel lines or planes—must be regarded as intersecting the surface of the sphere in the same point or the same great circle. The point of the surface of the sphere in which a number of parallel lines are conceived to meet is called the *vanishing point* of those lines; and, in like manner, the great circle in which a number of parallel planes are conceived to meet may be called the *vanishing circle* of those planes.

The *altitude* of a point of the celestial sphere is its distance from the horizon measured on a vertical circle, and its *azimuth* is the arc of the horizon intercepted between this vertical circle and any point of the horizon assumed as an origin. The origin from which azimuths are reckoned is arbitrary; so also is the direction in which they are reckoned; but astronomers usually take the south point of the horizon as the origin, and reckon towards the right hand, from 0° to 360°; that is, completely around the horizon in the direction expressed by writing the cardinal points of the horizon in the order S. W. N. E. We may, therefore, also define azimuth as the angle which the vertical plane makes with the plane of the meridian.

Navigators, however, usually reckon the azimuth from the north or south points, according as they are in north or south latitude, and towards the east or west, according as the point of the sphere considered is east or west of the meridian: so that the azimuth never exceeds 180°. Thus, an azimuth which is expressed according to the first method simply by 200° would be expressed by a navigator in north latitude by N. 20° E., and by a navigator in south latitude by S. 160° E., the letter prefixed denoting the origin, and the letter affixed denoting the direction in which the azimuth is reckoned, or whether the point considered is east or west of the meridian.

When the point considered is in the horizon, it is often referred to the east or west points, and its distance from the nearest of these points is called its *amplitude*. Thus, a point in the horizon whose azimuth is 110° is said to have an amplitude of W. 20° N.

Since by the diurnal motion the observer's horizon is made to change its position in the heavens, the co-ordinates, altitude and azimuth, are continually changing. Their values, therefore, will depend not only upon the observer's position on the earth, but upon the time reckoned at his meridian.

Instead of the altitude of a point, we frequently employ its *zenith distance*, which is the arc of the vertical circle between the point and the zenith. The altitude and zenith distance are, therefore, complements of each other.

We shall hereafter denote altitude by h, zenith distance by ζ, azimuth by A. We shall have then

$$\zeta = 90^\circ - h \qquad\qquad h = 90^\circ - \zeta$$

The value of ζ for a point below the horizon will be greater than 90°, and the corresponding value of h, found by the for mula $h = 90° - \zeta$, will be negative: so that a negative altitude will express the *depression* of a point below the horizon. Thus, a depression of 10° will be expressed by $h = -10°$, or $\zeta = 100°$.

4. *Second system of co-ordinates.—Declination and hour angle.*—In this system, the primitive circle is the *celestial equator*, or that great circle of the sphere whose plane is perpendicular to the axis of the earth and, consequently, coincides with the plane of the terrestrial equator. This circle is also sometimes called the *equinoctial.*

The diurnal motion of the earth does not change the position of the plane of the equator. The axis of the earth produced to the celestial sphere is called the *axis of the heavens:* the points in which it meets the sphere are the north and south poles of the equator, or the poles of the heavens.

Secondaries to the equator are called *circles of declination*, and also *hour circles*. Since the plane of the celestial meridian passes through the axis of the equator, it is also a secondary to the equator, and therefore also a circle of declination.

Parallels of declination are small circles parallel to the equator.

The *declination* of a point of the sphere is its distance from the equator measured on a circle of declination, and its *hour angle* is the angle at either pole between this circle of declination and the meridian. The hour angle is measured by the arc of the equator intercepted between the circle of declination and the meridian. As the meridian and equator intersect in two points, it is necessary to distinguish which of these points is taken as the origin of hour angles, and also to know in what direction the arc which measures the hour angle is reckoned. Astronomers reckon from that point of the equator which is on the meridian above the horizon, towards the west,—that is, in the direction of the apparent diurnal motion of the celestial sphere,—and from 0° to 360°, or from 0^h to 24^h, allowing 15° to each hour.

Of these co-ordinates, the declination is not changed by the diurnal motion, while the hour angle depends only on the time at the meridian of the observer, or (which is the same thing) on the position of his meridian in the celestial sphere. All the observers on the same meridian at the same instant will, for the same star, reckon the same declination and hour angle. We have

thus introduced co-ordinates of which one is wholly independent of the observer's position and the other is independent of his latitude.

We shall denote declination by δ, and north declination will be distinguished by prefixing to its numerical value the sign $+$, and south declination by the sign $-$.

We shall sometimes make use of the *polar distance* of a point, or its distance from one of the poles of the equator. If we denote it by P, the north polar distance will be found by the formula

$$P = 90^\circ - \delta$$

and the south polar distance by the formula

$$P = 90^\circ + \delta$$

The hour angle will generally be denoted by t. It is to be observed that as the hour angle of a celestial body is continually increasing in consequence of the diurnal motion, it may be conceived as having values greater than 360°, or 24^h, or greater than any given multiple of 360°. Such an hour angle may be regarded as expressing the time elapsed since some given passage of the body over the meridian. But it is usual, when values greater than 360° result from any calculation, to deduct 360°. Again, since hour angles reckoned towards the west are always positive, hour-angles reckoned towards the east must have the negative sign: so that an hour angle of 300°, or 20^h, may also be expressed by $-60°$, or -4^h.

5. *Third system of co-ordinates.—Declination and right ascension.*— In this system, the primitive plane is still the equator, and the first co-ordinate is the same as in the second system, namely, the declination. The second co-ordinate is also measured on the equator, but from an origin which is not affected by the diurnal motion. Any point of the celestial equator might be assumed as the origin; but that which is most naturally indicated is the vernal equinox, to define which some preliminaries are necessary.

The *ecliptic* is the great circle of the celestial sphere in which the sun appears to move in consequence of the earth's motion in its orbit. The position of the ecliptic is not absolutely fixed in space; but, according to the definition just given, its position at any instant coincides with that of the great circle in which the

sun appears to be moving at that instant. Its annual change is, however, very small, and its daily change altogether insensible.

The *obliquity of the ecliptic* is the angle which it makes with the equator.

The points where the ecliptic and equator intersect are called the *equinoctial points*, or the *equinoxes;* and that diameter of the sphere in which their planes intersect is the *line of equinoxes.*

The *vernal equinox* is the point through which the sun *ascends* from the southern to the northern side of the equator; and the *autumnal equinox* is that through which the sun *descends* from the northern to the southern side of the equator.

The *solstitial points*, or *solstices*, are the points of the ecliptic 90° from the equinoxes. They are distinguished as the northern and southern, or the summer and winter solstices.

The *equinoctial colure* is the circle of declination which passes through the equinoxes. The *solstitial colure* is the circle of declination which passes through the solstices. The equinoxes are the poles of the solstitial colure.

By the annual motion of the earth, its axis is carried very nearly parallel to itself, so that the plane of the equator, which is always at right angles to the axis, is very nearly a fixed plane of the celestial sphere. The axis is, however, subject to small changes of direction, the effect of which is to change the position of the intersection of the equator and the ecliptic, and hence, also, the position of the equinoxes. In expressing the positions of stars, referred to the vernal equinox, at any given instant, the actual position of the equinox at the instant is understood, unless otherwise stated.

The *right ascension* of a point of the sphere is the arc of the equator intercepted between its circle of declination and the vernal equinox, and is reckoned from the vernal equinox *eastward* from 0° to 360°, or, in time, from 0^h to 24^h.

The point of observation being supposed at the centre of the earth, neither the declination nor the right ascension will be affected by the diurnal motion: so that these co-ordinates are wholly independent of the observer's position on the surface of the earth. Their values, therefore, vary only with the time, and are given in the ephemerides as functions of the time reckoned at some assumed meridian.

We shall generally denote right ascension by α. As its value reckoned towards the east is positive, a negative value resulting

from any calculation would be interpreted as signifying an arc of the equator reckoned from the vernal equinox towards the west. Thus, a point whose right ascension is 300°, or 20^h, may also be regarded as in right ascension —60°, or -4^h; but such negative values are generally avoided by adding 360°, or 24^h. Again, in continuing to reckon eastward we may arrive at values of the right ascension greater than 24^h, or greater than 48^h, etc.; but in such cases we have only to reject 24^h, 48^h, etc. to obtain values which express the same thing.

6. *Fourth system of co-ordinates.—Celestial latitude and longitude.*—In this system the ecliptic is taken as the primitive circle, and the secondaries by which points of the sphere are referred to it are called *circles of latitude.* Parallels of latitude are small circles parallel to the ecliptic.

The *latitude* of a point of the sphere is its distance from the ecliptic measured on a circle of latitude, and its *longitude* is the arc of the ecliptic intercepted between this circle of latitude and the vernal equinox. The longitude is reckoned eastward from 0° to 360°. The longitude is sometimes expressed in *signs*, degrees, &c., a sign being equal to 30°, or one-twelfth of the ecliptic.

These co-ordinates are also independent of the diurnal motion. It is evident, however, that the system of declination and right ascension will be generally more convenient, since it is more directly related to our first and second systems, which involve the observer's position.

We shall denote celestial latitude by β; longitude by λ. Positive values of β belong to points on the same side of the ecliptic as the north pole; negative values, to those on the opposite side.

In connection with this system we may here define the *nonagesimal*, which is that point of the ecliptic which is at the greatest altitude above the horizon at any given time. That vertical circle of the observer which is perpendicular to the ecliptic meets it in the nonagesimal; and, being a secondary to the ecliptic, it is also a circle of latitude: it is the great circle which passes through the observer's zenith and the pole of the ecliptic.

7. *Co-ordinates of the observer's position.*—We have next to express the position of the observer on the surface of the earth, according to the different systems of co-ordinates. This will be

done by referring his zenith to the primitive circle in the same manner as in the case of any other point.

In the first system, the primitive circle being the horizon, of which the zenith is the pole, the altitude of the zenith is always 90°, and its azimuth is indeterminate.

In the second system, the declination of the zenith is the same as the terrestrial latitude of the observer, and its hour angle is zero. The declination of the zenith of a place is called the *geographical latitude*, or simply the latitude, and will be hereafter denoted by φ. North latitudes will have the sign $+$; south latitudes, the sign $-$.

In the third system, the declination of the zenith is, as before, the latitude of the observer, and its right ascension is the same as the hour angle of the vernal equinox.

In the fourth system, the celestial latitude of the zenith is the same as the zenith distance of the nonagesimal, and its celestial longitude is the longitude of the nonagesimal.

It is evident, from the definitions which have been given, that the problem of determining the latitude of a place by astronomical observation is the same as that of determining the declination of the zenith; and the problem of finding the longitude may be resolved into that of determining the right ascension of the meridian at a time when that of the prime meridian is also given, since the longitude is the arc of the equator intercepted between the two meridians, and is, consequently, the difference of their right ascensions.

8. The preceding definitions are exemplified in the following figures.

Fig. 1.

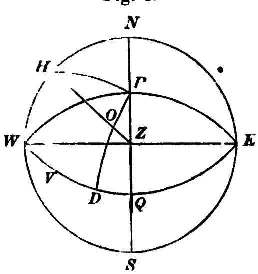


Fig. 1 is a stereographic projection of the sphere upon the plane of the horizon, the projecting point being the nadir. Since the planes of the equator and horizon are both perpendicular to that of the meridian, their intersection is also perpendicular to it; and hence the equator WQE passes through the east and west points of the horizon. All vertical circles passing through the projecting point will be projected into straight lines, as the meridian NZS, the prime vertical WZE, and the vertical circle ZOH drawn through any point O of the surface

of the sphere. We have then, according to the notation adopted in the first system of co-ordinates,

h = the altitude of the point $O = OH$,
ζ = the zenith distance " $= OZ$,
A = the azimuth " $= SH$, or
= the angle SZH.

If the declination circle POD be drawn, we have, in the second system of co-ordinates,

δ = the declination of $O = OD$,
P = the polar distance " $= PO$,
t = the hour angle " $= ZPD$, or $= QD$.

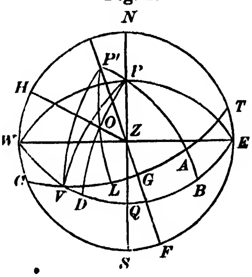
If V is the vernal equinox, we have, in the third system of co-ordinates,

δ = the declination of $O = OD$,
α = the right ascension $= VD$, or
= the angle VPD.

In this figure is also drawn the *six hour circle* EPW, or the declination circle passing through the east and west points of the horizon. The angle ZPW, or the arc QW, being 90°, the hour angle of a point on this circle is either $+ 6^h$ or -6^h, that is, either 6^h or 18^h.

Fig. 2 is a repetition of the preceding figure, with the addition of the ecliptic and the circles related to it. CVT represents the ecliptic, P' its pole, $P'OL$ a circle of latitude. Hence we have, in our third system of co-ordinates,

Fig. 2.



β = the celestial latitude of $O = OL$,
λ = " longitude " $= VL$,
= the angle $VP'L$.

We have also VP the equinoctial colure, $P'PAB$ the solstitial colure, $P'ZGF$ the vertical circle passing through P', which is therefore perpendicular to the ecliptic at G. The point G is the nonagesimal; ZG is its zenith distance, VG its longitude; or ZG is the celestial latitude and VG the celestial longitude of the zenith.

Finally, we have, in both Fig. 1 and Fig. 2,

φ = the geographical latitude of the observer
$= ZQ = 90° - PZ = PN$

Hence the latitude of the observer is always equal to the altitude of the north pole. For an observer in south latitude, the north pole is below the horizon, and its altitude is a negative quantity: so that the definition of latitude as the altitude of the north pole is perfectly general if we give south latitudes the negative sign. The south latitude of an observer considered independently of its sign is equal to the altitude of the south pole above his horizon, the elevation of one pole being always equal to the depression of the other.

9. *Numerical expression of hour angles.*—The equator, upon which hour angles are measured, may be conceived to be divided into 24 equal parts, each of which is the measure of one hour, and is equivalent to $\frac{1}{24}$ of 360°, or to 15°. The hour is divided sexagesimally into minutes and seconds of *time*, distinguished from minutes and seconds of *arc* by the letters ^{m} and ^{s} instead of the accents $'$ and $''$. We shall have, then,

$$1^h = 15^\circ \qquad 1^m = 15' \qquad 1^s = 15''$$

To convert an angle expressed in time into its equivalent in arc, multiply by 15 and change the denominations ^{h} ^{m} ^{s} into $^\circ$ $'$ $''$; and to convert arc into time, divide by 15 and change $^\circ$ $'$ $''$ into ^{h} ^{m} ^{s}. The expert computer will readily find ways to abridge these operations in practice. It is well to observe, for this purpose, that from the above equalities we also have,

$$1^\circ = 4^m \qquad 1' = 4^s$$

and that we may therefore convert degrees and minutes of arc into time by multiplying by 4 and changing $^\circ$ $'$ into ^{m} ^{s}; and reciprocally.

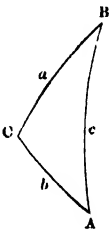
TRANSFORMATION OF SPHERICAL CO-ORDINATES.

10. *Given the altitude* (h) *and azimuth* (A) *of a star, or of any point of the sphere, and the latitude* (φ) *of the observer, to find the declination* (δ) *and hour angle* (t) *of the star or the point.* In other words, to transform the co-ordinates of the first system into those of the second.

This problem is solved by a direct application of the formulæ of Spherical Trigonometry to the triangle *POZ*, Fig. 1, in which, *O* being the given star or point, we have three parts given,

namely, ZO the zenith distance or complement of the given altitude, PZO the supplement of the given azimuth, and PZ the complement of the given latitude; from which we can find PO the polar distance or complement of the required declination, and ZPO the required hour angle. But, to avoid the trouble of taking complements and supplements, the formulæ are adapted so as to express the declination and hour angle directly in terms of the altitude, azimuth, and latitude.

Fig. 3.



To show as clearly as possible how the formulæ of Spherical Trigonometry are thus converted into formulæ of Spherical Astronomy, let us first consider a spherical triangle ABC, Fig. 3, in which there are given the angle A, and the sides b and c, to find the angle B and the side a. The general relations between these five quantities are [Sph. Trig. Art. 114]*

$$\left.\begin{aligned} \cos a &= \cos c \cos b + \sin c \sin b \cos \mathrm{A} \\ \sin a \cos \mathrm{B} &= \sin c \cos b - \cos c \sin b \cos \mathrm{A} \\ \sin a \sin \mathrm{B} &= \sin b \sin \mathrm{A} \end{aligned}\right\} \quad (\mathfrak{A})$$

Now, comparing the triangle ABC with the triangle PZO of Fig. 1, we have

$$\begin{aligned} \mathrm{A} &= PZO = 180° - A & a &= PO = 90° - \delta \\ b &= ZO = 90° - h & \mathrm{B} &= ZPO = t \\ c &= PZ = 90° - \varphi & & \end{aligned}$$

Substituting these values in the above equations, we obtain

$$\sin \delta = \sin \varphi \sin h - \cos \varphi \cos h \cos A \qquad (1)$$
$$\cos \delta \cos t = \cos \varphi \sin h + \sin \varphi \cos h \cos A \qquad (2)$$
$$\cos \delta \sin t = \cos h \sin A \qquad (3)$$

which are the required expressions of δ and t in terms of h and A.

If the zenith distance (ζ) of the star is given, the equations will be

$$\sin \delta = \sin \varphi \cos \zeta - \cos \varphi \sin \zeta \cos A \qquad (4)$$
$$\cos \delta \cos t = \cos \varphi \cos \zeta + \sin \varphi \sin \zeta \cos A \qquad (5)$$
$$\cos \delta \sin t = \sin \zeta \sin A \qquad (6)$$

Since, in Spherical Astronomy, we consider arcs and angles whose values may exceed 180°, it becomes necessary, in general,

* The references to Trigonometry are to the 5th edition of the author's "Treatise on Plane and Spherical Trigonometry."

to determine such arcs and angles by both the sine and the cosine, in order to fix the quadrant in which their values are to be taken. It has been shown in Spherical Trigonometry that when we consider the *general* triangle, or that in which values are admitted greater than 180°, there are two solutions of the triangle in every case, but that the ambiguity is removed and one of these solutions excluded "when, in addition to the other data, the sign of the sine or cosine of one of the required parts is given." [Sph. Trig. Art. 113.] In our present problem the sign of cos δ is given, since it is necessarily positive; for δ is always numerically less than 90°, that is, between the limits +90° and −90°. Hence cos t has the sign of the second member of (2) or (5), and sin t the sign of the second member of (3) or (6), and t is to be taken in the quadrant required by these two signs. Since h also falls between the limits +90° and −90°, or ζ between 0° and 180°, cos h, or sin ζ, is positive, and therefore by (3) or (6) sin t has the sign of sin A; that is, when $A < 180°$ we have $t < 180°$, and when $A > 180°$ we have $t > 180°$,—conditions which also follow directly from the nature of our problem, since the star is west or east of the meridian according as $A < 180°$ or $A > 180°$. The formula (1) or (4) fully determines δ, which will always be taken less than 90°, positive or negative according to the sign of its sine.*

To adapt the equations (4), (5), and (6) for logarithmic computation, let m and M be assumed to satisfy the conditions [Pl. Trig. Art. 174],

$$\left.\begin{aligned} m \sin M &= -\sin \zeta \cos A \\ m \cos M &= \cos \zeta \end{aligned}\right\} \quad (7)$$

the three equations may then be written as follows:

$$\left.\begin{aligned} \sin \delta &= m \sin (\varphi - M) \\ \cos \delta \cos t &= m \cos (\varphi - M) \\ \cos \delta \sin t &= \sin \zeta \sin A \end{aligned}\right\} \quad (8)$$

If we eliminate m from these equations, the solution takes the following convenient form:

---

* There are, however, special problems in which it is convenient to depart from this general method, and to admit declinations greater than 90°, as will be seen hereafter

$$\left.\begin{aligned} \tan M &= \tan \zeta \cos A \\ \tan t &= \frac{\tan A \sin M}{\cos(\varphi - M)} \\ \tan \delta &= \tan(\varphi - M) \cos t \end{aligned}\right\} \quad (9)$$

in the use of which, we must observe to take t greater or less than 180° according as A is greater or less than 180°, since the hour angle and the azimuth must fall on the same side of the meridian.

EXAMPLE.—In the latitude $\varphi = 38° 58' 53''$, there are given for a certain star $\zeta = 69° 42' 30''$, $A = 300° 10' 30''$; required δ and t. The computation by (9) may be arranged as follows:*

				log tan ζ	0.4320966		
$\phi =$	38°	58′	53″	log cos A	9.7012595	log tan A	n0.2355026
$M =$	53	39	41.98	log tan M	0.1333561	log sin M	9.9060828
$\phi - M =$	−14	40	48.98	log tan $(\phi - M)$	n9.4182633	log sec $(\phi - M)$	0.0144141
$t =$	304	55	26.49	log cos t	9.7577677	log tan t	n0.1559995
$\delta =$	−8	31	46.56	log tan δ	n9.1760310		

Converting the hour angle into time, we have $t = 20^h\ 19^m\ 41^s.766$.

11. The angle POZ, Fig. 1, between the vertical circle and the declination circle of a star, is frequently called the *parallactic angle*, and will here be denoted by q. To find its value from the data ζ, A, and φ, we have the equations

$$\left.\begin{aligned} \cos\delta \cos q &= \sin\zeta \sin\varphi + \cos\zeta \cos\varphi \cos A \\ \cos\delta \sin q &= \qquad\qquad\qquad\quad \cos\varphi \sin A \end{aligned}\right\} \quad (10)$$

which may be solved in the following form:

$$\left.\begin{aligned} f \sin F &= \sin\zeta \\ f \cos F &= \cos\zeta \cos A \\ \cos\delta \cos q &= f \cos(\varphi - F) \\ \cos\delta \sin q &= \cos\varphi \sin A \end{aligned}\right\} \quad (11)$$

or in the following:

$$\left.\begin{aligned} g \sin G &= \sin\varphi \\ g \cos G &= \cos\varphi \cos A \\ \cos\delta \cos q &= g \cos(\zeta - G) \\ \cos\delta \sin q &= \cos\varphi \sin A \end{aligned}\right\} \quad (12)$$

or again in the following:

* In this work the letter n prefixed to a logarithm indicates that the number to which it corresponds is to have the negative sign.

$$\left.\begin{aligned}\tan G &= \tan\varphi \sec A \\ \tan q &= \frac{\tan A \cos G}{\cos(\zeta - G)}\end{aligned}\right\} \quad (13)$$

and, in the use of the last form, it is to be observed that q is to be taken greater or less than 180° according as A is greater or less than 180°, as is evident from the preceding forms.

12. If, in a given latitude, the azimuth of a star of known declination is given, its hour angle and zenith distance may be found as follows. We have

$$\begin{aligned}\cos t \sin\varphi - \sin t \cot A &= \cos\varphi \tan\delta \\ \cos\zeta \sin\varphi - \sin\zeta \cos\varphi \cos A &= \sin\delta\end{aligned}$$

The solution of the first of these is effected by the equations

$$\begin{aligned}b \sin B &= \sin\varphi \\ b \cos B &= \cot A \\ \sin(B - t) &= \frac{\cos\varphi \tan\delta}{b}\end{aligned}$$

and that of the second by

$$\begin{aligned}c \sin C &= \sin\varphi \\ c \cos C &= \cos\varphi \cos A \\ \sin(C - \zeta) &= \frac{\sin\delta}{c}\end{aligned}$$

13. Finally, if from the given altitude and azimuth we wish to find the declination, hour angle, and parallactic angle at the same time, it will be convenient to use Gauss's Equations, which for the triangle ABC, Fig. 3, are

$$\left.\begin{aligned}\cos\tfrac{1}{2}a \sin\tfrac{1}{2}(\mathrm{B}+\mathrm{C}) &= \cos\tfrac{1}{2}(b-c)\cos\tfrac{1}{2}\mathrm{A} \\ \cos\tfrac{1}{2}a \cos\tfrac{1}{2}(\mathrm{B}+\mathrm{C}) &= \cos\tfrac{1}{2}(b+c)\sin\tfrac{1}{2}\mathrm{A} \\ \sin\tfrac{1}{2}a \sin\tfrac{1}{2}(\mathrm{B}-\mathrm{C}) &= \sin\tfrac{1}{2}(b-c)\cos\tfrac{1}{2}\mathrm{A} \\ \sin\tfrac{1}{2}a \cos\tfrac{1}{2}(\mathrm{B}-\mathrm{C}) &= \sin\tfrac{1}{2}(b+c)\sin\tfrac{1}{2}\mathrm{A}\end{aligned}\right\} \quad (\mathfrak{B})$$

which are to be solved in the usual manner [Sph. Trig. Art. 116] after substituting the values $\mathrm{A} = 180° - A$, $b = \zeta$, $c = 90° - \varphi$, $a = 90° - \delta$, $\mathrm{B} = t$, $\mathrm{C} = q$.

14. *Given the declination* (δ) *and hour angle* (t) *of a star, and the latitude* (φ), *to find the zenith distance* (ζ) *and azimuth* (A) *of the star.* That is, to transform the co-ordinates of the second system into those of the first.

We take the same general equations ($\mathfrak{A}$) of Spherical Trigonometry which have been employed in the solution of the pre-

ceding problem, Art. 10; but we now suppose the letters A, B, C, in Fig. 3, to represent respectively the pole, the zenith, and the star, so that we substitute

$$\begin{array}{ll} \mathrm{A} = t & a = \zeta \\ b = 90° - \delta & \mathrm{B} = 180° - A \\ c = 90° - \varphi & \end{array}$$

and the equations become

$$\left.\begin{array}{rl} \cos\zeta = & \sin\varphi\sin\delta + \cos\varphi\cos\delta\cos t \\ \sin\zeta\cos A = & -\cos\varphi\sin\delta + \sin\varphi\cos\delta\cos t \\ \sin\zeta\sin A = & \cos\delta\sin t \end{array}\right\} \tag{14}$$

which express ζ and A directly in terms of the data.

Adapting these for logarithmic computation, we have

$$\left.\begin{array}{rl} m\sin M = & \sin\delta \\ m\cos M = & \cos\delta\cos t \\ \cos\zeta = & m\cos(\varphi - M) \\ \sin\zeta\cos A = & m\sin(\varphi - M) \\ \sin\zeta\sin A = & \cos\delta\sin t \end{array}\right\} \tag{15}$$

in which m is a positive number.

Eliminating m, we deduce the following simple and accurate formulæ:

$$\left.\begin{array}{rl} \tan M = & \dfrac{\tan\delta}{\cos t} \\[2ex] \tan A = & \dfrac{\tan t\cos M}{\sin(\varphi - M)} \\[2ex] \tan\zeta = & \dfrac{\tan(\varphi - M)}{\cos A} \end{array}\right\} \tag{16}$$

where A is to be taken greater or less than 180° according as t is greater or less than 180°.

EXAMPLE 1.—In latitude $\varphi = 38° 58' 53''$, there are given for a certain star, $\delta = -8° 31' 46''.56$, $t = 20^h 19^m 41^s.766$; required A and ζ. By (16) we have:

		log tan δ	n0.1760310		
$\phi =$	38° 58′ 53″	log cos t	9.7577677	log tan t	n0.1559995
$M =$	−14 40 48.98	log tan M	n0.4182633	log cos M	9.9855859
$\varphi - M =$	53 39 41.98	log tan $(\varphi - M)$	0.1333561	log cosec $(\phi - M)$	0.0939172
$A =$	300 10 30	log cos A	9.7012595	log tan A	n0.2355026
$\zeta =$	69 42 30	log tan ζ	0.4320966		

For verification we can use the equation

$$\sin \zeta \sin A = \cos \delta \sin t$$

$\log \sin \zeta$	9.9721748	$\log \cos \delta$	9.9951697
$\log \sin A$	9.9367621	$\log \sin t$	9.9137672
	9.9089369		9.9089369

EXAMPLE 2.—In latitude $\varphi = -48° 32'$, there are given for a star, $\delta = 44° 6' 0''$, $t = 17^h 25^m 4^s$; required A and ζ.

We find $A = 241° 53' 33''.2$, $\zeta = 126° 25' 6''.6$; the star is below the horizon, and its negative altitude, or depression, is $h = -36° 25' 6''.6$.

If the zenith distance of the same star is to be frequently computed on the same night at a given place, it will be most readily done by the following method. In the first equation of (14) substitute

$$\cos t = 1 - 2 \sin^2 \tfrac{1}{2} t$$

then we have

$$\cos \zeta = \cos (\varphi \backsim \delta) - 2 \cos \varphi \cos \delta \sin^2 \tfrac{1}{2} t$$

where $\varphi \backsim \delta$ signifies either $\varphi - \delta$ or $\delta - \varphi$, and if $\delta > \varphi$ the latter form is to be used. Subtracting both members from unity, we obtain

$$\sin^2 \tfrac{1}{2} \zeta = \sin^2 \tfrac{1}{2} (\varphi \backsim \delta) + \cos \varphi \cos \delta \sin^2 \tfrac{1}{2} t$$

Now let

$$m = \sqrt{\cos \varphi \cos \delta}$$
$$n = \sin \tfrac{1}{2} (\varphi \backsim \delta)$$

then we have

$$\sin \tfrac{1}{2} \zeta = n \sqrt{1 + \frac{m^2 \sin^2 \tfrac{1}{2} t}{n^2}}$$

and hence, by taking an auxiliary N such that

$$\left.\begin{aligned} \tan N &= \frac{m}{n} \sin \tfrac{1}{2} t \\ \text{we have} \quad \sin \tfrac{1}{2} \zeta &= \frac{n}{\cos N} = \frac{m}{\sin N} \sin \tfrac{1}{2} t \end{aligned}\right\} \quad (17)$$

The second form for $\sin \frac{1}{2} \zeta$ will be more precise than the first when $\sin N$ is greater than $\cos N$.

The quantities m and n will be constant so long as the declination does not vary.

15. If the parallactic angle q (Art. 11) and the zenith distance

ζ are required from the data φ, δ, and t, they may be found from the equations

$$\left.\begin{aligned} \cos\zeta &= \sin\varphi\sin\delta + \cos\varphi\cos\delta\cos t \\ \sin\zeta\cos q &= \sin\varphi\cos\delta - \cos\varphi\sin\delta\cos t \\ \sin\zeta\sin q &= \cos\varphi\sin t \end{aligned}\right\} \quad (18)$$

which are adapted for logarithms as follows:

$$\left.\begin{aligned} n\sin N &= \cos\varphi\cos t \\ n\cos N &= \sin\varphi \\ \cos\zeta &= n\sin(\delta + N) \\ \sin\zeta\cos q &= n\cos(\delta + N) \\ \sin\zeta\sin q &= \cos\varphi\sin t \end{aligned}\right\} \quad (19)$$

or, eliminating n, thus:

$$\left.\begin{aligned} \tan N &= \cot\varphi\cos t \\ \tan\zeta\sin q &= \frac{\tan t\sin N}{\sin(\delta + N)} \\ \tan\zeta\cos q &= \cot(\delta + N) \end{aligned}\right\} \quad (20)$$

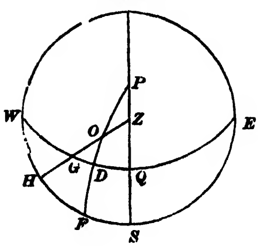
When this last form is employed in the case of a star which has been observed above the horizon, tan ζ is known to be positive, and there is no ambiguity in the determination of q. This form is, therefore, the most convenient in practice.

If ζ, A, and q are all required from the data δ, t, and φ, we have, by Gauss's equations,

$$\left.\begin{aligned} \sin\tfrac{1}{2}\zeta\sin\tfrac{1}{2}(A + q) &= \sin\tfrac{1}{2}t\cos\tfrac{1}{2}(\varphi + \delta) \\ \sin\tfrac{1}{2}\zeta\cos\tfrac{1}{2}(A + q) &= \cos\tfrac{1}{2}t\sin\tfrac{1}{2}(\varphi - \delta) \\ \cos\tfrac{1}{2}\zeta\sin\tfrac{1}{2}(A - q) &= \sin\tfrac{1}{2}t\sin\tfrac{1}{2}(\varphi + \delta) \\ \cos\tfrac{1}{2}\zeta\cos\tfrac{1}{2}(A - q) &= \cos\tfrac{1}{2}t\cos\tfrac{1}{2}(\varphi - \delta) \end{aligned}\right\} \quad (21)$$

16. When the altitude, azimuth, and parallactic angle of known stars are to be frequently computed at the same place, the labor of computation is much diminished by an auxiliary table prepared for the latitude of the place according to formulæ proposed by Gauss. A specimen of such a table computed for the latitude of the Altona Observatory will be found in "*Schumacher's Hülfstafeln*, neu herausg. v. Warnstorff." The requisite formulæ are readily deduced as follows:

Fig. 4.



Let the declination circle through the object O, Fig. 4, be produced to intersect the horizon in F. By the diurnal motion

the point F changes its position on the horizon with the time; but its position depends *only* on the time or the hour angle ZPO, and not upon the declination of O. The elements of the position of F may therefore be previously computed for successive values of t.

We have in the triangle PFS, right-angled at S, $FPS = t$, $PS = 180° - \varphi$; and if we put

$$\mathfrak{A} = FS, \qquad B = PF - 90°, \qquad \gamma = 180° - PFS.$$

we find

$$\tan \mathfrak{A} = \sin \varphi \tan t, \quad \tan B = \cot \varphi \cos t, \quad \cot \gamma = \sin B \tan t$$

We have now in the triangle HOF, right-angled at H,

$$B + \delta = OF, \qquad \gamma = HFO, \qquad h = OH,$$

and if we put

$$u = HF = HS - FS = A - \mathfrak{A},$$

we find

$$\tan u = \cos \gamma \tan (B + \delta) \qquad A = \mathfrak{A} + u$$
$$\sin h = \sin \gamma \sin (B + \delta) \qquad \text{or, } \tan h = \tan \gamma \sin u.$$

To find the parallactic angle $q = POZ$, we have in the triangle HOF

$$\tan q = \cot \gamma \sec (B + \delta)$$

In the Gaussian table for Altona as given in the "Hülfstafeln" we find five columns, which give for the argument t, the quantities $\mathfrak{A}$, B, $\log \cos \gamma$, $\log \sin \gamma$, $\log \cot \gamma$, the last three under the names $\log C$, $\log D$, and $\log E$, respectively. With the aid of this table, then, the labor of finding any one of the quantities h, A, q is reduced to the addition of two logarithms, namely:

$$\tan u = C \tan (B + \delta) \qquad \sin h = D \sin (B + \delta)$$
$$A = \mathfrak{A} + u \qquad \tan q = E \sec (B + \delta)$$

The formulæ for the inverse problem (of Art. 10) may also be found thus. Let G be the intersection of the equator and the vertical circle through O, and put $B = HG$, $u = DG$, $\mathfrak{A} = QG$, $\gamma = ZGQ$; then we readily find

$$\tan \mathfrak{A} = \sin \varphi \tan A, \quad \tan B = \cot \varphi \cos A, \quad \cot \gamma = \sin B \tan A$$

which are of the same form as those given above, with the exchange of A for t. Hence the same table gives also the elements of the point G, by entering with the argument "azimuth," expressed in time, instead of the hour angle. We then have $t =$

DQ, and if we here put $u = DG = \mathfrak{A} - t$, we have from the triangle GDO

$$\sin\delta = \sin\gamma \sin(h - B) \qquad \tan u = \cos\gamma \tan(h - B)$$

or, employing the notation of the table,

$$\tan u = C \tan(h - B) \qquad \sin\delta = D \sin(h - B)$$
$$t = \mathfrak{A} - u \qquad \tan q = E \sec(h - B)$$

17. *To find the zenith distance and azimuth of a star, when on the six hour circle.*—Since in this case $t = 6^h = 90°$, the triangle PZO, Fig. 4, is right-angled at P, and gives immediately

$$\cos ZO = \cos PZ \cos PO \qquad \cot PZO = \sin PZ \cot PO$$

or, since $PZO = 180° - A$, and $\cot PZO = -\cot A$,

$$\cos\zeta = \sin\varphi \sin\delta \qquad \cot A = -\cos\varphi \tan\delta$$

But if the star is on the six hour circle east of the meridian we must put $t = 18^h = 270°$ and $PZO = A - 180°$; hence for this case

$$\cot A = + \cos\varphi \tan\delta$$

A more general solution, however, is obtained from the equations (14), by putting $\cos t = 0$, $\sin t = \pm 1$, whence

$$\left.\begin{aligned} \cos\zeta &= \sin\varphi \sin\delta \\ \sin\zeta \cos A &= -\cos\varphi \sin\delta \\ \sin\zeta \sin A &= \pm \cos\delta \end{aligned}\right\} \quad (22)$$

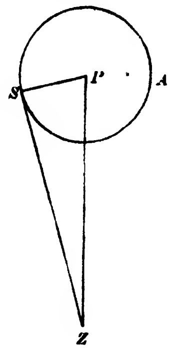
the lower sign in the last equation being used when the star is east of the meridian.

Example.—Required the zenith distance and azimuth of *Sirius*, $\delta = -16° 31' 20''$, when on the six hour circle east of the meridian at the Cape of Good Hope, $\varphi = -33° 56' 3''$. We find

$$\log(-\cos\delta) = \log\sin\zeta \sin A = n9.9816870$$
$$\log(-\cos\varphi \sin\delta) = \log\sin\zeta \cos A = 9.3728204$$
$$A = 283° 49' 34''.9$$
$$\log\sin A = 9.9872302$$
$$\log\sin\zeta = 9.9944568$$
$$\log\sin\varphi \sin\delta = \log\cos\zeta = 9.2007309$$
$$\zeta = 80° 51' 55''$$

18. *To find the hour angle, azimuth, and zenith distance of a given star at its greatest elongation.*—In this case the vertical circle ZS, Fig. 5, is tangent to the diurnal circle, SA, of the star, and is, therefore, perpendicular to the declination circle PS. The right triangle PZS gives, therefore,

Fig. 5.



$$\left.\begin{aligned}\cos t &= \frac{\tan\varphi}{\tan\delta}\\ \sin A &= \frac{\cos\delta}{\cos\varphi}\\ \cos\zeta &= \frac{\sin\varphi}{\sin\delta}\end{aligned}\right\} \tag{23}$$

If δ and φ are nearly equal, each of the quantities $\cos t$, $\sin A$, and $\cos\zeta$ will be nearly equal to unity, and a more accurate solution for that case will then be as follows:

Subtract the square of each from unity; then we have

$$\sin^2 t = \frac{\tan^2\delta - \tan^2\varphi}{\tan^2\delta} = \frac{\sin(\delta+\varphi)\sin(\delta-\varphi)}{\cos^2\varphi\sin^2\delta}$$

$$\cos^2 A = \frac{\cos^2\varphi - \cos^2\delta}{\cos^2\varphi} = \frac{\sin(\delta+\varphi)\sin(\delta-\varphi)}{\cos^2\varphi}$$

$$\sin^2\zeta = \frac{\sin^2\delta - \sin^2\varphi}{\sin^2\delta} = \frac{\sin(\delta+\varphi)\sin(\delta-\varphi)}{\sin^2\delta}$$

Hence if we put

$$k = \sqrt{[\sin(\delta+\varphi)\sin(\delta-\varphi)]}$$

we shall have

$$\sin t = \frac{k}{\cos\varphi\sin\delta} \qquad \cos A = \frac{k}{\cos\varphi} \qquad \sin\zeta = \frac{k}{\sin\delta} \tag{24}$$

19. *To find the hour angle, zenith distance, and parallactic angle of a given star on the prime vertical of a given place.*

In this case, the point O in Fig. 1 being in the circle WZE, the angle PZO is 90°, and the right triangle PZO gives

$$\left.\begin{aligned}\cos t &= \frac{\tan\delta}{\tan\varphi}\\ \cos\zeta &= \frac{\sin\delta}{\sin\varphi}\\ \sin q &= \frac{\cos\varphi}{\cos\delta}\end{aligned}\right\} \tag{25}$$

If δ is but little less than φ, the star will be near the zenith, and, as in the preceding article, we shall obtain a more accurate solution as follows:

Put

$$k = \sqrt{[\sin (\varphi + \delta) \sin (\varphi - \delta)]}$$

then

$$\sin t = \frac{k}{\sin \varphi \cos \delta} \qquad \sin \zeta = \frac{k}{\sin \varphi} \qquad \cos q = \frac{k}{\cos \delta} \qquad (26)$$

We may also deduce the following convenient and accurate formulæ for the case where the star's declination is nearly equal to the latitude [see Sph. Trig. Arts. 60, 61, 62]:

$$\left.\begin{aligned} \tan \tfrac{1}{2} t &= \sqrt{\left(\frac{\sin (\varphi - \delta)}{\sin (\varphi + \delta)}\right)} \\ \tan \tfrac{1}{2} \zeta &= \sqrt{\left(\frac{\tan \frac{1}{2} (\varphi - \delta)}{\tan \frac{1}{2} (\varphi + \delta)}\right)} \\ \tan (45^\circ - \tfrac{1}{2} q) &= \sqrt{[\tan \tfrac{1}{2} (\varphi + \delta) \tan \tfrac{1}{2} (\varphi - \delta)]} \end{aligned}\right\} \qquad (27)$$

If $\delta > \varphi$, these values become imaginary; that is, the star cannot cross the prime vertical.

Example.—Required the hour angle and zenith distance of the star 12 *Canum Venaticorum* ($\delta = + 39^\circ\ 5'\ 20''$) when on the prime vertical of Cincinnati ($\varphi = + 39^\circ\ 5'\ 54''$).

$\varphi - \delta =$	$0^\circ\ 0'\ 34''$	$\frac{1}{2}(\varphi - \delta) =$	$0^\circ\ 0'\ 17''$
$\varphi + \delta =$	78 11 14	$\frac{1}{2}(\varphi - \delta) =$	39 5 37
$\log \sin (\varphi - \delta)$	6.21705	$\log \tan \frac{1}{2}(\varphi - \delta)$	5.91602
$\log \sin (\varphi + \delta)$	9.99070	$\log \tan \frac{1}{2}(\varphi + \delta)$	9.90982
	2)6.22635		2)6.00620
$\log \tan \frac{1}{2} t$	8.11318	$\log \tan \frac{1}{2} \zeta$	8.00310
$\frac{1}{2} t =$	$0^\circ\ 44'\ 36''.6$	$\frac{1}{2} \zeta =$	$0^\circ\ 34'\ 37''.3$
$t =$	$1^\circ\ 29'\ 13''.2$	$\zeta =$	$1^\circ\ 9'\ 14''.6$
$=$	$0^h\ 5^m\ 56^s.88$		

20. *To find the amplitude and hour angle of a given star when in the horizon.*—If the star is at H, Fig. 1, we have in the triangle PHN, right-angled at N, $PN = \varphi$, $HPN = 180^\circ - t$, $PH = 90^\circ - \delta$; and if the amplitude WH is denoted by a, we have $HN = 90^\circ - a$. This triangle gives, therefore,

$$\left.\begin{aligned} \sin a &= \sec \varphi \sin \delta \\ \cos t &= - \tan \varphi \tan \delta \end{aligned}\right\} \qquad (28)$$

21. *Given the hour angle* (t) *of a star, to find its right ascension* (α). —Transformation from our second system of co-ordinates to the third.

There must evidently be given also the position of the meridian with reference to the origin of right ascensions. Suppose then in Fig. 1 we know the right ascension of the meridian, or $VQ = \Theta$, then we have $VD = VQ - DQ$, that is,

$$\alpha = \Theta - t$$

Conversely, if α and Θ are known, we have

$$t = \Theta - \alpha$$

The methods of finding Θ at a given time will be considered hereafter.

22. *Given the zenith distance of a known star at a given place, to find the star's hour angle, azimuth, and parallactic angle.*

In this case there are given in the triangle POZ, Fig. 1, the three sides $ZO = \zeta$, $PO = 90° - \delta$, $PZ = 90° - \varphi$, to find the angles $ZPO = t$, $PZO = 180° - A$, and $POZ = q$. The formula for computing an angle B of a spherical triangle ABC, whose sides are a, b, c, is either

$$\sin \tfrac{1}{2} B = \sqrt{\left(\frac{\sin (s - a) \sin (s - c)}{\sin a \sin c}\right)}$$

$$\cos \tfrac{1}{2} B = \sqrt{\left(\frac{\sin s \sin (s - b)}{\sin a \sin c}\right)}$$

or $$\tan \tfrac{1}{2} B = \sqrt{\left(\frac{\sin (s - a) \sin (s - c)}{\sin s \sin (s - b)}\right)}$$

in which $s = \frac{1}{2} (a + b + c)$. We have then only to suppose B to represent one of the angles of our astronomical triangle, and to substitute the above corresponding values of the sides, to obtain the required solution.

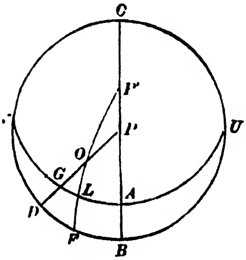
This substitution will be carried out hereafter in those cases where the problem is practically applied.

23. *Given the declination* (δ) *and the right ascension* (α) *of a star, and the obliquity of the ecliptic* (ε), *to find the latitude* (β) *and the longitude* (λ) *of the star.*—Transformation from the third system of co-ordinates to the fourth.

The solution of this problem is similar to that of Art. 10.

The analogy of the two will be more apparent if we here represent the sphere projected on the plane of the equator as in Fig. 6, where $VBUC$ is the equator, P its pole; VAU the ecliptic, P' its pole, and consequently $CP'PB$ the solstitial colure; POD, $P'OL$, circles of declination and latitude drawn through the star O. Since the angle which two great circles make with each other is equal to the angular distance of their poles, we have $PP' = \varepsilon$; and since the angle $P'PO$ is measured by CD and $PP'O$ by AL, we have in the triangle $PP'O$.

Fig. 6.



$$
\begin{array}{ccccc}
P'PO, & PP'O, & P'O, & PO, & PP' \\
90^\circ + \alpha, & 90^\circ - \lambda, & 90^\circ - \beta, & 90^\circ - \delta, & \varepsilon
\end{array}
$$

which, substituted respectively for

$$
\begin{array}{ccccc}
\mathrm{A}, & \mathrm{B} & a, & b, & c,
\end{array}
$$

in the general equations ($\mathfrak{A}$), Art. 10, give

$$
\left.
\begin{aligned}
\sin\beta &= \cos\varepsilon\sin\delta - \sin\varepsilon\cos\delta\sin\alpha \\
\cos\beta\sin\lambda &= \sin\varepsilon\sin\delta + \cos\varepsilon\cos\delta\sin\alpha \\
\cos\beta\cos\lambda &= \cos\delta\cos\alpha
\end{aligned}
\right\} \quad (29)
$$

which are the required formulæ of transformation. Adapting for logarithmic computation, we have

$$
\left.
\begin{aligned}
m\sin M &= \sin\delta \\
m\cos M &= \cos\delta\sin\alpha \\
\sin\beta &= m\sin(M - \varepsilon) \\
\cos\beta\sin\lambda &= m\cos(M - \varepsilon) \\
\cos\beta\cos\lambda &= \cos\delta\cos\alpha
\end{aligned}
\right\} \quad (30)
$$

in which m is a positive number.

A still more convenient form is obtained by substituting

$$
k = \frac{m}{\cos\delta} \qquad\qquad k' = \frac{\cos\beta}{m}
$$

by which we find

$$\left.\begin{array}{rl} k\sin M &= \tan\delta \\ k\cos M &= \sin\alpha \\ k'\sin\lambda &= \cos(M-\epsilon) \\ k'\cos\lambda &= \cos M\cot\alpha \\ \tan\beta &= \sin\lambda\tan(M-\epsilon) \\ \textit{For verification:}\quad \dfrac{\cos\beta\sin\lambda}{\cos\delta\sin\alpha} &= \dfrac{\cos(M-\epsilon)}{\cos M} \end{array}\right\} \quad (31)$$

EXAMPLE.—Given δ, α, and ϵ as below, to find β and λ. Computation by (31).

$\delta =$	$-16°\ 22'\ 35''.45$	$\log\sin\lambda$	$n8.0897286$
$\alpha =$	$6\ \ 33\ \ 29\ .30$	$\log\tan(M-\epsilon)$	1.4114658
$\epsilon =$	$23\ \ 27\ \ 31\ .72$	$\log\tan\beta$	$n9.5011944$
$\log\tan\delta = \log k\sin M$	$n9.4681562$	$\beta =$	$-17°\ 35'\ 37''.51$
$\log\sin\alpha = \log k\cos M$	9.0577093		
$M =$	$-68°\ 45'\ 41''.87$	*Verification.*	
$M-\epsilon =$	$-92\ \ 13\ \ 13\ .59$	$\log\cos\beta\sin\lambda$	$n8.0689234$
		$\log\cos\delta\sin\alpha$	9.0397224
$\log\cos M$	9.5590070	$\log\dfrac{\cos(M-\epsilon)}{\cos M}$	$n9.0292010$
$\log\cot\alpha$	0.9394396		
$\log k'\cos\lambda$	0.4984466		
$\log\cos(M-\epsilon) = \log k'\sin\lambda$	$n8.5882080$		
$\lambda =$	$359°\ 17'\ 43''.91$		

Tables for facilitating the above transformation, based upon the same method as that employed in Art. 16, are given in the American Ephemeris* and Berlin Jahrbuch. The formulæ there used may be obtained from Fig. 6, in which the points F and G are used precisely as in Fig. 4 of Art. 16.

24. If we denote the angle at the star, or $P'OP$, by $90° - E$, the solution of the preceding problem by Gauss's Equations is

$$\left.\begin{array}{l} \cos(45° - \frac{1}{2}\beta)\sin\frac{1}{2}(E+\lambda) = \sin[45° - \frac{1}{2}(\epsilon-\delta)]\sin(45° + \frac{1}{2}\alpha) \\ \cos(45° - \frac{1}{2}\beta)\cos\frac{1}{2}(E+\lambda) = \cos[45° - \frac{1}{2}(\epsilon+\delta)]\cos(45° + \frac{1}{2}\alpha) \\ \sin(45° - \frac{1}{2}\beta)\sin\frac{1}{2}(E-\lambda) = \sin[45° - \frac{1}{2}(\epsilon+\delta)]\cos(45° + \frac{1}{2}\alpha) \\ \sin(45° - \frac{1}{2}\beta)\cos\frac{1}{2}(E-\lambda) = \cos[45° - \frac{1}{2}(\epsilon-\delta)]\sin(45° + \frac{1}{2}\alpha) \end{array}\right\} (32)$$

25. If the angle at the star is required when the Gaussian Equations have not been employed, we have from the triangle POP', Fig. 6, putting $P'OP = \eta = 90° - E$,

$$\cos\beta\cos\eta = \cos\varepsilon\cos\delta + \sin\varepsilon\sin\delta\sin\alpha$$
$$\cos\beta\sin\eta = \sin\varepsilon\cos\alpha$$

or, adapted for logarithms,

$$\left.\begin{aligned} n\sin N &= \sin\varepsilon\sin\alpha \\ n\cos N &= \cos\varepsilon \\ \cos\beta\cos\eta &= n\cos(N-\delta) \\ \cos\beta\sin\eta &= \sin\varepsilon\cos\alpha \end{aligned}\right\} \qquad (33)$$

26. *Given the latitude (β) and longitude (λ) of a star, and the obliquity of the ecliptic (ε), to find the declination and right ascension of the star.*

By the process already employed, we derive from the triangle $PP'O$, Fig. 6, for this case,

$$\left.\begin{aligned} \sin\delta &= \cos\varepsilon\sin\beta + \sin\varepsilon\cos\beta\sin\lambda \\ \cos\delta\sin\alpha &= -\sin\varepsilon\sin\beta + \cos\varepsilon\cos\beta\sin\lambda \\ \cos\delta\cos\alpha &= \cos\beta\cos\lambda \end{aligned}\right\} \qquad (34)$$

which, it will be observed, may be obtained from (29) by interchanging α with λ, and δ with β, and at the same time changing the sign of ε, that is, putting $-\varepsilon$ for ε, and, consequently, $-\sin\varepsilon$ for $\sin\varepsilon$.

For logarithmic computation, we have

$$\left.\begin{aligned} m\sin M &= \sin\beta \\ m\cos M &= \cos\beta\sin\lambda \\ \sin\delta &= m\sin(M+\varepsilon) \\ \cos\delta\sin\alpha &= m\cos(M+\varepsilon) \\ \cos\delta\cos\alpha &= \cos\beta\cos\lambda \end{aligned}\right\} \qquad (35)$$

or the following, analogous to (31):

$$\left.\begin{aligned} k\sin M &= \tan\beta \\ k\cos M &= \sin\lambda \\ k'\sin\alpha &= \cos(M+\varepsilon) \\ k'\cos\alpha &= \cos M\cot\lambda \\ \tan\delta &= \sin\alpha\tan(M+\varepsilon) \\ \text{For verification: } \frac{\cos\delta\sin\alpha}{\cos\beta\sin\lambda} &= \frac{\cos(M+\varepsilon)}{\cos M} \end{aligned}\right\} \qquad (36)$$

27. The angle at the star, POP', being denoted, as in Art. 24,

by $90° - E$, the solution of this problem by the Gaussian Equations is

$$\left.\begin{array}{l}\sin(45°-\tfrac{1}{2}\delta)\sin\tfrac{1}{2}(E+\alpha)=\sin[45°-\tfrac{1}{2}(\varepsilon+\beta)]\sin(45°+\tfrac{1}{2}\lambda)\\ \sin(45°-\tfrac{1}{2}\delta)\cos\tfrac{1}{2}(E+\alpha)=\cos[45°-\tfrac{1}{2}(\varepsilon-\beta)]\cos(45°+\tfrac{1}{2}\lambda)\\ \cos(45°-\tfrac{1}{2}\delta)\sin\tfrac{1}{2}(E-\alpha)=\sin[45°-\tfrac{1}{2}(\varepsilon-\beta)]\cos(45°+\tfrac{1}{2}\lambda)\\ \cos(45°-\tfrac{1}{2}\delta)\cos\tfrac{1}{2}(E-\alpha)=\cos[45°-\tfrac{1}{2}(\varepsilon+\beta)]\sin(45°+\tfrac{1}{2}\lambda)\end{array}\right\} \quad (37)$$

28. But if the angle $\eta = 90° - E$ is required when the Gaussian Equations have not been employed, we have directly

$$\cos\delta\cos\eta = \cos\varepsilon\cos\beta - \sin\varepsilon\sin\beta\sin\lambda$$
$$\cos\delta\sin\eta = \sin\varepsilon\cos\lambda$$

or, adapted for logarithms,

$$\left.\begin{array}{r}n\sin N = \sin\varepsilon\sin\lambda\\ n\cos N = \cos\varepsilon\\ \cos\delta\cos\eta = n\cos(N+\beta)\\ \cos\delta\sin\eta = \sin\varepsilon\cos\lambda\end{array}\right\} \quad (38)$$

29. *For the sun*, we may, except when extreme precision is desired, put $\beta = 0$, and the preceding formulæ then assume very simple forms. Thus, if in (34) we put $\sin\beta = 0$, $\cos\beta = 1$, we find

$$\sin\delta = \sin\varepsilon\sin\lambda$$
$$\cos\delta\sin\alpha = \cos\varepsilon\sin\lambda$$
$$\cos\delta\cos\alpha = \cos\lambda$$

whence if any two of the four quantities δ, α, λ, ε are given, we can deduce the other two.

RECTANGULAR CO-ORDINATES.

30. By means of spherical co-ordinates we have expressed only a star's *direction.* To define its position in space completely, another element is necessary, namely, its *distance.* In Spherical Astronomy we consider this element of distance only so far as may be necessary in determining the changes of apparent direction of a star resulting from a change in the point from which it is viewed. For this purpose the rectangular co-ordinates of analytical geometry may be employed.

Three planes of reference are taken at right angles to each other, their common intersection, or origin, being the point of

observation: and the star's distances from these planes are denoted by x, y, and z respectively. These co-ordinates are respectively parallel to the three axes (or mutual intersections of the planes, taken two and two), and hence these axes are called, respectively, the axis of x, the axis of y, and the axis of z. The planes are distinguished by the axes they contain, as "the plane of xy," the "plane of xz," the "plane of yz." The co-ordinates may be conceived to be measured on the axes to which they belong, from the origin, in two opposite directions, distinguished by the algebraic signs of *plus* and *minus*, so that the numerical values of the co-ordinates of a star, together with their algebraic signs, fully determine the position of the star in space without ambiguity.

Of the eight solid angles formed by the planes of reference, that in which a star is placed will always be known by the signs of the three co-ordinates, and in one only of these angles will the three signs all be *plus*. This angle is usually called the *first angle*. To simplify the investigations of a problem, we may, if we choose, assume all the points considered to lie in the first angle, and then treat the equations obtained for this simplest case as entirely general; for, by the principles of analytical geometry, negative values of the co-ordinates which satisfy such equations also satisfy a geometrical construction in which these co-ordinates are drawn in the negative direction.

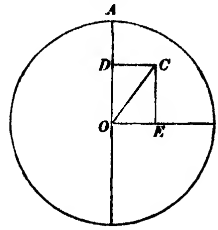
The *polar co-ordinates* of analytical geometry (of three dimensions) when applied to astronomy are nothing more than the spherical co-ordinates we have already treated of, combined with the element distance; and the formulæ of transformation from rectangular to polar co-ordinates are nothing more than the values of the rectangular co-ordinates in terms of the distance and the spherical co-ordinates. For the convenience of reference, we shall here recapitulate these formulæ, with special reference to our several systems of spherical co-ordinates.

31. We shall find it useful to premise the following

LEMMA.—*The distance of a point in space from the plane of any great circle of the celestial sphere is equal to its distance from the centre of the sphere multiplied by the cosine of its angular distance from the pole of that circle; and its distance from the axis of the circle is equal to its distance from the centre of the sphere multiplied by the sine of its angular distance from the pole.*

For, let AB, Fig. 7, be the given great circle orthographically projected upon a plane passing through its axis OP and the given point C; P its pole. The distance of the point C from the plane of the great circle is the perpendicular CD; CE is its distance from the axis; CO its distance from the centre of the sphere; and the angle COP the angular distance from the pole. The truth of the Lemma is, therefore, obvious from the figure.

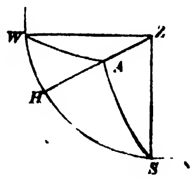
Fig. 7.



32. The values of the rectangular co-ordinates in our several systems may be found as follows:

First system.—Altitude and azimuth.—Let the primitive plane, or that of the horizon, be the plane of xy; that of the meridian, the plane of xz; that of the prime vertical, the plane of yz. The meridian line is then the axis of x; the east and west line, the axis of y; and the vertical line, the axis of z. Positive x will be reckoned from the origin towards the south, positive y towards the west, and positive z towards the zenith. The *first angle*, or angle of positive values, is therefore the southwest quarter of the hemisphere above the plane of the horizon. Let Z, Fig. 8, be the zenith, S the south point, W the west point of the horizon. These points are respectively the poles of the three great circles of reference; if, then, A is the position of a star on the surface of the sphere as seen from the centre of the earth, and if we put

Fig. 8.



h = altitude of the star = AH,
A = azimuth " = SH,
Δ = its distance from the centre of the sphere

we have immediately, by the preceding Lemma,

$$x = \Delta \cos AS, \qquad y = \Delta \cos AW, \qquad z = \Delta \cos AZ,$$

which, by considering the right triangles AHS, AHW, become

$$\left.\begin{aligned} x &= \Delta \cos h \cos A \\ y &= \Delta \cos h \sin A \\ z &= \Delta \sin h \end{aligned}\right\} \qquad (39)$$

These equations determine the rectangular co-ordinates x, y, z.

when the polar co-ordinates Δ, h, A are given. Conversely, if x, y, and z are given, we may find Δ, h, and A; for the first two equations give

$$\tan A = \frac{y}{x}$$

and then we have

$$\Delta \sin h = z$$

$$\Delta \cos h = \frac{x}{\cos A} = \frac{y}{\sin A}$$

whence Δ and h. Or, by adding the squares of the first two equations, we have

$$\Delta \cos h = \sqrt{x^2 + y^2}$$

whence

$$\tan h = \frac{z}{\sqrt{(x^2 + y^2)}}$$

and the sum of the squares of the three equations gives

$$\Delta = \sqrt{(x^2 + y^2 + z^2)}$$

Second system.—Declination and hour angle.—Let the plane of the equator be the plane of xy; that of the meridian, the plane of xz; that of the six hour circle, the plane of yz. In the preceding figure, let Z now denote the north pole, S that point of the equator which is on the meridian above the horizon and from which hour angles are reckoned, W the west point. Positive x will be reckoned towards S, positive y towards the west, positive z towards the north. If then A is the place of a star on the sphere as seen from the centre, and we put

δ = the star's declination = AH,
t = " hour angle = SH,
Δ = " distance from the centre,

and denote the rectangular co-ordinates in this case by x', y', z', we have

$$\left.\begin{aligned} x' &= \Delta \cos \delta \cos t \\ y' &= \Delta \cos \delta \sin t \\ z' &= \Delta \sin \delta \end{aligned}\right\} \qquad (40)$$

Third system.—Declination and right ascension.—Let the plane of the equator be the plane of xy; that of the equinoctial colure, the plane of xz; that of the solstitial colure, the plane of yz.

The axis of x is the intersection of the planes of the equator and equinoctial colure, positive towards the vernal equinox; the axis of y is the intersection of the planes of the equator and solstitial colure, positive towards that point whose right ascension is $+90°$; and the axis of z is the axis of the equator, positive towards the north. If then, in Fig. 8, Z is the north pole, W the vernal equinox, A a star in the first angle, projected upon the celestial sphere, and we put

$$\begin{aligned}\delta &= \text{declination of the star} = AH,\\ \alpha &= \text{right ascension} \quad \text{"} \quad = WH,\\ \Delta &= \text{distance from the centre,}\end{aligned}$$

while x'', y'', z'' denote the rectangular co-ordinates, we have

$$x'' = \Delta \cos AW, \qquad y'' = \Delta \cos AS, \qquad z'' = \Delta \cos AZ,$$

which become

$$\left.\begin{aligned}x'' &= \Delta \cos \delta \cos \alpha\\ y'' &= \Delta \cos \delta \sin \alpha\\ z'' &= \Delta \sin \delta\end{aligned}\right\} \qquad (41)$$

Fourth system.—Celestial latitude and longitude.—Let the plane of the ecliptic be the plane of xy; the plane of the circle of latitude passing through the equinoctial points, the plane of xz; the plane of the circle of latitude passing through the solstitial points, the plane of yz. The positive axis of x is here also the straight line from the centre towards the vernal equinox; the positive axis of y is the straight line from the centre towards the north solstitial point, or that whose longitude is $+90°$; and the positive axis of z is the straight line from the centre towards the north pole of the ecliptic.

If then, in Fig. 8, Z now denotes the north pole of the ecliptic, W the vernal equinox, A the star's place on the sphere, and we put

$$\begin{aligned}\beta &= \text{latitude of the star} = AH,\\ \lambda &= \text{longitude of the star} = WH,\\ \Delta &= \text{distance of the star from the centre,}\end{aligned}$$

and x''', y''', z''', denote the rectangular co-ordinates for this system, we have

$$\left.\begin{aligned}x''' &= \Delta \cos \beta \cos \lambda\\ y''' &= \Delta \cos \beta \sin \lambda\\ z''' &= \Delta \sin \beta\end{aligned}\right\} \qquad (42)$$

TRANSFORMATION OF RECTANGULAR CO-ORDINATES.

33. For the purposes of Spherical Astronomy, only the most simple cases of the general transformations treated of in analytical geometry are necessary. We mostly consider but two cases:

First. Transformation of rectangular co-ordinates to a new origin, without changing the system of spherical co-ordinates.

The general planes of reference which have been used in this chapter may be supposed to be drawn through any point in space without changing their directions, and therefore without changing the great circles of the infinite celestial sphere which represent them. We thus repeat the same *system* of spherical co-ordinates with various origins and different systems of rectangular co-ordinates, the planes of reference, however, remaining always parallel to the planes of the primitive system.

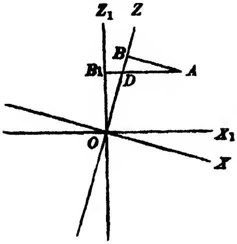
The transformation from one system of rectangular co-ordinates to a parallel system is evidently effected by the formulæ

$$\left.\begin{aligned} x_1 &= x_2 + a \\ y_1 &= y_2 + b \\ z_1 &= z_2 + c \end{aligned}\right\} \qquad (48)$$

in which x_1, y_1, z_1 are the co-ordinates of a point in the primitive system; x_2, y_2, z_2 the co-ordinates of the same point in the new system; and a, b, c are the co-ordinates of the new origin taken in the first system.

As we have shown how to express the values of x_1, y_1, z_1 and of x_2, y_2, z_2 in terms of the spherical co-ordinates, we have only to substitute these values in the preceding formulæ to obtain the general relations between the spherical co-ordinates corresponding to the two origins. This is, indeed, the most general method of determining the effect of *parallax*, as will appear hereafter.

Fig. 9.



Second. Transformation of rectangular co-ordinates when the system of spherical co-ordinates is changed but the origin is unchanged. This amounts to changing the directions of the axes. The cases which occur in practice are chiefly those in which the two systems have one plane in common. Suppose this plane is that of xz, and let OX, OZ, Fig. 9, be the axes of x and z in the first system; OX_1,

OZ_1, the axes of x_1 and z_1 in the new system. Let A be the projection of a point in space upon the common plane; and let $x = AB$, $z = OB$, $x_1 = AB_1$, $z_1 = OB_1$. The distance of the point from the common plane being unchanged, we have

$$y = y_1.$$

The axis of y may be regarded as an axis of revolution about which the planes of yx and yz revolve in passing from the first to the second system; and if u denotes the angular measure of this revolution, or $u = XOX_1 = ZOZ_1 = BAB_1$, we readily derive from the figure the equation

$$x \sec u = x_1 - z_1 \tan u$$

or, multiplying by $\cos u$,

$$x = x_1 \cos u - z_1 \sin u$$

and

$$z = x \tan u + z_1 \sec u$$

or, substituting in this the preceding value of x,

$$z = x_1 \sin u + z_1 \cos u$$

Thus, to pass from the first to the second system, we have the formulæ

$$\left.\begin{aligned} x &= x_1 \cos u - z_1 \sin u \\ y &= y_1 \\ z &= x_1 \sin u + z_1 \cos u \end{aligned}\right\} \quad (44)$$

And to pass from the second to the first, we obtain with the same ease,

$$\left.\begin{aligned} x_1 &= x \cos u + z \sin u \\ y_1 &= y \\ z_1 &= -x \sin u + z \cos u \end{aligned}\right\} \quad (45)$$

As an example, let us apply these to transforming from our second system of spherical co-ordinates to the first; that is, from declination and hour angle to altitude and azimuth. The meridian is the common plane; the axis of z in the system of declination and hour angle is the axis of the equator, and the axis of z_1 in the system of altitude and azimuth is the vertical line; the angle between these axes is the complement of the latitude, or $u = 90° - \varphi$. Substituting this value of u in (44), and also the values of x, y, z, x_1, y_1, z_1, given by (39) and (40), we have, after omitting the common factor Δ,

$$\begin{aligned}\cos h \cos A &= \sin \varphi \cos \delta \cos t - \cos \varphi \sin \delta \\ \cos h \sin A &= \cos \delta \sin t \\ \sin h &= \cos \varphi \cos \delta \cos t + \sin \varphi \sin \delta\end{aligned}$$

which agree with (14). We see that when the element of distance is left out of view (as it must necessarily be when the origin is not changed), the transformation by means of rectangular co-ordinates leads to the same forms as the direct application of Spherical Trigonometry. With regard to the entire generality of these formulæ in their application to angles of all possible magnitudes, see Sph. Trig. Chap. IV.

DIFFERENTIAL VARIATIONS OF CO-ORDINATES.

34. It is often necessary in practical astronomy to determine what effect given variations of the data will produce in the quantities computed from them. Where the formulæ of computation are derived directly from a spherical triangle, we can employ for this purpose the equations of *finite differences* [Sph. Trig. Chap. VI.] if we wish to obtain rigorously exact relations, or the simpler differential equations if the variations considered are extremely small. As the latter case is very frequent, I shall deduce here the most useful differential formulæ, assuming as well known the fundamental ones [Sph. Trig. Art. 153],

$$\left.\begin{aligned} da - \cos C\, db - \cos B\, dc &= \sin b \sin C\, dA \\ -\cos C\, da + db - \cos A\, dc &= \sin c \sin A\, dB \\ -\cos B\, da - \cos A\, db + dc &= \sin a \sin B\, dC \end{aligned}\right\} \quad (46)$$

From these we obtain the following by eliminating da:

$$\left.\begin{aligned} \sin C\, db - \cos a \sin B\, dc &= \sin b \cos C\, dA + \sin a\, dB \\ -\cos a \sin C\, db + \sin B\, dc &= \sin c \cos B\, dA + \sin a\, dC \end{aligned}\right\} \quad (47)$$

and by eliminating db from these:

$$\sin a \sin B\, dc = \cos b\, dA + \cos a\, dB + dC \quad (48)$$

If we eliminate dA from (47), we find

$$\cos b \sin C\, db - \cos c \sin B\, dc = \sin c \cos B\, dB - \sin b \cos C\, dC$$

the terms of which being divided either by $\sin b \sin C$, or by its equivalent $\sin c \sin B$, we obtain

$$\cot b\, db - \cot c\, dc = \cot B\, dB - \cot C\, dC \quad (49)$$

35. As an example, take the spherical triangle formed by the zenith, the pole, and a star, Art. 10, and put

$$\begin{aligned} \mathrm{A} &= 180^\circ - A & a &= 90^\circ - \delta \\ \mathrm{B} &= t & b &= \zeta \\ \mathrm{C} &= q & c &= 90^\circ - \varphi \end{aligned}$$

then the first equations of (46) and (47) give

$$\left.\begin{aligned} d\delta &= -\cos q\, d\zeta + \sin q \sin \zeta\, dA + \cos t\, d\varphi \\ \cos \delta\, dt &= \sin q\, d\zeta + \cos q \sin \zeta\, dA + \sin \delta \sin t\, d\varphi \end{aligned}\right\} \quad (50)$$

which determine the errors $d\delta$ and dt in the values of δ and t computed according to the formulæ (4), (5), and (6), when ζ, A, and φ are affected by the small errors $d\zeta$, dA, and $d\varphi$ respectively.

In a similar manner we obtain

$$\left.\begin{aligned} d\zeta &= -\cos q\, d\delta + \sin q \cos \delta\, dt + \cos A\, d\varphi \\ \sin \zeta\, dA &= \sin q\, d\delta + \cos q \cos \delta\, dt - \cos \zeta \sin A\, d\varphi \end{aligned}\right\} \quad (51)$$

which determine the errors $d\zeta$ and dA in the values of ζ and A computed by (14), when δ, t, and φ are affected by the small errors $d\delta$, dt, and $d\varphi$ respectively.

36. As a second example, take the triangle formed by the pole of the equator, the pole of the ecliptic, and a star, Art. 23. Denoting the angle at the star by η, we find

$$\left.\begin{aligned} d\beta &= \cos \eta\, d\delta - \sin \eta \cos \delta\, d\alpha - \sin \lambda\, d\varepsilon \\ \cos \beta\, d\lambda &= \sin \eta\, d\delta + \cos \eta \cos \delta\, d\alpha + \sin \beta \cos \lambda\, d\varepsilon \end{aligned}\right\} \quad (52)$$

and reciprocally,

$$\left.\begin{aligned} d\delta &= \cos \eta\, d\beta + \sin \eta \cos \beta\, d\lambda + \sin \alpha\, d\varepsilon \\ \cos \delta\, d\alpha &= -\sin \eta\, d\beta + \cos \eta \cos \beta\, d\lambda - \sin \delta \cos \alpha\, d\varepsilon \end{aligned}\right\} \quad (53)$$

CHAPTER II.

TIME—USE OF THE EPHEMERIS—INTERPOLATION—STAR CATALOGUES.

37. Transit.—The instant when any point of the celestial sphere is on the meridian of an observer is designated as the *transit* of that point over the meridian; also the *meridian passage*, and *culmination*. In one complete revolution of the sphere about its axis, every point of it is twice on the meridian, at points which are 180° distant in right ascension. It is therefore necessary to distinguish between the two transits. The meridian is bisected at the poles of the equator: the transit over that half of the meridian which contains the observer's zenith is the *upper* transit, or culmination; that over the half of the meridian which contains the nadir is the *lower* transit, or culmination. At the upper transit of a point its hour angle is zero, or 0^h; at the lower transit, its hour angle is 12^h.

38. The motion of the earth about its axis is perfectly uniform. If, then, the axis of the earth preserved precisely the same direction in space, the apparent diurnal motion of the celestial sphere would also be perfectly uniform, and the intervals between the successive transits of any assumed point of the sphere would be perfectly equal. The effect of changes in the position of the earth's axis upon the transit of stars is most perceptible in the case of stars near the vanishing points of the axis, that is, near the poles of the heavens. We obtain a measure of time *sensibly* uniform by employing the successive transits of a point of the equator. The point most naturally indicated is the *vernal equinox* (also called the First point of Aries, and denoted by the symbol for Aries, ♈).

39. A *sidereal day* is the interval of time between two successive (upper) transits of the true vernal equinox over the same meridian.

The effect of precession and nutation upon the time of transit

of the vernal equinox is so nearly the same at two successive transits, that sidereal days thus defined are *sensibly* equal. (See Chapter XI. Art. 411.)

The *sidereal time* at any instant is the hour angle of the vernal equinox at that instant, reckoned from the meridian westward from 0^h to 24^h.

When ♈ is on the meridian, the sidereal time is $0^h\ 0^m\ 0^s$; and this instant is sometimes called *sidereal noon.*

40. A *solar day* is the interval of time between two successive upper transits of the sun over the same meridian.

The *solar time* at any instant is the hour angle of the sun at that instant.

In consequence of the earth's motion about the sun from west to east, the sun appears to have a like motion among the stars, or to be constantly increasing its right ascension; and hence a solar day is longer than a sidereal day.

41. *Apparent and mean solar time.*—If the sun changed its right ascension uniformly, solar days, though not equal to sidereal days, would still be equal to each other. But the sun's motion in right ascension is not uniform, and this for two reasons:

1st. The sun does not move in the equator, but in the ecliptic, so that, even were the sun's motion in the ecliptic uniform, its equal changes of longitude would not produce equal changes of right ascension; 2d. The sun's motion in the ecliptic is not uniform.

To obtain a uniform measure of time depending on the sun's motion, the following method is adopted. A fictitious sun, which we shall call the *first mean sun*, is supposed to move uniformly at such a rate as to return to the perigee at the same time with the true sun. Another fictitious sun, which we shall call the *second mean sun* (and which is often called simply the mean sun), is supposed to move uniformly in the equator at the same rate as the first mean sun in the ecliptic, and to return to the vernal equinox at the same time with it. Then the time denoted by this second mean sun is perfectly uniform in its increase, and is called *mean time.*

The time which is denoted by the true sun is called the *true* or, more commonly, the *apparent time.*

The instant of transit of the true sun is called *apparent noon*, and the instant of transit of the second mean sun is called *mean noon.*

The *equation of time* is the difference between apparent and mean time; or, in other words, it is the difference between the hour angles of the true sun and the second mean sun. The greatest difference is about 16^m

The equation of time is also the difference between the right ascensions of the true sun and the second mean sun. The right ascension of the second mean sun is, according to the preceding definitions, equal to the longitude of the first mean sun, or, as it is commonly called, the sun's mean longitude. To compute the equation of time, therefore, we must know how to find the longitude of the first mean sun; and this is deduced from a knowledge of the true sun's apparent motion in the ecliptic, which belongs to Physical Astronomy. Here it suffices us that its value is given for each day of the year in the Ephemeris, or Nautical Almanac.

42. *Astronomical time.*—The solar day (apparent or mean) is conceived by astronomers to commence at noon (apparent or mean), and is divided into twenty-four hours, numbered successively from 0 to 24.

Astronomical time (apparent or mean) is, then, the hour angle of the sun (apparent or mean), reckoned on the equator *westward* throughout its entire circumference from 0^h to 24^h.

43. *Civil time.*—For the common purposes of life, it is more convenient to begin the day at midnight, that is, when the sun is on the meridian at its *lower transit*

The civil day is divided into two periods of twelve hours each, namely, from midnight to noon, marked A.M. (Ante Meridiem), and from noon to midnight, marked P.M. (Post Meridiem)

44. *To convert civil into astronomical time.*—The civil day begins 12^h *before* the astronomical day of the same date. This remark is the only precept that need be given for the conversion of one of these kinds of time into the other.

EXAMPLES.

Ast. T. May 10, 15^h = Civ. T. May 11, 3^h A.M.
Ast. T. Jan. 3, 7^h = Civ. T. Jan. 3, 7^h P.M.
Ast. T. Aug. 31, 20^h = Civ T. Sept. 1, 8^h A.M.

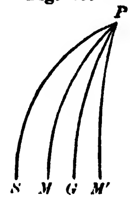
45. *Time at different meridians.*—The hour angle of the sun at any meridian is called the *local* (solar) time at that meridian.

The hour angle of the sun at the Greenwich meridian at the same instant is the corresponding *Greenwich time.* This time we shall have constant occasion to use, both because longitudes in this country and England are reckoned from the Greenwich meridian, and because the American and British Nautical Almanacs are computed for Greenwich time.*

The difference between the local time at any meridian and the Greenwich time is equal to the longitude of that meridian from Greenwich, expressed in time, observing that $1^h = 15°$.

The difference between the local times of any two meridians is equal to the difference of longitude of those meridians.

Fig. 10.



In comparing the corresponding times at two different meridians, the most easterly meridian may be distinguished as that at which the time is greatest; that is, latest.

If then PM, Fig. 10, is any meridian (referred to the celestial sphere), PG the Greenwich meridian, PS the declination circle through the sun, and if we put

$$
\begin{aligned}
T_0 &= \text{the Greenwich time} &= GPS,\\
T &= \text{the local time} &= MPS,\\
L &= \text{the west longitude of the meridian } PM &= GPM,
\end{aligned}
$$

we have

$$
\left.\begin{aligned}
L &= T_0 - T\\
T_0 &= T + L
\end{aligned}\right\} \qquad (54)
$$

If the given meridian were east of Greenwich, as PM', we should have its east longitude $= T - T_0$; but we prefer to use the general formula $L = T_0 - T$ in all cases, observing that *east longitudes are to be regarded as negative.*

In the formula (54), T_0 and T are supposed to be reckoned always westward from their respective meridians, and from 0^h to 24^h; that is, T_0 and T are the *astronomical times*, which should, of course, be used in all astronomical computations.

As in almost every computation of practical astronomy we are dependent for some of the data upon the ephemeris,—and these

* What we have to say respecting the Greenwich time is, however, equally applicable to the time at any other meridian for which the ephemeris may be computed.

are commonly given for Greenwich,—it is generally the first step in such a computation to deduce an exact or, at least, an approximate value of the Greenwich astronomical time. It need hardly be added that the Greenwich time should never be otherwise expressed than astronomically.*

Examples.

1. In Longitude 76° 32′ W. the local time is 1856 April 1, $9^h\ 3^m\ 10^s$ A.M.: what is the Greenwich time?

Local Ast. T. March 31,	21^h	3^m	10^s
Longitude	+ 5	6	8
Greenwich T. April 1,	26	9	18

2. In Long. 105° 15′ E. the local time is August 21, $4^h\ 3^m$ P.M; what is the Greenwich time?

Local Ast. T. Aug. 21,	4^h	3^m
Longitude	— 7	1
Greenwich T. Aug. 20,	21	2

3. Long. 175° 30′ W. Loc. T. Sept. 30, $8^h\ 10^m$ A.M. = G. T. Sept. 30, $7^h\ 52^m$.

4. Long. 165° 0′ E. Loc. T. Feb. 1, $7^h\ 11^m$ P.M. = G. T. Jan. 31, $20^h\ 11^m$.

5. Long. 64° 30′ E. Loc. T. June 1, 0^h M. (Noon) = G. M. T. May 31, $19^h\ 42^m$.

46. In nautical practice the observer is provided with a chronometer which is regulated to Greenwich time, before sailing, at a place whose longitude is well known. Its error on Greenwich time is carefully determined, as well as its daily gain or loss, that is, its *rate*, so that at any subsequent time the Greenwich time may be known from the indication of the chronometer corrected for its error and the accumulated rate since the date of sailing. As, however, the chronometer has usually only 12^h marked on the dial, it is necessary to distinguish whether it indicates A.M. or P.M. at Greenwich. This is always readily done by means of the observer's *approximate* longitude and local

* On this account, chronometers intended for nautical and astronomical purposes should always be marked from 0^h to 24^h, instead of from 0^h to 12^h as is now usual. It is surprising that navigators have not insisted upon this point.

time. As this is a daily operation at sea, it may be well to illustrate it by a few examples.

1. In the approximate longitude 5^h W. about 3^h P.M. on August 3, the Greenwich Chronometer marks $8^h\ 11^m\ 7^s$, and is fast on G. T. $6^m\ 10^s$; what is the Greenwich astronomical time?

Approx. Local T. Aug. 3,	3^h	Gr. Chronom.	8^h	11^m	7^s
" Longitude,	$+5$	Correction,		-6	10
Approx. G. T. Aug. 3,	8	Gr. Ast. T. Aug. 3,	8	4	57

2. In Long. 10^h E. about 1^h A.M. on Dec. 7, the Greenwich Chronometer marks $3^h\ 14^m\ 13^s.5$, and is fast $25^m\ 18^s.7$; what is the G. T.?

Approx. Local T. Dec. 6,	13^h	Gr. Chronom.	3^h	14^m	$13^s.5$
" Long.	-10	Correction,		-25	18.7
Approx. G. T. Dec. 6,	3	G. A. T. Dec. 6,	2	48	54.8

3. In Long. $9^h\ 12^m$ W. about 2^h A.M. on Feb. 13, the Gr. Chron. marks $10^h\ 27^m\ 13^s.3$, and is slow $30^m\ 30^s.3$; what is the G. T.?

Approx. Local T. Feb. 12,	14^h	Gr. Chronom.	10^h	37^m	$13^s.3$
" Long.	$+9$	Correction.		$+30$	30.3
Approx. G. T. Feb. 12,	23	G. A. T. Feb. 12,	23	7	43.6

The computation of the approximate Greenwich time may, of course, be performed mentally.

47. The formula (54), $L = T_0 - T$, is true not only when T_0 and T are solar times, but also when they are any kinds of time whatever, or, in general, when T_0 and T express the hour angles of any point whatever of the sphere at the two meridians whose difference of longitude is L. This is evident from Fig. 10, where S may be any point of the sphere.

48. *To convert the apparent time at a given meridian into the mean time, or the mean into the apparent time.*

If M = the mean time,
A = the corresponding apparent time.
E = the equation of time,

we have

$$M = A + E$$

or

$$A = M - E$$

in which E is to be regarded as a positive quantity when it is *additive to apparent time.* The value of E is to be taken from the Nautical Almanac for the Greenwich instant corresponding to the given local time. If apparent time is given, find the Gr. apparent time and take E from page I of the month in the Nautical Almanac; if mean time is given, find the Gr. mean time and take E from page II of the month.

EXAMPLE 1.—In longitude 60° W., 1856 May 24, $3^h\ 12^m\ 10^s$ P.M., apparent time; what is the mean time?

We have first

Local time May 24,	$3^h\ 12^m\ 10^s$
Longitude,	4 0 0
Gr. app. time May 24,	7 12 10

We must, therefore, find E for the Gr. time, May 24, $7^h\ 12^m\ 10^s$, or $7^h.21$. By the Nautical Almanac for 1856, we have E at apparent Greenwich noon May 24 $= -3^m\ 25^s.43$, and the hourly difference $+ 0^s.224$. Hence at the given time

$$E = -3^m\ 25^s.43 + 0^s.224 \times 7.21 = -3^m\ 23^s.81$$

and the required mean time is

$$M = 3^h\ 12^m\ 10^s - 3^m\ 23^s.81 = 3^h\ 8^m\ 46^s.19.$$

EXAMPLE 2.—In longitude 60° W., 1856 May 24, $3^h\ 8^m\ 46^s.19$ mean time; what is the apparent time?

Gr. mean time, May 24, $7^h\ 8^m\ 46^s.19$ $(= 7^h.15)$

E at mean noon May 24	$= -\ 3^m\ 25^s.41$	Hourly diff.	$= 0^s.224$
Correction for $7^h.15$	$= \quad + 1.60$		7.15
	$E = -\ 3\ \ 23.81$		1.60

and hence

$$\begin{aligned} M &= 3^h\ 8^m\ 46^s.19 \\ -E &= +\ 3\ \ 23.81 \\ A &= 3\ \ 12\ \ 10.00 \end{aligned}$$

As the equation of time is not a uniformly varying quantity, it is not quite accurate to compute its correction as above, by multiplying the given hourly difference by the number of hours in the Greenwich time, for that process assumes that this hourly difference is the same for each hour. The variations in the hourly difference are, however, so small that it is only when

extreme precision is required that recourse must be had to the more exact method of interpolation which will be given hereafter.

49. *To determine the relative length of the solar and sidereal units of time.*

According to BESSEL, the length of the tropical year (which is the interval between two successive passages of the sun through the mean vernal equinox) is 365.24222 mean solar days;* and since in this time the mean sun has described the whole arc of the equator included between the two positions of the equinox, it has made one transit less over any given meridian than the vernal equinox; so that we have

$$366.24222 \text{ sidereal days} = 365.24222 \text{ mean solar days}$$

whence we deduce

$$1 \text{ sid. day} = \frac{365.24222}{366.24222} \text{ sol. day} = 0.99726957 \text{ sol. day}$$

or

$$24^h \text{ sid. time} = 23^h\ 56^m\ 4^s.091 \text{ solar time}$$

Also,

$$1 \text{ sol. day} = \frac{366.24222}{365.24222} \text{ sid. day} = 1.00273791 \text{ sid. day}$$

or

$$24^h \text{ sol. time} = 24^h\ 3^m\ 56^s.555 \text{ sid. time}$$

If we put

$$\mu = \frac{366.24222}{365.24222} = 1.00273791$$

and denote by I an interval of mean solar time, by I' the equivalent interval of sidereal time, we always have

$$\left.\begin{aligned} I' &= \mu I = I + (\mu - 1)\, I = I + .00273791\, I \\ I &= \frac{I'}{\mu} = I' - \left(1 - \frac{1}{\mu}\right) I' = I' - .00273043\, I' \end{aligned}\right\} \quad (55)$$

Tables are given in the Nautical Almanacs to save the labor of computing these equations. In some of these tables, for each solar interval I there is given the equivalent sidereal interval $I' = \mu I$, and reciprocally: in others there are given the correction to be added to I to find I' (*i.e.* the correction $.00273791\, I$),

* The length of the tropical year is not absolutely constant. The value given in the text is for the year 1800 Its decrease in 100 years is about $0^s.6$ (Art. 407).

and the correction to be subtracted from I' to find I (*i.e.* the correction $.00273043\, I'$). The latter form is the most convenient, and is adopted in the American Ephemeris. The correction $(\mu - 1)\, I$ is frequently called the *acceleration of the fixed stars* (relatively to the sun). The daily acceleration is $3^m\ 56^s.555$.

50. ***To convert the mean solar time at a given meridian into the corresponding sidereal time.***

In Fig. 1, page 25, if PQ is the given meridian, VQ the equator, D the mean sun, V the vernal equinox, and if we put

$$\begin{aligned} T &= DQ = \text{the mean solar time,} \\ \Theta &= VQ = \text{the sidereal time,} \\ &= \text{the right ascension of the meridian,} \\ V &= \text{the right ascension of the mean sun,} \end{aligned}$$

we have

$$\Theta = T + V \tag{56}$$

The right ascension of the mean sun, or V, is given in the American Ephemeris, on page II of the month, for each Greenwich mean noon. It is, however, there called the "Sidereal Time," because at mean noon the second mean sun is on the meridian, and its right ascension is also the right ascension of the meridian, or the sidereal time. But this quantity V is uniformly increasing* at the rate of $3^m\ 56^s.555$ in 24 mean solar hours, or of $9^s.8565$ in one mean hour. To find its value at the given time T, we may first find the Greenwich mean time T_0 by applying the longitude; then, if we put

$$\begin{aligned} V_0 &= \text{the value of } V \text{ at Gr. mean noon,} \\ &= \text{the "sidereal time" in the ephemeris for the given date,} \end{aligned}$$

we have

$$V = V_0 + 9^s.8565 \times T_0$$

in which T_0 must be expressed in hours and decimal parts. It is easily seen that $9^s.8565$ is the acceleration of sidereal time on solar time in one solar hour, and therefore the term $9^s.8565 \times T_0$ is the correction to add to T_0 to reduce it from a solar to a sidereal interval. This term is identical with $(\mu - 1)\, T_0$ as given by

* The sidereal time at mean noon is equal to the *true* R.A. of the mean sun, or it is the R.A. of the mean sun referred to the *true* equinox, and therefore involves the nutation, so that its rate of increase is not, strictly, uniform. But it is sufficiently so for 24 hours to be so regarded in all practical computations. See Chapter XI.

the preceding article, if T_0 in the latter expression is expressed in seconds, since we have

$$\frac{9^s.8565}{3600^s} \quad 0.00273791 = \mu - 1$$

We may then write (56) in the following form, putting $L =$ the west longitude of the given meridian, and $T_0 = T + L$:

$$\Theta = T + V_0 + (\mu - 1)(T + L) \qquad (57)$$

The term $(\mu - 1)(T + L)$ is given in the tables of the American Ephemeris for converting "Mean into Sidereal Time," and may be found by entering the table with the argument $T + L$, or by entering successively with the arguments T and L and adding the corrections found, observing to give the correction for the longitude the negative sign when the longitude is east. If no tables are at hand, the direct computation of this term will be more convenient under the form $9^s.8565 \times T_0$.

EXAMPLE 1.—In Longitude 165° W. 1856 May 17, 4^h A.M.; what is the sidereal time?

The Greenwich time is May 17, 3^h; and the computation may be arranged as follows:

Local Ast. Time	$T =$	$16^h\ 0^m\ 0^s.$
At Gr. Noon May 17.	$V_0 =$	3 41 28 .32
Correction of V_0 for 3^h $= 9^s.8565 \times 3$		29 .57
	$\Theta =$	19 41 57 .89

EXAMPLE 2.—In Longitude 25° 17′ E. 1856 March 13, about $9^h\ 30^m$ P.M., an observation is noted by a Greenwich chronometer which gives $7^h\ 51^m\ 12^s.3$ and is slow $3^m\ 13^s.4$; what is the local sidereal time?

Gr. mean date, March 13,		$7^h\ 54^m\ 25^s.7$
Longitude,		1 41 8 $E.$
	$T =$	9 35 33 .7
March 13,	$V_0 =$	23 25 12 .26
Tabular corr. for $7^h\ 54^m\ 25^s.7$	$=$	1 17 .94
	$\Theta =$	9 2 3 .90

EXAMPLE 3.—In Longitude $7^h\ 25^m\ 12^s$ E. 1856 March 13, $13^h\ 15^m\ 47^s.3$ mean local astronomical time; what is the sidereal time?

		T =	$13^h\ 15^m\ 47^s.3$
		V_0 =	23 25 12 .26
Tabular corr. for	$13^h\ 15^m\ 47^s.3$	=	+ 2 10 .73
Tab. corr. for long. —	$7^h\ 25^m\ 12^s$.	=	— 1 13 .14
		Θ =	12 41 57 .15

51. *To convert the apparent solar time at a given meridian into the sidereal time at that meridian.*

Find the mean time by Art. 48, and then the sidereal time by Art. 50.

Or, more directly, *to the given apparent time add the true sun's right ascension.* For if in Fig. 1 we take D as the true sun, we have DQ = apparent solar time, VD = R. A. of true sun, and VQ, the sidereal time, is the sum of these two.

The right ascension of the true sun is called in the Ephemeris the "sun's apparent right ascension," and is there given for each apparent noon. It is not a uniformly increasing quantity; but for many purposes it will be sufficiently accurate to consider the hourly increase given in the Ephemeris as constant for 24^h, and to add to the app. R. A. of the Ephemeris the correction found by multiplying the hourly difference by the number of hours in the Greenwich time.

EXAMPLE.—In Longitude 98° W. 1856 June 3, $4^h\ 10^m$ P.M. app. time; what is the sidereal time?

Gr. app. date June 3, $10^h\ 42^m$ (= $10^h.7$)	Local app. t.	=	$4^h\ 10^m\ 0^s$.
	☉'s App. R. A. App. noon June 3	=	4 46 22 .04
Hourly diff. = $10^s.271$	Corr. = $10^s.271 \times 10.7$	=	1 49 .90
	Sidereal time	=	8 58 11 .94

52. *To convert the sidereal time at a given meridian into the mean time at that meridian.*

First method.—When the Greenwich mean time is also given, as is frequently the case, we have only to find V as in Art. 50 by adding to V_0 given in the Ephemeris the correction for the Greenwich time taken from the table "Mean into Sidereal Time," and then we have, by transposing equation (56),

$$T = \Theta - V$$

Example.—In Longitude 165° W., the Greenwich mean time being 1856 May 17, 3^h, the local sidereal time $19^h\ 41^m\ 57^s.89$, what is the local mean time?

$$\begin{array}{rcrrr} V_0 & = & 3^h & 41^m & 28^s.32 \\ \text{Corr. for } 3^h & = & & + & 29\ .57 \\ \hline V & = & 3 & 41 & 57\ .89 \\ \Theta & = & 19 & 41 & 57\ .89 \\ \hline \Theta - V = T & = & 16 & 0 & 0\ .00 \end{array}$$

The longitude being 11^h W., the local date is May 16.

Second method.—When the Greenwich mean time is not given, we can find T from (57), all the other quantities in that equation being known. We find

$$T = \frac{\Theta - V_0 + L}{\mu} - L$$

or, in a more convenient form for use,

$$T = \Theta - V_0 - \left(1 - \frac{1}{\mu}\right)(\Theta - V_0 + L) \qquad (58)$$

in which the term multiplied by $1 - \frac{1}{\mu}$ is the *retardation* of mean time on sidereal in the interval $\Theta - V_0 + L$, and is given in the table "Sidereal into Mean Time." It is convenient to enter the table first with the argument $\Theta - V_0$ and then with the argument L, and to subtract the two corrections from $\Theta - V_0$, observing that the correction for the longitude becomes additive if the longitude is east.

Example.—In Longitude 165° W. 1856 May 16, the sidereal time is $19^h\ 41^m\ 57^s.89$; what is the mean local time?

$$\begin{array}{lrcrrr} & \Theta & = & 19^h & 41^m & 57^s.89 \\ & \text{May 16, } V_0 & = & 3 & 37 & 31\ .76 \\ \hline & \Theta - V_0 & = & 16 & 4 & 26\ .13 \\ \text{Table, "Sidereal into} & \text{Corr. for } 16^h\ 4^m\ 26^s.13 & = & & -\ 2 & 38\ .00 \\ \text{Mean Time"} & \text{" \ " longitude } 11^h & = & & -\ 1 & 48\ .13 \\ \hline & T & = & 16 & 0 & 0\ .00 \end{array}$$

53. The following method of converting the sidereal into the mean time is preferred by some. In the last column of page III of the month in the American Naut. Alm. is given the "Mean Time of Sidereal 0^h." This quantity, which we may denote by V', is the number of hours the mean sun is *west* of the vernal

equinox, and is merely the difference between 24^h and the mean sun's right ascension. The hour angle of the mean sun at any instant is then the hour angle of the vernal equinox increased by the value of V' at that instant. To find this value of V', we first reduce the Almanac value to the given meridian by correcting it for the longitude by the table for converting sidereal into mean time; then reduce it to the given sidereal time Θ (which is the elapsed sidereal time since the transit of the vernal equinox over the given meridian) by further correcting it by the same table for this time Θ. We then have the mean time T by the formula

$$T = \Theta + V'$$

It is necessary to observe, however, that if $\Theta + V'$ exceed 24^h it will increase our date by one day; and in that case V' should be taken from the Almanac for a date one day less than the given date; that is, we must in every case take that value which belongs to the Greenwich transit of the vernal equinox immediately *preceding* that over the given meridian.

Example.—Same as in Art. 52.

Θ	=	19^h	41^m	$57^s.89$
May 15, V_0'	=	20	23	3 .88
Corr. for long. 11^h W.	=		− 1	48 .13
Corr. for $19^h\ 41^m\ 58^s$	=		− 3	13 .64
T	=	16	0	0 .00

54. *To find the hour angle of a star* at a given time at a given meridian.*

In Fig. 1, we have for the star at O, $DQ = VQ - VD$; that is, if we put

Θ = the sidereal time,
α = the right ascension of the star,
t = the hour angle " " "

then $$t = \Theta - \alpha \tag{59}$$

If α exceeds Θ, this formula will give a negative value of t which will express the hour angle east of the meridian: in that case, if we increase Θ by 24^h before subtracting α, we shall find

* We shall use "star," for brevity, to denote any celestial body.

the value of t reckoned in the usual manner, west of the meridian.

According to this formula, then, we have first to convert the given time into the sidereal time, from which we then subtract the right ascension of the star, increasing the sidereal time by 24^h when necessary; the remainder is the required hour angle west of the meridian.

In the case of the sun, however, the apparent time is at once the required hour angle, and we only have to apply to the given mean time the equation of time.

EXAMPLE.—In Longitude 165° W. 1856 May 16, $16^h\ 0^m\ 0^s$ mean time, find the hour angles of the sun, the moon, Jupiter, and the star Fomalhaut.

The Greenwich mean date is 1856 May 17, 3^h, and the local sidereal time is (see Example 1, Art. 50) $\Theta = 19^h\ 41^m\ 57^s.89$. For the Greenwich date we find from the Naut. Alm. the equation of time E, and the right ascensions α of the moon, Jupiter, and Fomalhaut, as below:

$$\begin{array}{rrrr}
T = & 16^h & 0^m & 0^s \\
-E = & + & 3 & 49.85 \\
\hline
\odot\text{'s } t = & 16 & 3 & 49.85
\end{array}
\qquad
\begin{array}{rrrr}
\Theta = & 19^h & 41^m & 57^s.89 \\
☽\text{'s } \alpha = & 13 & 50 & 21.35 \\
\hline
☽\text{'s } t = & 5 & 51 & 36.54
\end{array}$$

$$\begin{array}{rrrr}
\Theta = & 19^h & 41^m & 57^s.89 \\
♃\text{'s } \alpha = & 0 & 7 & 57.52 \\
\hline
♃\text{'s } t = & 19 & 34 & 0.37
\end{array}
\qquad
\begin{array}{rrrr}
\Theta = & 19^h & 41^m & 57^s.89 \\
\text{Fomalh. } \alpha = & 22 & 49 & 40.18 \\
\hline
\text{Fomalh. } t = & 20 & 52 & 17.71
\end{array}$$

If the sidereal time had been given at first, we should have found the hour angle of the sun by subtracting its apparent right ascension as in the case of any other body.

55. *Given the hour angle of a star at a given meridian on a given day, to find the local mean time.*

By transposing the formula (59), we have

$$\Theta = t + \alpha \tag{60}$$

so that, the right ascension of the star being given, we have only to add it to the given hour angle to obtain the local sidereal time, whence the mean time is found by Art. 52. When the sum $t + \alpha$ exceeds 24^h, we must, of course, deduct 24^h. If the body is the sun, however, the given hour angle is at once the apparent time, whence the mean time as before. But if the body is the moon

or a planet, its right ascension can be found from the Ephemeris only when we know the Greenwich time. If then the Greenwich time is not given, we must find an approximate value of the local time by formula (60), using for α a value taken for a Greenwich time as nearly estimated as possible; from this local time deduce a more exact value of the Greenwich time, with which a more exact value of α may be found; and so repeating as often as may be necessary to reach the required degree of precision.

EXAMPLE 1.—In Longitude 165° W. 1856 May 16, the hour angle of Fomalhaut is $20^h\ 52^m\ 17^s.71$; what is the mean time?

	$t =$	$20^h\ 52^m\ 17^s.71$
May 16, Fomalh.	$\alpha =$	$22\ \ 49\ \ 40\ .18$
	$\Theta =$	$19\ \ 41\ \ 57\ .89$

whence the mean time is found to be $T = 16^h\ 0^m\ 0^s$.

EXAMPLE 2.—In Longitude 165° W. 1856 May 16, the moon's hour angle is $5^h\ 51^m\ 36^s.54$, and the Greenwich date is given May 17, 3^h; what is the mean time?

	$t =$	$5^h\ 51^m\ 36^s.54$
For May 17, 3^h,	$\alpha =$	$13\ \ 50\ \ 21\ .35$
	$\Theta =$	$19\ \ 41\ \ 57\ .89$
" May 17, 3^h,	$V =$	$3\ \ 41\ \ 57\ .89$
	$T =$	$16\ \ 0\ \ 0\ .00$

EXAMPLE 3.—In Longitude 30° E. 1856 August 10, the moon's hour angle is $4^h\ 10^m\ 53^s.2$; what is the mean time?

For a first approximation, we observe that the moon passes the meridian on August 10 at about 7^h mean time (Am. Eph. page IV of the month), and when it is west of the meridian 4^h the mean time is about 4^h later, or 11^h, from which subtracting the longitude 2^h we have, as a rough value of the Greenwich time Aug. 10, 9^h. We then have

	$t =$	$4^h\ 11^m$
For Aug. 10, 9^h,	$\alpha =$	$16\ \ 29$
	$\Theta =$	$20\ \ 40$
" Aug. 10, 9^h,	$V =$	$9\ \ 18$
1st approx. value	$T =$	$11\ \ 22$

Hence the more exact Greenwich date is Aug. 10, $9^h\ 22^m$; and with this we now repeat:

	$t =$	$4^h\ 10^m\ 53^s.2$
For Aug. 10, $9^h\ 22^m$	$\alpha' =$	$16\ \ 29\ \ 26.8$
	$\Theta =$	$20\ \ 40\ \ 20.0$
" "	$V =$	$9\ \ 18\ \ 8.1$
2d approx. value	$T =$	$11\ \ 22\ \ 11.9$

A third approximation, setting out from this value of T, gives us $T = 11^h\ 22^m\ 12^s.32$.

56. The mean time of the meridian passage not only of the moon but of each of the planets is given in the Ephemeris. This quantity is nothing more than the arc of the equator intercepted between the mean sun and the moon's or planet's declination circle. If we denote it by M, we may regard M as the equation between mean time and the *lunar* or *planetary* time, these terms being used instead of "hour angle of the moon" or "hour angle of a planet," just as we use "solar time" to signify "hour angle of the sun." This quantity M is given in the Ephemeris for the instant when the lunar or planetary time is 0^h, and its variation in 1^h of such time is also given in the adjacent column. If, then, when the moon's or a planet's hour angle at a given meridian $= t$, we take out from the Almanac the value of M for the corresponding Greenwich value of t, we shall find the mean time T by simply adding M to t; that is,

$$T = t + M \tag{61}$$

This is, in fact, the *direct* solution of the problem of the preceding article, and neither requires a previous knowledge of the Greenwich mean time nor introduces the sidereal time. But the Almanac values of M are not given to seconds; and therefore we can use (61) only for making our first approximation to T, after which we proceed as in the last article. The Greenwich value of t with which we take out M is equal to $t + L$, denoting by L the longitude of the given meridian (to be taken with the negative sign when east), and the required value of M is the Almanac value increased by the hourly diff. multiplied by $(t + L)$ in hours. As the hourly diff. of M in the case of the moon is itself variable, we should use that value of it which corresponds to the middle of the interval $t + L$; that is, we should first correct the hourly diff. by the product of *its* hourly change into $\frac{1}{2}(t + L)$ in hours.

Example.—Same as Example 3, Art. 55. We have

$t + L = 2^h\ 10^m\ 53^s.2 = 2^h.18$		$t = 4^h\ 10^m\ 53^s.2$
At Gr. trans. Hour. Diff. $= 2^m.17$	At Gr. trans. Aug. 10, $M =$	$7\ \ 6\ \ 30$
Variation of H. D. in $1^h 5^m = \ .01$	$2^m.18 \times 2.18$	$= +\ \ 4\ \ 45$
Corrected Hourly Diff. $= 2\ .18$		$T = 11\ \ 22\ \ \ 8\ .2$

which agrees within 4^s with the true value. Taking it as a first approximation, and proceeding as in Art. 55, a second approximation gives $T = 11^h\ 22^m\ 12^s.19$.

THE EPHEMERIS, OR NAUTICAL ALMANAC.

57. We have already had occasion to refer to the Ephemeris; but we propose here to treat more particularly of its arrangement and use.

The Astronomical Ephemeris expresses in numbers the actual state of the celestial sphere at given instants of time; that is, it gives for such instants the numerical values of the co-ordinates of the principal celestial bodies, referred to circles whose positions are independent of the diurnal motion of the earth, as declination and right ascension, latitude and longitude; together with the elements of position of the circles of reference themselves. It also gives the effects of changes of position of the observer upon the co-ordinates, or, rather, numbers from which such changes can be readily computed (namely, the parallax, which will be fully considered hereafter), the apparent angular magnitude of the sun, moon, and planets, and, in general, all those phenomena which depend on the time; that is, which may be regarded simply as *functions of the time.*

The *American Ephemeris* is composed of two parts, the first computed for the meridian of Greenwich, in conformity with the *British Nautical Almanac*, especially for the use of navigators; the second computed for the meridian of Washington for the convenience of American astronomers. The French Ephemeris, *La Connaissance des Temps*, is computed for the meridian of Paris; the German, *Berliner Astronomisches Jahrbuch*, for the meridian of Berlin. All these works are published annually several years in advance.

58. In what follows, we assume the Ephemeris to be computed for the Greenwich meridian, and, consequently, that it contains the right ascensions, declinations, equation of time, &c. for given equidistant instants of Greenwich time.

Before we can find from it the values of any of these quantities for a given local time, we must find the corresponding Greenwich time (Arts. 45, 46). When this time is exactly one of the instants for which the required quantity is put down in the Ephemeris, nothing more is necessary than to transcribe the quantity as there put down. But when, as is mostly the case, the time falls between two of the times in the Ephemeris, we must obtain the required quantity by interpolation. To facilitate this interpolation, the Ephemeris contains the rate of change, or difference of each of the quantities in some unit of time.

To use the difference columns with advantage, the Greenwich time should be expressed in that unit of time for which the difference is given: thus, when the difference is for one hour, our time must be expressed in hours and decimal parts of an hour; when the difference is for one minute, the time should be expressed in minutes and decimal parts, &c.

59. *Simple interpolation.*—In the greater number of cases in practice, it is sufficiently exact to obtain the required quantities by *simple* interpolation; that is, by assuming that the differences of the quantities are proportional to the differences of the times, which is equivalent to assuming that the differences given in the Ephemeris are constant. This, however, is never the case; but the error arising from the assumption will be smaller the less the interval between the times in the Ephemeris; hence, those quantities which vary most irregularly, as the moon's right ascension and declination, are given for every hour of Greenwich time; others, as the moon's parallax and semidiameter, for every twelfth hour, or for noon and midnight; others, as the sun's right ascension, &c., for each noon; others, as the right ascensions and declinations of the fixed stars, for every tenth day of the year. Thus, for example, the *greatest* errors in the right ascensions and declinations found from the American Ephemeris by simple interpolation are nearly as follows:—

	Error in R. A.	Error in Decl.
Sun	$0^s.1$	$3''.5$
Moon	0 .1	1 .5
Jupiter	0 .1	0 .6
Mars	0 .4	2 .4
Venus	0 .2	5 .4

To illustrate simple interpolation when the Greenwich time is given, we add the following

EXAMPLES.

For the Greenwich mean time 1856 March 30, $17^h\ 11^m\ 12^s$, find the following quantities from the American Ephemeris: the Equation of time, the Right Ascension, Declination, Horizontal Parallax, and Semidiameter of the Sun, the Moon, and Jupiter.

1. *The Equation of time.*—The Gr. T. = March 30, $17^h\ 11^m.2$ = March 30, $17^h.187$.

(Page II) E at mean noon =	$+\ 4^m\ 27^s.11$	H. D. =	$-\ 0^s.763$
Corr. for $17^h.19$ =	$-\ 13\ .11$		17.19
E =	$+\ 4\ \ 14\ .00$		$-\ 13.11$

NOTE.—Observe to mark E always with the sign which denotes how it is to be applied to *apparent* time. If *increasing*, the H. D. (hourly difference) should have the same sign as E; otherwise, the contrary sign.

2. *Sun's R. A. and Dec.*

(P. II.) α at 0^h =	$0^h\ 36^m\ 40^s.78$	H. D.	$+\ 9^s.094$
Corr. for $17^h.187$ =	$+\ \ 2\ \ 36\ .29$		17.187
α =	$0\ \ 39\ \ 17\ .07$		156.29
δ at 0^h =	$+\ 3°\ 57'\ 21''.9$	H. D.	$+\ 58''.15$
Corr. for $17^h.187$ =	$+\ \ 16\ \ 39\ .4$		17.187
δ =	$+\ 4\ \ 14\ \ 1\ .3$		999.4

3. *Moon's R. A. and Dec.*

α at 17^h =	$20^h\ 18^m\ 9^s.80$	Diff. 1^m	$+\ 2^s.4975$
Corr. for $11^m.2$ =	$+\ \ 27\ .97$		11.2
α =	$20\ \ 18\ \ 37\ .77$		27.97
δ at 17^h =	$-\ 25°\ 3'\ 10''.9$	Diff. 1^m	$+\ 8''.275$
Corr. for $11^m.2$ =	$+\ 1\ \ 32\ .7$		11.2
δ =	$-\ 25\ \ 1\ \ 38\ .2$		92.68

4. *Moon's Hor. Par.* ($= \pi$) *and Semid.* ($= S$).

π at 12^h =	$58'\ 44''.1$	H. D.	$+\ 2''.17$
Corr. for $5^h.2$ =	$+\ 11\ .3$		5.2
π =	$58\ \ 55\ .4$		11.28

$$S \text{ at } 12^h = 16' \; 2''.0 \qquad \text{Diff. in } 12^h = +\,7''.1$$
$$\text{Corr. for } 5^h.2 = +\;\;3\;.1$$
$$S = 16\;\;5\;.1$$

5. *Jupiter's R. A. and Dec.*

α at 0^h =	$23^h\ 29^m\ 49^s.95$	H. D. $+\ 2^s.175$
Corr. for $17^h.187$ =	$+\ 37.38$	17.187
α =	$23\ \ 30\ \ 27.33$	37.38

δ at 0^h =	$-\ 4°\ 22'\ 45''.6$	H. D. $+\ 13''.74$
Corr. for $17^h.187$ =	$+\ \ 3\ \ 56\ .1$	17.187
δ =	$-\ 4\ \ 18\ \ 49\ .5$	236.1

6. *Jupiter's Hor. Par. and Semid.*—At the bottom of page 231, we find for the nearest date March 31, without interpolation:

$$\pi = 1''.5 \qquad\qquad S = 15''.7$$

NOTE.—It may be observed that we mark hourly differences of declination *plus*, when the body is moving *northward*, and *minus* when it is moving *southward*.

In the above we have carried the computation to the utmost degree of precision ever necessary in simple interpolation.

60. *To find the right ascension and declination of the sun at the time of its transit over a given meridian, and also the equation of time at the same instant.*

When the sun is on a meridian in *west* longitude, the Greenwich *apparent* time is precisely equal to the longitude, that is, the Gr. App. T. is *after* the noon of the same date with the local date, by a number of hours equal to the longitude. When the sun is on a meridian in *east* longitude, the Gr. App. T. is *before* the noon of the same date as the local date, by a number of hours equal to the longitude. Hence, to obtain the sun's right ascension and declination and the equation of time for apparent noon at any meridian, take these quantities from the Ephemeris (page I of the month) for Greenwich Apparent Noon of the same date as the local date, and apply a correction equal to the hourly difference multiplied by the number of hours in the longitude, observing to *add* or *subtract* this correction, according as the numbers in the Ephemeris may indicate, for a time *before* or *after* noon.

EXAMPLE 1.—Longitude 167° 31′ W. 1856 March 20, App. Noon, find ☉'s R. A., ☉'s Dec., and Eq. of T.

Longitude $= + 11^h\ 10^m\ 4^s = + 11^h.17$

α at App. $0^h =$	$0^h\ 0^m\ 20^s.94$	H. D. $+ 9^s.098$
Corr. for $+ 11^h.17 =$	$+ 1\ \ 41\ .62$	$+ 11.17$
$\alpha =$	$0\ \ 2\ \ \ 2\ .56$	$+ 101.62$
δ at App. $0^h =$	$+ 0°\ 2'\ 16''.5$	H. D. $+ 59''.21$
Corr. for $+ 11^h.17 =$	$+ \ 11\ \ 1\ .4$	$+ 11.17$
$\delta =$	$+ 0\ 13\ 17\ .9$	$+ 661.4$
E at App. $0^h =$	$+ 7^m\ 31^s.57$	H. D. $- 0^s.759$
Corr. for $+ 11^h.17 =$	$- \ 8\ .48$	$+ 11.17$
$E =$	$+ 7\ 23\ .09$	$- 8.48$

EXAMPLE 2.—Longitude 167° 31′ E. 1856 March 20, App Noon, find ☉'s R.A., ☉'s Dec., and Eq. of T.

Longitude $= - 11^h\ 10^m\ 4^s = - 11^h.17$

α at App. $0^h =$	$0^h\ 0^m\ 20^s.94$	H. D. $+ 9^s.098$
Corr. for $- 11^h.17 =$	$- 1\ \ 41\ .62$	$- 11.17$
$\alpha =$	$23\ 58\ 39\ .32$	$- 101.62$
δ at App. $0^h =$	$+ 0°\ 2'\ 16''.5$	H. D. $+ 59''.21$
Corr. for $- 11^h.17 =$	$- 11\ \ 1\ .4$	$- 11.17$
* $\delta =$	$- 0\ \ 8\ 44\ .9$	$- 661.4$
E at App. $0^h =$	$+ 7^m\ 31^s.57$	H. D. $- 0^s.759$
Corr. for $- 11^s.17 =$	$+ \ 8\ .48$	$- 11.17$
$E =$	$+ 7\ 40\ .05$	$+ 8.48$

61. *To find the mean local time of the moon's or a planet's transit over a given meridian.*

This is the same as the problem of Art. 55, in the special case where the hour angle of the moon or planet at the given meridian is 0^h. We can, however, obtain the required time directly from the Ephemeris, with sufficient accuracy for many purposes,

* In this example the sun crosses the equator between the times of its transits over the local and the Greenwich meridians. The case must be noted, as it is a frequent occasion of error among navigators. The same case can occur on September 22 or 23.

by simple interpolation. On page IV of the month (*Am. Ephem.* and *British Naut. Alm.*) we find the mean time of transit of the moon over the Greenwich meridian on each day. This mean time is nothing more than the hour angle of the mean sun at the instant, or the difference of the right ascensions of the moon and the mean sun; and if this difference did not change, the mean *local* time of moon's transit would be the same for all meridians; but as the moon's right ascension increases more rapidly than the sun's, the moon is apparently retarded from transit to transit. The difference between two successive times of transit given in the Ephemeris is the retardation of the moon in passing over 24^h *of longitude*, and the hourly difference given is the retardation in passing from the Greenwich meridian to the meridian 1^h from that of Greenwich. Hence, to find the local time of the moon's transit on a given day, take the time of meridian passage from the Ephemeris for the same date (astronomical account) and apply a correction equal to the hourly difference multiplied by the longitude in hours; adding the correction when the longitude is west, subtracting it when east. The same method applies to planets whose mean times of transit are given in the Ephemeris as in the case of the moon.

EXAMPLE.—Longitude 130° 25′ E. 1856 March 22; required local time of moon's transit.

Gr. Merid. Passage March 22,		$13^h.\ 2^m.7$	H. D.	$+\ 1^m.59$
Corr. for Long. $-\ 8^h.7$	=	$-\ 13.8$		$-\ 8.7$
Local M. T. of transit	=	$12\ 48.9$		$-\ 13.8$

62. *To find the moon's or a planet's right ascension, declination, &c., at the time of transit over a given meridian.*

Find the local time of transit by the preceding article, deduce the Greenwich time, and take out the required quantities from the Ephemeris for this time. This is the usual nautical method, and is accurate enough even for the moon, as meridian observations of the moon at sea are not susceptible of great precision. For greater precision, find the local time by Art. 55 for $t = 0^h$, and thence the Greenwich time. See also *Moon Culminations*, Chapter VII.

63. INTERPOLATION BY SECOND DIFFERENCES.—The differences between the successive values of the quantities given in the

Ephemeris as functions of the time, are called the *first differences;* the differences between these successive differences are called the *second differences;* the differences of the second differences are called the *third differences*, &c. In simple interpolation we assume the function to vary uniformly; that is, we regard the first difference as constant, neglecting the second difference, which is, consequently, assumed to be zero. In interpolation by second differences we take into account the variation in the first difference, but we assume *its* variations to be constant; that is, we assume the second differences to be constant and the third differences to be zero.

When the American Ephemeris is employed, we can take the second differences into account in a very simple manner. In this work, the difference given for a unit of time is always the difference belonging to the instant of Greenwich time against which it stands, and it expresses, therefore, the rate at which the function is changing *at that instant.* This difference, which we may here call the first difference, varies with the Greenwich time, and (the second difference being constant) it varies uniformly, so that its value for any intermediate time may be found by simple interpolation, using the second differences as first differences. Now, in computing a correction for a given interval of Greenwich time, we should employ the *mean*, or average value, of the first difference for the interval, and this mean value, when we regard the second differences as constant, is that which belongs to the middle of the interval. Hence, to take into account the second differences, we have only to observe the very simple rule—*employ that (interpolated) value of the first difference which corresponds to the middle of the interval for which the correction is to be computed.*

EXAMPLE.—For the Greenwich time 1856 March 2, $12^h\ 29^m\ 36^s$, find the moon's declination.

March 2, $12^h(\delta) = -27°\ 10'\ 41''.8$	Diff. $1^m = +\ 4''.814$	2d Diff. $= +0''.189$
Corr. for $29^m\ 6$ $\quad +\ \ 2\ 23\ .9$	Corr. for 2d diff. $+\ .047$	0.25
$\delta = -27\ \ 8\ 17\ .9$	$+\ 4.861$	$+\ 0.047$
	29.6	
	$+\ 143.89$	

Here the "diff. for 1^m" increases $0''.189$ in 1^h; the half of the interval for which the correction is to be computed is $14^m\ 48^s$ =

$0^h.25$; we therefore find the value of the first difference at 12^h 14^m 48^s, by adding to its value taken for 12^h the quantity $0''.189 \times 0.25$, and then proceed as in simple interpolation. This example suffices to illustrate the method in all cases where the first difference is given in the Ephemeris for the time against which it stands. In using the British Nautical Almanac and other works of the same kind, interpolation by second differences may be performed by the general interpolation formula hereafter given.

64. *To find the Greenwich time corresponding to a given right ascension of the moon on a given day.*

Let T' = the Greenwich time corresponding to the given right ascension α',
T = the Greenwich hour preceding T' and corresponding to the right ascension α,
$\Delta\alpha$ = the diff. of R. A. in 1^m at the time T,

then we have, approximately,

$$T' - T = \frac{\alpha' - \alpha}{\Delta\alpha}$$

To correct for second differences, we have now only to find

$\Delta_0\alpha$ = diff. of R.A. in 1^m for the middle instant of the interval $T'-T$,

and then we have, accurately,

$$T' - T = \frac{\alpha' - \alpha}{\Delta_0\alpha}$$

These formulæ give $T' - T$ in minutes of time.

65. *To find the distance of the moon from a given object at a given Greenwich time.*

In the American Ephemeris and the British Nautical Almanac, the "lunar distances" are given at every 3d hour of Greenwich time, together with the *proportional logarithms* of the differences between the successive distances.

The proportional logarithm of an angle expressed in hours, &c. is the logarithm of the quotient of 3^h divided by the angle; that of an angle expressed in degrees, &c. is the logarithm of the quotient of 3° divided by the angle. Thus, if A is the angle, in hours,

$$\text{P. L. } A = \log \frac{3^h}{A} = \log 3^h - \log A$$

or, if A is in degrees,

$$\text{P. L. } A = \log \frac{3^\circ}{A} = \log 3^\circ - \log A$$

The angle is always supposed to be reduced to seconds; so that, whether A is in seconds of time or of arc, we have

$$\text{P. L. } A = \log 10800 - \log A$$

Tables of such logarithms are given in works on Navigation.

If now we wish to interpolate a value of a lunar distance for a time $T + t$ which falls between the two times of the Ephemeris T and $T + 3^h$, we are to compute the correction for the interval t and apply it to the distance given for the time T; and if we put

$\varDelta$ = the difference of the distances in the Ephemeris,
$\varDelta'$ = the difference in the interval t,

we shall have, by simple interpolation,

$$\varDelta' = \varDelta \times \frac{t}{3^h}$$

or, by logarithms,

$$\log \varDelta' = \log t + \log \varDelta - \log 3^h$$

or, supposing $\varDelta$, $\varDelta'$, and t all reduced to seconds,

$$\log \varDelta' = \log t - \text{P. L. } \varDelta \tag{62}$$

Subtracting both members of this from log 10800, we have

$$\text{P. L. } \varDelta' = \text{P.L. } t + \text{P. L. } \varDelta \tag{63}$$

which is computed by the tables above mentioned. By (62), however, only the common logarithmic table is required.

But the first differences of the lunar distance cannot be assumed as constant when the intervals of time are as great as 3^h. If we put

$$\text{P. L. } \varDelta = Q$$

we observe that Q is variable, and the value given in the Ephemeris is to be regarded as its value at the middle instant of the interval to which it belongs. If then

Q' = the value of Q for the middle of the interval t,
ΔQ = the increase of Q in 3^h (found from the successive values in the Ephemeris),

we have

$$Q' = Q - \left(\frac{1^h.5 - \frac{1}{2}t}{3^h}\right)\Delta Q \tag{64}$$

in which t is in hours and decimal parts. We find then, with regard to second differences,

$$\log \varDelta' = \log t - Q'$$

EXAMPLE.—Find the distance d of the moon's centre from the star Fomalhaut at the Greenwich time 1856 March 30, $13^h\ 20^m\ 24^s$.

Here $T = 12^h$, $t = 1^h\ 20^m\ 24^s = 1^h.34$; $\dfrac{1^h.5 - \frac{1}{2}t}{3^h} = 0.28$; and from the Ephemeris:

March 30, 12^h (d)	36° 17′ 53″	Q,	.2993	ΔQ,	+ .0041
$\varDelta'$	−0 40 28		−.0011		.28
At $13^h\ 20^m\ 24^s$ d =	35 37 25	Q',	.2982		+ .0011
		$\log t$,	3.6834		
		$\log \varDelta'$,	3.3852		

66. *To find the Greenwich time corresponding to a given lunar distance on a given day.*

We find in the Ephemeris for the given day the two distances between which the given one falls; and if $\varDelta'$ = difference between the first of these and the given one, $\varDelta$ = difference of the distances in the Ephemeris, we find the interval t, to be added to the preceding Greenwich time, by simple interpolation, from the formula

$$t = 3^h \times \frac{\varDelta'}{\varDelta}$$

or

$$\log t = \log \varDelta' + \text{P. L. } \varDelta = \log \varDelta' + Q \tag{65}$$

and, with regard to second differences, the *true* interval, t', by the formula

$$\log t' = \log \varDelta' + Q' \tag{66}$$

where Q' has the value given in the preceding article.

But to find Q' by (64) we must first find an approximate value of t. To avoid this double computation, it is usual to find t by (65), and to give a correction to reduce it to t' in a small table which is computed as follows. We have from (64), (65), and (66)

$$\log t' - \log t = Q' - Q = -\left(\frac{1^h.5 - \frac{1}{2}t}{3^h}\right)\Delta Q$$

By the theory of logarithms, we have, M being the modulus of the common system,

$$\log x = M[(x-1) - \tfrac{1}{2}(x-1)^2 + \&c.]$$

so that

$$\log t' - \log t = \log\frac{t'}{t} = M\left[\frac{t'-t}{t} - \frac{1}{2}\left(\frac{t'-t}{t}\right)^2 + \&c.\right]$$

or, neglecting the square and higher powers of the small fraction $\frac{t'-t}{t}$,

$$\log t' - \log t = M\left(\frac{t'-t}{t}\right)$$

This, substituted above, gives

$$t' - t = -\frac{t(1^h.5 - \frac{1}{2}t)}{M \times 3^h}\Delta Q = -\frac{t(3^h - t)}{2M \times 3^h}\Delta Q$$

by which a table is readily computed giving the value of $t' - t$ [or the correction of t found by (65)], with the arguments ΔQ and t. In this formula t and $t' - t$ are supposed to be expressed in hours; and to obtain $t' - t$ in seconds we must multiply the second member by 3600; this will be effected if we multiply each of the factors t and $3^h - t$ by 60, that is, reduce them each to minutes, so that if we substitute the value of $M = .434294$ the formula becomes

$$t' - t = -\frac{t(180^m - t)}{2.60576}\Delta Q \tag{67}$$

in which t is expressed in minutes, and $t' - t$ in seconds.

EXAMPLE.—1856 March 30, the distance of the moon and Fomalhaut is 35° 37′ 25″; what is the Greenwich time?

March 30, 12^h 0^m 0^s	$(d) = 36° 17' 53''$	$Q = .2993$	$\Delta Q = +41$
$t =$ 1 20 36	$d = 35$ 37 25	$\log \Delta' = 3.3852$	
Ap. Gr. time = 13 20 36	Δ' 40 28	$\log t = 3.6845$	
By (67)*, $t' - t =$ — 12			
True Gr. time = 13 20 24			

* Or from the "Table showing the correction required on account of the second differences of the moon's motion in finding the Greenwich time corresponding to a corrected lunar distance," which is given in the American Ephemeris, and is also included in the Tables for Correcting Lunar Distances given in Vol. II. of this work.

INTERPOLATION BY DIFFERENCES OF ANY ORDER.

67. When the exact value of any quantity is required from the Ephemeris, recourse must be had to the general interpolation formulæ which are demonstrated in analytical works. These enable us to determine intermediate values of a function from tabulated values corresponding to equidistant values of the variable on which they depend. In the Ephemeris the data are in most cases to be regarded as functions of the time considered as the variable or argument.

Let T, $T+w$, $T+2w$, $T+3w$, &c., express equidistant values of the variable; F, F', F'', F''', &c., corresponding values of the given function; and let the differences of the first, second, and following orders be formed, as expressed in the following table:—

Argument.	Function.	1st Diff.	2d Diff.	3d Diff.	4th Diff.	5th Diff.	6th Diff.
T	F						
		a					
$T+w$	F'		b				
		a'		c			
$T+2w$	F''		b'		d		
		a''		c'		e	
$T+3w$	F'''		b''		d'		f
		a'''		c''		e'	
$T+4w$	F^{iv}		b'''		d''		
		a^{iv}		c'''			
$T+5w$	F^{v}		b^{iv}				
		a^{v}					
$T+6w$	F^{vi}						

The differences are to be found by subtracting *downwards*, that is, each number is subtracted from the number below it, and the proper algebraic sign must be prefixed. The differences of any order are formed from those of the preceding order in the same manner as the first differences are formed from the given functions. The *even* differences (2d, 4th, &c.) fall in the same lines with the argument and function; the *odd* differences (1st, 3d, &c.) between the lines.

Now, denoting the value of the function corresponding to a value of the argument $T+nw$ by $F^{(n)}$, we have, from algebra,

$$F^{(n)}=F+na+\frac{n\,(n-1)}{1\,.\,2}\,b+\frac{n\,(n-1)\,(n-2)}{1\,.\,2\,.\,3}\,c+\frac{n\,(n-1)\,(n-2)\,(n-3)}{1\,.\,2\,.\,3\,.\,4}\,d+\text{\&c.} \quad (68)$$

in which the coefficients are those of the n^{th} power of a binomial.

In this formula the interpolation sets out from the first of the given functions, and the differences used are the first of their respective orders. If n be taken successively equal to 0, 1, 2, 3, &c., we shall obtain the functions F, F', F'', F''', &c., and intermediate values are found by using fractional values of n. We usually apply the formula only to interpolating between the function from which we set out and the next following one, in which case n is less than unity. To find the proper value of n in each case, let $T + t$ denote the value of the argument for which we wish to interpolate a value of the function: then

$$nw = t \qquad\qquad n = \frac{t}{w}$$

that is, n is the value of t reduced to a fraction of the interval w.

EXAMPLE.—Suppose the moon's right ascension had been given in the Ephemeris for every twelfth hour as follows:

	☽'s R. A.	1st. Diff.	2d Diff	3d Diff.	4th Diff.	5th Diff.
1856 March 5, 0^h	$21^h\ 58^m\ 28^s.39$					
		$+\,28^m\ 47^s.04$				
" 5, 12	22 27 15.43		$-\,36^s.97$			
		28 10.07		$+\,4^s.79$		
" 6, 0	22 55 25.50		32.18		$+\,1^s.74$	
		27 37.89		6.53		$-\,0^s.66$
" 6, 12	23 23 3.39		25.65		1.08	
		27 12.24		7.61		
" 7, 0	23 50 15.63		18.04			
		26 54.20				
" 7, 12	0 17 9.83					

Required the moon's right ascension for March 5, 6^h.

Here $T =$ March 5, 0^h, $t = 6^h$, $w = 12^h$, $n = \frac{6^h}{12^h} = \frac{1}{2}$; and if we denote the coefficients of a, b, c, d, e in (68) by A, B, C, D, E, we have

$$F = 21^h\ 58^m\ 28^s.39$$

$$a = +\,28^m\ 47^s.04, \quad A = n = \tfrac{1}{2}, \quad Aa = +\ 14\ \ 23.52$$

$$b = -\ 36.97, \quad B = A\,.\,\frac{n-1}{2} = -\,\tfrac{1}{8}, \quad Bb = +\ 4.62$$

$$c = +\ 4.79, \quad C = B\,.\,\frac{n-2}{3} = +\,\tfrac{1}{16}, \quad Cc = +\ 0.30$$

$$d = +\ 1.74, \quad D = C\,.\,\frac{n-3}{4} = -\,\tfrac{5}{128}, \quad Dd = -\ 0.07$$

$$e = -\ 0.66, \quad E = D\,.\,\frac{n-4}{5} = +\,\tfrac{7}{256}, \quad Ee = -\ 0.02$$

☽'s R. A. 1856 March 5, 6^h . . . . $F^{(\frac{1}{2})} = 22\ \ 12\ \ 56.74$

which agrees precisely with the value given in the American Ephemeris.

68. The formula (68) may also be written as follows:

$$F^{(n)} = F + n\left(a + \frac{n-1}{2}\left(b + \frac{n-2}{3}\left(c + \frac{n-3}{4}\left(d + \frac{n-4}{5}\left(e + \&c.\right)\right)\right)\right)\right) \qquad (68^*)$$

Thus, in the preceding example, we should have

$$\begin{array}{llll}
\frac{n-4}{5} = -\frac{7}{10}, & -\frac{7}{10} \times -0^s.66 & = & +0^s.46 \\
\frac{n-3}{4} = -\frac{5}{8}, & -\frac{5}{8}\,(+\ 1^s.74 + 0^s.46) & = & -1.38 \\
\frac{n-2}{3} = -\frac{1}{2}, & -\frac{1}{2}\,(+\ 4^s.79 - 1^s.38) & = & -1.71 \\
\frac{n-1}{2} = -\frac{1}{4}, & -\frac{1}{4}\,(-\ 36^s.97 - 1^s.71) & = & +9.67 \\
n = \frac{1}{2}, & \frac{1}{2}\,(+\ 28^m 47^s.04 + 9^s.67) & = & +14^m\,28^s.35
\end{array}$$

and adding this last quantity, $14^m\ 28^s.35$, to $21^h\ 58^m\ 28^s.39$, we obtain the same value as before, or $22^h\ 12^m\ 56^s.74$.

69. A more convenient formula, for most purposes, may be deduced from (68), if we use not only values of the functions following that from which we set out, but also preceding values; that is, also values corresponding to the arguments $T-w$, $T-2w$, &c. We then form a table according to the following schedule:

Argument.	Function.	1st Diff.	2d Diff.	3d Diff.	4th Diff.	5th Diff.	6th Diff.
$T-3w$	$F_{\prime\prime\prime}$						
		$a_{\prime\prime\prime}$					
$T-2w$	$F_{\prime\prime}$		$b_{\prime\prime}$				
		$a_{\prime\prime}$		$c_{\prime\prime}$			
$T-w$	$F_{\prime}$		$b_{\prime}$		$d_{\prime}$		
		$a_{\prime}$		$c_{\prime}$		$e_{\prime}$	
T	F		b		d		f
		a'		c'		e'	
$T+w$	F'		b'		d'		
		a''		c''			
$T+2w$	F''		b''				
		a'''					
$T+3w$	F'''						

According to the formula (68), if we set out from the function F, we employ the differences denoted in this table by a', b', c'', &c., and hence for the argument $T + nw$ we find the value of $F^{(n)}$ by the formula

$$F^{(n)} = F + na' + \frac{n(n-1)}{1.2} b' + \frac{n(n-1)(n-2)}{1.2.3} c'' + \frac{n(n-1)(n-2)(n-3)}{1.2.3.4} d'' + \&c.$$

But we have

$$\begin{aligned}
b' &= b + c' \\
c'' &= c' + d' = c' + d + e' \\
d'' &= d' + e'' = d + e' + e' + f' = d + 2e' + f' \\
\&c. & \quad \&c.
\end{aligned}$$

in which b', c'', &c. are expressed in terms of the differences that lie on each side of a horizontal line drawn in the table immediately under the function from which we set out. These values substituted in the formula give

$$F^{(n)} = F + na' + \frac{n(n-1)}{1.2} b + \frac{(n+1)(n)(n-1)}{1.2.3} c' + \frac{(n+1)(n)(n-1)(n-2)}{1.2.3.4} d + \&c. \tag{69}$$

in which the law of the coefficients is that one new factor is introduced into the numerator *alternately after and before* the other factors, observing always that the factors decrease by unity from left to right. The new factor in the denominator, as in the original formula (68), denotes the order of difference.

The interpolation by this formula is rendered somewhat more accurate by using, instead of the last difference, the mean of the two values that lie nearest the horizontal line drawn under the middle function: thus, if we stop at the fourth difference, we use a mean between d and d' instead of d. We thus take into account a part of the term involving the fifth difference.

EXAMPLE.—Find the moon's right ascension for 1856 March 5, 6^h, employing the values given in the Ephemeris for every twelfth hour. This is the same as the example under Art. 67, where it is worked by the primitive formula (68). But here we take from the Ephemeris three values *preceding* that for March 5, 0^h, and three values *following* it, and form our table as follows:

	☽'s R. A.	1st Diff.	2d Diff.	3d Diff.	4th Diff.	5th Diff.
1856 March 3, 12^h	$20^h\ 28^m\ 17^s.88$					
		$+\ 30^m\ 39^s.20$				
" 4, 0	20 58 57.08		$-34^s.27$			
		30 4.93		$-4^s.28$		
" 4, 12	21 29 2.01		38.55		$+3^s.49$	
		29 26.38		-0.79		$-0^s.33$
" 5, 0	21 58 28.39		39.34		3.16	
		28 47.04		$+2.37$		-0.74
" 5, 12	22 27 15.43		36.97		2.42	
		28 10.07		$+4.79$		
" 6, 0	22 55 25.50		32.18			
		27 37.89				
" 6, 12	23 23 3.39					

Drawing a horizontal line under the function from which we set out, the differences required in the formula (69) stand next to this line, alternately below and above it.

$$
\begin{array}{llll}
 & & F & = 21^h\ 58^m\ 28^s.39 \\
a' = +\ 28^m\ 47^s.04, & A = n = \tfrac{1}{2}, & Aa' & = +\ 14\ \ 23.52 \\
b = -\ 39.34, & B = A\,.\,\dfrac{n-1}{2} = -\tfrac{1}{8}, & Bb & = +\ 4.92 \\
c' = +\ 2.37, & C = B\,.\,\dfrac{n+1}{3} = -\tfrac{1}{16}, & Cc' & = -\ 0.15 \\
d = +\ 3.16, & D = C\,.\,\dfrac{n-2}{4} = +\tfrac{3}{128}, & Dd & = +\ 0.07 \\
e' = -\ 0.74, & E = D\,.\,\dfrac{n+2}{5} = +\tfrac{3}{256}, & Ee' & = -\ 0.01 \\
\end{array}
$$

☽'s R. A. 1856 March 5, 6^h = $F^{(n)} = 22\ \ 12\ \ 56.74$

69*. If in (69) we substitute the values

$$
\begin{aligned}
a' &= a_, + b \\
c' &= c_, + d \\
&\&c.
\end{aligned}
$$

we find

$$
F^{(n)} = F + na_, + \frac{(n+1)\,n}{1\,.\,2}\,b + \frac{(n+1)\,(n)\,(n-1)}{1\,.\,2\,.\,3}\,c_,
$$
$$
+ \frac{(n+2)\,(n+1)\,(n)\,(n-1)}{1\,.\,2\,.\,3\,.\,4}\,d + \&c. \tag{70}
$$

in which the law of the coefficients is that one new factor is introduced into the numerator alternately *before and after* the other factors, observing still that the factors decrease by unity from left to right. The differences employed are those which lie on each side of the horizontal line drawn immediately above the function from which we set out.

If in the preceding formulæ we employ a negative value of n less than unity, we shall obtain a value of the function between F and $F_{,}$, and in that case (70) is more convergent than (69). In general, if we set out from that function which is nearest to the required one, we shall always have values of n numerically less than $\frac{1}{2}$, and we should prefer (69) for values of n between 0 and $+\frac{1}{2}$, and (70) for values of n between 0 and $-\frac{1}{2}$.

70. If we take the mean of the two formulæ (69) and (70), and denote the means of the odd differences that lie above and below the horizontal lines of the table, by letters without accents, that is, if we put

$$a = \tfrac{1}{2}(a_{,} + a'), \quad c = \tfrac{1}{2}(c_{,} + c') \text{ \&c.}$$

we have

$$F^{(n)} = F + na + \frac{n^2}{2}b + \frac{(n+1)(n)(n-1)}{2 \cdot 3}c + \frac{(n+1)(n^2)(n-1)}{2 \cdot 3 \cdot 4}d + \text{\&c.} \quad (71)$$

The quantities a, c, &c. may be inserted in the table, and will thus complete the row of differences standing in the same line with the function from which we set out.

The law of the coefficients in (71) is that the coefficient of any odd difference is obtained from that of the preceding odd difference by introducing two factors, one at the beginning and the other at the end of the line of factors, observing as before that these factors are respectively greater and less by unity than those next to which they are placed; and the coefficients of the even differences are obtained from the next preceding even differences in the same manner. The factors in the denominator follow the same law as in the other formulæ.

Example.—Find the moon's right ascension for 1856 March 5, 6^h, from the values given in the Ephemeris for noon and midnight

The table will be as below:

	☽'s R. A.	1st Diff.	2d Diff.	3d Diff.	4th Diff.	5th Diff.
Mar. 3, 12^h	$20^h 28^m 17^s.88$					
		$+30^m\ 29^s.20$				
" 4, 0	20 58 57 .08		$-34^s.27$			
		30 4 .93		$-4^s.28$		
" 4, 12	21 29 2 .01		38 .55		$+3^s.49$	
		29 26 .38		-0.79		$-0^s.33$
" 5, 0	21 58 28 .39	[+29 6 .71]	-39.34	[+0 .79]	$+3.16$	[−0 .54]
		28 47 .04		$+2.37$		-0.74
" 5, 12	22 27 15 .43		36 .97		2 .42	
		28 10 .07		$+4.79$		
" 6, 0	22 55 25 .50		32 .18			
		27 37 .89				
" 6, 12	23 23 3 .39					

Drawing two lines, one above and the other below the function from which we set out, and then filling the blanks by the means of the odd differences above and below these lines (which means are here inserted in brackets), we have presented in the same line all the differences required in the formula (71); and we then have

$$F = 21^h\ 58^m\ 28^s.39$$

$$a = +29^m\ 6^s.71,\quad A = n = \tfrac{1}{2},\qquad Aa = +\ 14\ \ 33\ .36$$

$$b = -\ 39\ .34,\quad B = \frac{n^2}{2} = +\tfrac{1}{8},\qquad Bb = -\ \ 4\ .92$$

$$c = +\ 0\ .79,\quad C = A\,.\,\frac{n^2-1}{6} = -\tfrac{1}{16},\qquad Cc = -\ \ 0\ .05$$

$$d = +\ 3\ .16,\quad D = B\,.\,\frac{n^2-1}{12} = -\tfrac{1}{128},\qquad Dd = -\ \ 0\ .02$$

$$e = -\ 0\ .54,\quad E = C\,.\,\frac{n^2-4}{20} = +\tfrac{3}{256},\qquad Ee = -\ \ 0\ .01$$

$$F^{(\frac{1}{2})} = 22\ \ 12\ \ 56\ .75$$

agreeing within $0^s.01$ with the value found in the preceding article. Hansen has given a table for facilitating the use of this formula. (See his *Tables de la Lune*).

71. Another form, considered by Bessel as more accurate than any of the preceding, is found by employing the odd differences that fall next below the horizontal line drawn below the function from which we set out, and the means of the even differences that fall next above and next below this line. Thus, if we put

$$b_0 = \tfrac{1}{2}(b + b'),\quad d_0 = \tfrac{1}{2}(d + d'),\ \&c.$$

and combine these with the expressions

$$\tfrac{1}{2}\, c' = \tfrac{1}{2}\,(b' - b), \quad \tfrac{1}{2}\, e' = \tfrac{1}{2}(\, d' - d), \text{ \&c.}$$

we deduce

$$b = b_0 - \tfrac{1}{2}\, c', \qquad d = d_0 - \tfrac{1}{2}\, e', \text{ \&c.}$$

which substituted in (69) give

$$F^{(n)} = F + na' + \frac{n(n-1)}{1\,.\,2}\, b_0 + \frac{n(n-1)(n-\frac{1}{2})}{1\,.\,2\,.\,3}\, c' + \frac{(n+1)n(n-1)(n-2)}{1\,.\,2\,.\,3\,.\,4}\, d_0$$
$$+ \frac{(n+1)\,n\,(n-1)\,(n-2)\,(n-\frac{1}{2})}{1\,.\,2\,.\,3\,.\,4\,.\,5\,.}\, e' + \text{\&c.} \qquad (72)$$

To facilitate the application of this formula, draw a horizontal line under the function from which the interpolation sets out, and another over the next following function; these lines will embrace the odd differences a', c', &c. If we then insert in the blank spaces between these lines the means of the even differences that fall above and below them, we shall have presented in a row all the differences to be employed in the formula.

EXAMPLE.—Find the right ascension of the moon's second limb at the instant of its transit over the meridian whose longitude is $4^h\ 42^m\ 19^s$ west from Greenwich, on May 15, 1851.

The right ascensions of the moon's bright limb at the instant of its upper and lower transits over the Greenwich meridian, are given in the Ephemeris, under the head of "Moon Culminations." The argument in this case is the longitude, and the intervals of the argument are 12^h. The value for any meridian is therefore to be obtained by interpolation, taking for n the quotient obtained by dividing the given longitude (in hours) by 12^h.

We take from the British Nautical Almanac the following values:

	R. A. ☽'s 2d limb	1st Diff.	2d Diff.	3d Diff.	4th Diff.	5th Diff.
May 14, U. C.	$15^h\ 12^m\ 39^s.04$					
		$+\ 28^m\ 24^s.37$				
" 15, L. C.	15 41 3 .41		$+\ 12^s.11$			
		28 36 .48		$-\ 2^s.62$		
" 15, U. C.	16 9 39 .89		$+\ 9.49$		$-\ 1^s.58$	
		28 45 .97	[+ 7 .39]	$-\ 4.20$	[$-$1 .42]	$+\ 0^s.33$
" 16, L. C.	16 38 25 .86		$+\ 5.29$		$-\ 1.25$	
		28 51 .26		$-\ 5.45$		
" 16, U. C.	17 7 17 .12		$-\ 0.16$			
		28 51 .10				
" 17, L. C.	17 36 8 .22					

For interpolation by formula (72) we draw a horizontal line below the function from which we set out, and one above the next following function. These lines enclose the odd differences regularly occurring in the table. Inserting in the blanks in the columns of even differences the means of the numbers above and below, all the differences to be employed in the formula stand in the same line, namely:

$$a' = +1725^s.97,\ b_0 = +7^s.39,\ c' = -4^s.20,\ d_0 = -1^s.42,\ e' = +0^s.33$$

As n is here not a simple fraction, the computation will be most conveniently performed by logarithms, as follows:

$$4^h\ 42^m\ 19^s = 16939^s \qquad \log 4.2288878$$
$$12^h = 43200 \qquad \log 4.6354837$$
$$\log A = \log n = 9.5934041$$

$n =$	0.3921065	9.59340	9.5934	9.5934	9.5934
$n-1 = -$	0.60789	n9.78383	n9.7838	n9.7838	n9.7838
$n-\frac{1}{2} = -$	0.10789		n9.0330		n9.0330
$n-2 = -$	1.6079			n0.2063	n0.2063
$n+1 = +$	1.3921			0.1437	0.1437
		$(\frac{1}{2})$ 9.69897	$(\frac{1}{6})$ 9.2218	$(\frac{1}{24})$ 8.6198	$(\frac{1}{120})$ 7.9208
(A)	9.5934041	$(B)n$9.07620	(C) 7.6320	(D) 8.3470	$(E)n$6.6810
(a')	3.2370332	(b_0) 0.86864	$(c')n$0.6232	$(d_0)n$0.1523	(e') 9.5185
	2.8304373	n9.94484	n8.2552	n8.4993	n6.1995

Aa'	$=$	$11^m\ 16^s.764$
Bb_0	$= -$	0.879
Cc'	$= -$	0.018
Dd_0	$= -$	0.032
Ee'	$=$	0.000
Increase of R. A.	$=$	11 15.835
R. A. Greenwich Culm.	$=$	$16^h\ 9^m\ 39^s.890$
R. A. on given meridian	$=$	$16^h\ 20^m\ 55^s.725$

The use of Bessel's formula of interpolation is facilitated by a table in which the values of the coefficients above denoted by A, B, C, D, &c., and also their logarithms, are given with the argument n.

72. *Interpolation into the middle.*—When a value of the function is sought corresponding to a value of the argument which is a

mean between two values for which the function is given, that is, when $n = \frac{1}{2}$, we have by (72), since $n - \frac{1}{2} = 0$,

$$F^{(\frac{1}{2})} = F + \tfrac{1}{2}\,a' - \tfrac{1}{8}\,b_0 + \tfrac{3}{128}\,d_0 - \tfrac{5}{1024}\,f_0 + \&c.$$

or, since $F + \frac{1}{2}\,a' = \frac{1}{2}\,(F + F')$,

$$F^{(\frac{1}{2})} = \tfrac{1}{2}\,(F + F') - \tfrac{1}{8}\left[b_0 - \tfrac{3}{16}\,[d_0 - \tfrac{5}{24}\,(f_0 - \&c.)]\right] \qquad (73)$$

which is known as the formula for *interpolating into the middle*.

When the third differences are constant, d_0, f_0, &c. are zero, and the rule for interpolating into the middle between two functions is simply: *From the mean of the two functions subtract one-eighth the mean of the second differences which stand against the functions.* Interpolation by this rule is correct to third differences inclusive.

The formula (73) is especially convenient in computing tables. Values of the function to be tabulated are directly computed for values of the argument differing by $2^m w$; then interpolating a value into the middle between each two of these, the arguments now differ by $2^{m-1} w$; again interpolating into the middle between each two of the resulting series, we obtain a series with arguments differing by $2^{m-2} w$; and so on, until the interval of the argument is reduced to $2^{m-m} w$ or w.

Example.—Find the moon's right ascension for 1856 March 5, 6^h, from the values of the Ephemeris for noon and midnight.

This is the same as the example of Art. 69; but, as 6^h is the middle instant between noon and midnight, the result will be obtained by the formula (73) in the following simple manner. We have from the table in Art. 69

$$\begin{array}{lrlr}
 & b_0 = -\,38^s.16 & \tfrac{1}{2}(F + F') = & 22^h\ 12^m\ 51^s.91 \\
d_0 = +\,2^s.79, & -\tfrac{3}{16}\,d_0 = -\quad 0.52 & 38.68 \times \tfrac{1}{8} = & +\ 4.83 \\
 & -\,38.68 & F^{(\frac{1}{2})} = & 22\ \ 12\ \ 56.74
\end{array}$$

73. In case we have to interpolate between the last two values of a given series, we may consider the series in inverse order, the arguments being T, $T - w$, $T - 2w$, &c., T being the last argument. The signs of the odd differences will then be changed, and, taking the last differences in the several columns as a, b, c, d, &c., the interpolation will be effected by (68).

74. *The interpolation formulæ arranged according to the powers of the fractional part of the argument.*

When several values of the function are to be inserted between two of the given series, it is often convenient to employ the formula arranged according to the powers of n. Performing the multiplications of the factors indicated in (68), and arranging the terms, we obtain

$$\begin{aligned} F^{(n)} = F &+ n\left(a - \tfrac{1}{2}b + \tfrac{1}{3}c - \tfrac{1}{4}d + \tfrac{1}{5}e - \&c.\right) \\ &+ \frac{n^2}{1\,.\,2}\left(b - c + \tfrac{11}{12}d - \tfrac{5}{6}e + \&c.\right) \\ &+ \frac{n^3}{1\,.\,2\,.\,3}\left(c - \tfrac{3}{2}d + \tfrac{7}{4}e - \&c.\right) \\ &+ \frac{n^4}{1\,.\,2\,.\,3\,.\,4}\left(d - 2e + \&c.\right) \\ &+ \frac{n^5}{1\,.\,2\,.\,3\,.\,4\,.\,5}\left(e - \&c.\right) \\ &+ \&c. \qquad \ldots \qquad (74) \end{aligned}$$

where the differences are obtained according to the schedule in Art. 67.

Transforming (71) in the same manner, we have

$$\begin{aligned} F^{(n)} = F &+ n\left(a - \tfrac{1}{6}c + \tfrac{1}{30}e - \&c.\right) \\ &+ \frac{n^2}{1\,.\,2}\left(b - \tfrac{1}{12}d + \&c.\right) \\ &+ \frac{n^3}{1\,.\,2\,.\,3}\left(c - \tfrac{1}{4}e + \&c.\right) \\ &+ \frac{n^4}{1\,.\,2\,.\,3\,.\,4}\left(d - \&c.\right) \\ &+ \frac{n^5}{1\,.\,2\,.\,3\,.\,4\,.\,5}\left(e - \&c.\right) \\ &+ \&c. \qquad \ldots \qquad (75) \end{aligned}$$

where the differences a, c, e, are the mean interpolated odd differences in the line of the function F of the schedule Art. 69.

75. *Derivatives of a tabulated function.*—When the analytical expression of a function is given, its derivatives may be directly found by successive differentiation; but when this expression is not known, or when it is very complicated, we may obtain values of the derivatives, for particular values of the variable, from the tabulated values of the functions by means of their differences.

Denoting the argument by $T + nw$, its corresponding function

by $f(T' + nw)$, the successive derivatives of this function corresponding to the same value of the argument will be denoted by $f'(T + nw)$, $f''(T + nw)$, $f'''(T + nw)$, &c., and $f(T)$, $f'(T)$, $f''(T)$, &c., will denote the values of the function and its derivatives corresponding to the argument T, or when $n = 0$. Hence, if we regard nw as the variable, we shall have, by Maclaurin's Theorem,

$$f(T + nw) = f(T) + f'(T)\, nw + f''(T) \frac{n^2 w^2}{1 \,.\, 2} + \&c.$$

Comparing the coefficients of the several powers of n in this formula with those in (74), we have

$$\begin{aligned}
f'(T) &= \frac{1}{w} (a - \tfrac{1}{2}\, b + \tfrac{1}{3}\, c - \tfrac{1}{4}\, d + \tfrac{1}{5}\, e - \&c.) \\
f''(T) &= \frac{1}{w^2} (b - c + \tfrac{11}{12}\, d - \tfrac{5}{6}\, e + \&c.) \\
f'''(T) &= \frac{1}{w^3} (c - \tfrac{3}{2}\, d + \tfrac{7}{4}\, e - \&c.) \\
f^{\text{iv}}(T) &= \frac{1}{w^4} (d - 2\, e + \&c.) \\
f^{\text{v}}(T) &= \frac{1}{w^5} (e - \&c.) \\
\&c. \qquad & \&c. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (76)
\end{aligned}$$

the differences being taken as in Art. 67.

Still more convenient expressions are found by comparing Maclaurin's Theorem with (75); namely:

$$\begin{aligned}
f'(T) &= \frac{1}{w} (a - \tfrac{1}{6}\, c + \tfrac{1}{30}\, e - \&c.) \\
f''(T) &= \frac{1}{w^2} (b - \tfrac{1}{12}\, d + \&c.) \\
f'''(T) &= \frac{1}{w^3} (c - \tfrac{1}{4}\, e + \&c.) \\
f^{\text{iv}}(T) &= \frac{1}{w^4} (d - \&c.) \\
f^{\text{v}}(T) &= \frac{1}{w^5} (e - \&c.) \\
\&c. \qquad & \&c. \qquad\qquad\qquad\qquad (77)
\end{aligned}$$

the differences being found according to the schedule in Art. 69, and the odd differences, a, c, e, &c., being interpolated means.

The preceding formulæ determine the derivatives for the value T of the argument. To find them for any other value, we have, by differentiating Maclaurin's Formula with reference to nw,

$$f'(T + nw) = f'(T) + f''(T) \,.\, nw + \tfrac{1}{2} f'''(T) \,.\, n^2w^2 + \&c. \qquad (78)$$

in which we may substitute the values of $f'(T)$, $f''(T)$, &c. from (76) or (77).

In like manner, by successive differentiations of (78) we obtain

$$\begin{aligned} f''\ (T + nw) &= f''\ (T) + f'''(T) \,.\, nw + \tfrac{1}{2} f^{\text{iv}}(T) \,.\, n^2w^2 + \&c. \\ f'''(T + nw) &= f'''(T) + f^{\text{iv}}(T) \,.\, nw + \&c. \\ \&c. \qquad & \qquad \&c. \end{aligned}$$

76. An immediate application of (76) or (77) is the computation of *the differences in a unit of time* of the functions in the Ephemeris; for this difference is nothing more than the first derivative, denoted above by the symbol f'.

Example.—Find the difference of the moon's right ascension in one minute for 1856 March 5, 0^h.

We have in Art. 70, for T = March 5, 0^h, $a = 29^m\ 6^s.71$, $c = +\ 0^s.79$, $e = -\ 0^s.54$, and $w = 12^h = 720^m$. Hence, by the first equation of (77),

$$f'(T) = \tfrac{1}{720}\,(29^m\ 6^s.71 - 0^s.13 - 0^s.02) = 2^s.4258$$

On interpolation, consult also Encke in the Jahrbuch for 1830 and 1837.

STAR CATALOGUES.

77. The Nautical Almanac gives the position of only a small number of stars. The positions of others are to be found in the *Catalogues of stars*. These are lists of stars arranged in the order of their right ascensions, with the data from which their apparent right ascensions and declinations may be obtained for any given date.

The right ascension and declination of the so-called *fixed* stars are, in fact, ever changing: 1st, by precession, nutation, and aberration (hereafter to be specially treated of), which are not changes in the absolute position of the stars, but are either changes in the circles to which the stars are referred by spherical co-ordinates (precession and nutation), or apparent changes arising from the observer's motion (aberration); 2d, by the

proper motion of the stars themselves, which is a real change of the star's absolute position.

In the catalogues, the stars are referred to a mean equator and a mean equinox at some assumed epoch. The place of a star so referred at any time is called its *mean* place at that time; that of a star referred to the true equator and true equinox, its *true* place; that in which the star appears to the observer in motion, its *apparent* place. The mean place at any time will be found from that of the catalogue simply by applying the precession and the proper motion for the interval of time from the epoch of the catalogue. The true place will then be found by correcting the mean place for nutation; and finally the apparent place will be found by correcting the true place for aberration.

To facilitate the application of these corrections, BESSEL proposed the following very simple arrangement. He showed that if

$\alpha_0, \delta_0 =$ the star's mean right asc. and dec. at the beginning of the year,

$\alpha, \delta =$ the apparent right asc. and dec. at a time τ of that year,

$\tau =$ the time from the beginning of the year expressed in decimal parts of a year,

$\mu, \mu' =$ the annual proper motion of the star in right asc. and dec. respectively,

then,

$$\left.\begin{aligned} \alpha &= \alpha_0 + \tau\mu + Aa + Bb + Cc + Dd + E \\ \delta &= \delta_0 + \tau\mu' + Aa' + Bb' + Cc' + Dd' \end{aligned}\right\} \quad (79)$$

in which a, b, c, d, a', b', c', d' are functions of the star's right ascension and declination, and may, therefore, be computed for each star and given with it in the catalogue; A, B, C, D, E are functions of the sun's longitude, the moon's longitude, the longitude of the moon's ascending node, and the obliquity of the ecliptic, all of which depend on the time, so that A, B, C, D, E may be regarded simply as functions of the time, and given in the Nautical Almanac for the given year and day; E is a very small correction, usually neglected, as it can never exceed $0''.05$.

If the catalogue does not give the constants a, b, c, d, a', b', c', d', they may be computed, for the year 1850, by the following formulæ (see Chap. XI. p. 648):

$$
\begin{aligned}
a &= 46''.077 + 20''.056 \sin\alpha \tan\delta & a' &= 20''.056 \cos\alpha \\
b &= \cos\alpha \tan\delta & b' &= -\sin\alpha \\
c &= \cos\alpha \sec\delta & c' &= \tan\varepsilon \cos\delta - \sin\alpha \sin\delta \\
d &= \sin\alpha \sec\delta & d' &= \cos\alpha \sin\delta
\end{aligned}
$$

in which ε = obliquity of the ecliptic. Or we may resort to what are usually called the *independent constants*, and dispense with the a, b, c, d, a', b', c', d' altogether, proceeding then by the formula

$$
\left.
\begin{aligned}
\alpha &= \alpha_0 + \tau\mu + f \qquad\quad + g \sin(G + \alpha) \tan\delta + h \sin(H + \alpha) \sec\delta \\
\delta &= \delta_0 + \tau\mu' + i \cos\delta + g \cos(G + \alpha) \qquad\quad + h \cos(H + \alpha) \sin\delta
\end{aligned}
\right\} \quad (80)
$$

the independent constants f, g, G, h, H, i being given in the Ephemeris, together with the value of τ for the given date, expressed decimally.

It should be observed that the constants a, b, c, d, a', b', c', d' are not absolutely constant, since they depend on the right ascension and declination, which are slowly changing: unless, therefore, the catalogue which contains them gives also their variations, or unless the time to which we wish to reduce is not very remote from the epoch of the catalogue, it may be preferable to use the independent constants.

In forming the products Aa, Bb, &c., attention must of course be paid to the algebraic signs of the factors. The signs of A, B, C, D are, in the Ephemerides, prefixed to their logarithms; and the signs of a, b, c, &c. are in some catalogues (as that of the British Association) also prefixed to their logarithms; but I shall here, as elsewhere in this work, mark only the logarithms of *negative* factors, prefixing to them the letter *n*.

It should be remarked, also, that the B. A. C.* gives the

---

* B. A. C.—*British Association Catalogue*, containing 8377 stars, distributed in all parts of the heavens; a very useful work, but not of the highest degree of precision. The Greenwich Catalogues, published from time to time, are more reliable, though less comprehensive. For the places of certain fundamental stars, see BESSEL'S *Tabulæ Regiomontanæ* and its continuation by WOLFERS and ZECH.

LALANDE'S *Histoire Céleste* contains nearly 50,000 stars, most of which are embraced in a catalogue published by the British Association, reduced, under the direction of F. Baily, from the original work of Lalande. The Königsberg Observations embrace the series known as BESSEL'S ZONES, the most extensive series of observations of small stars yet published. The original observations are given with data for their reduction, but an important part of them is given in WEISSE'S *Positiones Mediae Stellarum fixarum in Zonis Regiomontanis a* BESSELIO *inter* —15° *et* +15° *declin. observat.*, containing nearly 32,000 stars.

See also STRUVE'S *Catal. generalis*, and the catalogues of ARGELANDER, RÜMKER,

north polar distance instead of the declination, or $\pi_0 = 90° - \delta_0$; and, since π decreases when δ increases, the corrections change their sign. This has been provided for by changing the signs of μ', a', b', c', d' in the catalogue itself. Moreover, in this catalogue, a, b, a', b' denote BESSEL's c, d, c', d', and *vice versa;* and to correspond with this, the A, B, C, D of the British Almanac denote BESSEL's C, D, A, B. The same inversion also exists in the American Ephemeris prior to the year 1865, but in the volume for 1865 the original notation is restored.

EXAMPLE.—Find the apparent right ascension and declination of α Tauri for June 15, 1865, from Argelander's Catalogue.

This star is Argel. 108; whence we take for

Jan. 1, 1830.	Mean R. A. $= 4^h\ 26^m\ 10^s.43$	Mean Decl.	$= +\ 16°\ \ 9'\ 36''.0$
Ann. prec. $= +\ 3^s.428$	for 35 yrs.	$+\ 7''.90$	$\times\ 35$
Prop. motion $= +\ 0\ .005$		$-\ 0\ .17$	
	$= +\ \ 2\ \ \ 0.155$		$= \ \ +\ \ \ 4\ 30\ .55$
Jan. 1, 1865,	$\alpha_0 = 4\ \ 28\ \ 10.585$		$\delta_0 = +\ 16\ \ 14\ \ \ 6\ .55$

We next take the logarithms

from the Catal.	logs.	a 0.5352	b 7.8794	c 8.4329	d 8.8058
from Am. Ephem. for June 15, 1865,	logs.	A 9.7877	B 0.9437	C n0.2125	D n1.3089
from the Catal.	logs.	a' 0.8934	b' n9.9607	c' 9.2019	d' 9.0378
	logs.	Aa 0.3229	Bb 8.8231	Cc n8.6454	Dd n0.1147
	logs.	Aa' 0.6811	Bb' n0.9044	Cc' n9.4144	Dd' n0.3467

Corr. of α_0,	$Aa = +\ 2^s.103$,	$Bb = +\ 0^s.067$,	$Cc = -\ 0^s.044$,	$Dd = -\ 1^s.302$
Corr. of δ_0,	$Aa' = +\ 4''.80$,	$Bb' = -\ 8''.02$,	$Cc' = -\ 0''.26$,	$Dd' = -\ 2''.22$

We have also from the catalogue $\mu = +\ 0^s.005$, $\mu' = -\ 0''.17$. The fraction of a year for June 15, 1865, is $\tau = 0.46$; and hence

Jan. 1, 1865,	$\alpha_0 = 4^h\ 28^m\ 10^s.585$		$\delta_0 = +\ 16°\ 14'\ 6''.55$
Sum of corr. of α_0	$= \ \ +\ \ \ 0\ .824$	Sum of corr. of $\delta_0 =$	$-\ 5\ .70$
	$\tau\mu = \ \ +\ \ \ 0\ .002$	$\tau\mu' =$	$-\ 0\ .08$
June 15, 1865	$\alpha = 4\ \ 28\ \ 11\ .411$		$\delta = +\ 16\ \ 14\ \ 0\ .77$

78. When the greatest precision is required, we should consider the change in the star's place even in a fraction of a day, and therefore also the change while the star is passing from one meridian to another; also the secular variation and the changes

PIAZZI, SANTINI; and the published observations of the principal observatories. See also a list of catalogues in the introduction to the B. A. C.

in the precession and in the logarithms of the constants. Further, it is to be observed that the annual precession of the catalogues is for a mean year of $365^d\ 5^h.8$. But for a fuller consideration of this subject see Chapter XI.

CHAPTER III.

FIGURE AND DIMENSIONS OF THE EARTH.

79. The apparent positions of those heavenly bodies which are within *measurable* distances from the earth are different for observers on different parts of the earth's surface, and, therefore, before we can compare observations taken in different places we must have some knowledge of the form and dimensions of the earth. I must refer the reader to geodetical works for the methods by which the exact dimensions of the earth have been obtained, and shall here assume such of the results as I shall have occasion hereafter to apply.

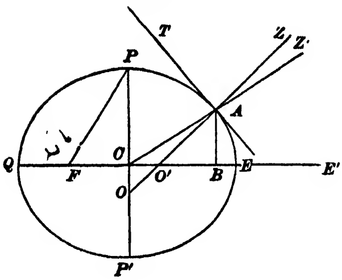
The figure of the earth is very nearly that of an *oblate spheroid*, that is, an ellipsoid generated by the revolution of an ellipse about its minor axis. The section made by a plane through the earth's axis is nearly an ellipse, of which the major axis is the equatorial and the minor axis the polar diameter of the earth. Accurate geodetical measurements have shown that there are small deviations from the regular ellipsoid; but it is sufficient for the purposes of astronomy to assume all the meridians to be ellipses with the mean dimensions deduced from all the measures made in various parts of the earth.

80. Let $EPQP'$, Fig. 11, be one of the elliptical meridians of the earth, EQ the diameter of the equator, PP' the polar diameter, or axis of the earth, C the centre, F a focus of the ellipse. Let

a = the semi-major axis, or equatorial radius, $= CE$,
b = the semi-minor axis, or polar radius, $= CP$,
c = the compression of the earth,
e = the eccentricity of the meridian.

By the *compression* is meant the difference of the equatorial and polar radii expressed in parts of the equatorial radius as unity, or

Fig. 11.



$$c = \frac{a-b}{a} = 1 - \frac{b}{a}$$

The *eccentricity of the meridian* is the distance of either focus from the centre, also expressed in parts of the equatorial radius, or, in Fig. 11.

$$e = \frac{CF}{CE}$$

But, since $PF = CE$, we have,

$$\frac{\overline{CF}^2}{\overline{CE}^2} = \frac{\overline{PF}^2 - \overline{PC}^2}{\overline{CE}^2} = 1 - \frac{\overline{PC}^2}{\overline{CE}^2}$$

that is,

$$e^2 = 1 - \frac{b^2}{a^2} = 1 - (1-c)^2$$

or

$$e = \sqrt{2c - c^2} \tag{81}$$

By a combination of all the most reliable measures, BESSEL deduced the most probable form of the spheroid, or that which most nearly represents all the observations that have been made in different parts of the world. He found*

$$\frac{b}{a} = 1 - c = \frac{298.1528}{299.1528}$$

or

$$c = \frac{1}{299.1528}$$

whence, by (81),

$$e = .0816967$$

$$\log e = 8.912205 \qquad \log \sqrt{(1-ee)} = 9.9985458$$

---

* *Astronomische Nachrichten*, No. 438. See also Encke's Tables of the dimensions of the terrestrial spheroid in the *Jahrbuch* for 1852.

The absolute lengths of the semi-axes, according to BESSEL, are,

$a = 6377397.15$ metres $= 6974532.34$ yds. $= 3962.802$ miles
$b = 6356078.96$ " $= 6951218.06$ " $= 3949.555$ "

81. *To find the reduction of the latitude for the compression of the earth.*

Let A, Fig. 11, be a point on the surface of the earth; AT the tangent to the meridian at that point; AO, perpendicular to AT, the normal to the earth's surface at A. A plane touching the earth's surface at A is the plane of the horizon at that point (Art. 3), and therefore AO, which is perpendicular to that plane, represents the *vertical line* of the observer at A. This vertical line does not coincide with the radius, except at the equator and the poles. If we produce CE, OA, and CA to meet the celestial sphere in E', Z, and Z' respectively, the angle $ZO'E'$ is the declination of the zenith, or (Art. 7) the *geographical latitude*, and Z is the *geographical zenith;* the angle $Z'CE'$ is the declination of the geocentric zenith Z', and is called the *geocentric or reduced latitude;* and $ZAZ' = CAO$ is called the *reduction of the latitude.* It is evident that the geocentric is always less than the geographical latitude.

Now, if we take the axes of the ellipse as the axes of co-ordinates, the centre being the origin, and denote by x the abscissa, and by y the ordinate of any point of the curve, by a and b the semi-major and semi-minor axes respectively, the equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

If we put

$$\varphi = \text{the geographical latitude,}$$
$$\varphi' = \text{the geocentric} \quad \text{"}$$

we have, since φ is the angle which the normal makes with the axis of abscissæ,

$$\tan \varphi = -\frac{dx}{dy}$$

and from the triangle ACB,

$$\tan \varphi' = \frac{y}{x}$$

Differentiating the equation of the ellipse, we have

$$\frac{y}{x} = -\frac{b^2}{a^2} \cdot \frac{dx}{dy}$$

or

$$\tan \varphi' = \frac{b^2}{a^2} \tan \varphi = (1 - e^2) \tan \varphi \tag{82}$$

which determines the relation between φ and φ'.

To find the difference $\varphi - \varphi'$, or the reduction of the latitude, we have recourse to the general development in series of an equation of the form

$$\tan x = p \tan y$$

which [Pl. Trig. Art. 254] is

$$x - y = q \sin 2y + \tfrac{1}{2} q^2 \sin 4y + \&c.$$

in which

$$q = \frac{p-1}{p+1}$$

Applying this to the development of (82), we find, after dividing by sin 1″ to reduce the terms of the series to seconds,

$$\varphi - \varphi' = -\frac{q}{\sin 1''} \sin 2\varphi - \frac{q^2}{2 \sin 1''} \sin 4\varphi - \&c. \tag{83}$$

in which

$$q = \frac{p-1}{p+1} = \frac{1 - e^2 - 1}{1 - e^2 + 1} = -\frac{e^2}{2 - e^2}$$

Employing Bessel's value of e, we find

$$-\frac{q}{\sin 1''} = 690''.65 \qquad -\frac{q^2}{2 \sin 1''} = -1''.16$$

and, the subsequent terms being insensible,

$$\varphi - \varphi' = 690''.65 \sin 2\varphi - 1''.16 \sin 4\varphi \tag{83*}$$

by which $\varphi - \varphi'$ is readily computed for given values of φ. Its value will be found in our Table III. Vol. II. for any given value of φ.

Example.—Find the reduced latitude when $\varphi = 35°$. We find by (83), or Table III.,

$$\varphi - \varphi' = 648''.25 = 10' \, 48''.25$$

and hence the reduced or geocentric latitude

$$\varphi' = 34° \, 49' \, 11''.75$$

82. *To find the radius of the terrestrial spheroid for a given latitude.* Let

$$\rho = \text{the radius for the latitude } \varphi = AC.$$

We have

$$\rho = \sqrt{x^2 + y^2}$$

To express x and y in terms of φ, we have from the equation of the ellipse and its differential equation, after substituting $1 - e^2$ for $\frac{b^2}{a^2}$,

$$x^2 + \frac{y^2}{1 - e^2} = a^2$$

$$\frac{y}{x} = (1 - e^2) \tan \varphi$$

from which by a simple elimination we find

$$x = \frac{a \cos \varphi}{\sqrt{(1 - e^2 \sin^2 \varphi)}}$$

$$y = \frac{(1 - e^2)\, a \sin \varphi}{\sqrt{(1 - e^2 \sin^2 \varphi)}}$$

and hence

$$\rho = a \sqrt{\left[\frac{1 - 2\, e^2 \sin^2 \varphi + e^4 \sin^2 \varphi}{1 - e^2 \sin^2 \varphi}\right]} \tag{84}$$

by which the value of ρ may be computed. The logarithm of ρ, putting $a = 1$, is given in our Table III. Vol. II.

But the logarithm of ρ may be more conveniently found by a series. If in (84) we substitute

$$e^2 = 1 - f^2$$

$$\sin^2 \varphi = \tfrac{1}{2}(1 - \cos 2\varphi)$$

we find, putting $a = 1$,

$$\rho = \sqrt{\left[\frac{1 + f^4 + (1 - f^4) \cos 2\varphi}{1 + f^2 + (1 - f^2) \cos 2\varphi}\right]}$$

$$= \frac{1 + f^2}{1 + f} \sqrt{\left[\frac{1 + \left(\frac{1 - f^2}{1 + f^2}\right)^2 + 2\left(\frac{1 - f^2}{1 + f^2}\right) \cos 2\varphi}{1 + \left(\frac{1 - f}{1 + f}\right)^2 + 2\left(\frac{1 - f}{1 + f}\right) \cos 2\varphi}\right]}$$

Now (Pl. Trig. Art. 260) if we have an expression of the form

$$X = \sqrt{(1 + m^2 - 2m \cos C)} \tag{A}$$

we have, if M = the modulus of the common system of logarithms,

$$\log X = -M\left(m \cos C + \frac{m^2 \cos 2C}{2} + \frac{m^3 \cos 3C}{3} + \&c.\right) \qquad (B)$$

by which we may develop the logarithms of the numerator and denominator of the above radical.

Hence we find

$$\log \rho = \log \frac{1+f^2}{1+f} + M\left((m - m') \cos 2\varphi - \frac{m^2 - m'^2}{2} \cos 4\varphi \right.$$
$$\left. + \frac{m^3 - m'^3}{3} \cos 6\varphi - \&c. \right)$$

in which we have put for brevity

$$m = \frac{1-f^2}{1+f^2} \qquad m' = \frac{1-f}{1+f}$$

Restoring the value of $f = \sqrt{(1 - e^2)}$ and computing the numerical values of the coefficients, we find

$$\log \rho = 9.9992747 + 0.0007271 \cos 2\varphi - 0.0000018 \cos 4\varphi \qquad (85)$$

as given by ENCKE in the Jahrbuch for 1852.

The values of ρ and φ' may also be determined under another form which will hereafter be found useful.

We have in Fig. 11, $\rho \sin \varphi' = y$, $\rho \cos \varphi' = x$, or

$$\left.\begin{aligned} \rho \sin \varphi' &= \frac{a\,(1 - e^2) \sin \varphi}{\sqrt{(1 - e^2 \sin^2 \varphi)}} \\ \rho \cos \varphi' &= \frac{a \cos \varphi}{\sqrt{(1 - e^2 \sin^2 \varphi)}} \end{aligned}\right\} \qquad (86)$$

which may be put under a simple form by introducing an auxiliary ψ, so that

$$\left.\begin{aligned} \sin \psi &= e \sin \varphi \\ \rho \sin \varphi' &= a\,(1 - e^2) \sin \varphi \sec \psi \\ \rho \cos \varphi' &= a \cos \varphi \sec \psi \end{aligned}\right\} \qquad (87)$$

We can also deduce from these,

$$\left.\begin{aligned} \rho \sin (\varphi - \varphi') &= \tfrac{1}{2}\, ae^2 \sin 2\varphi \sec \psi \\ \rho \cos (\varphi - \varphi') &= a \cos \psi \end{aligned}\right\} \qquad (88)$$

Hence, also, the following:

$$\rho = a \sqrt{\left(\frac{\cos\varphi}{\cos\varphi'\cos(\varphi-\varphi')}\right)} \tag{89}$$

83. *To find the length of the normal terminating in the axis, for a given latitude.*

Putting

$$N = \text{the normal} = AO \text{ (Fig. 11)},$$

we have evidently

$$N = \frac{\rho\cos\varphi'}{\cos\varphi} = \frac{a}{\sqrt{(1-e^2\sin^2\varphi)}} \tag{90}$$

or, employing the auxiliary ψ of the preceding article,

$$N = a\sec\psi$$

84. *To find the distance from the centre to the intersection of the normal with the axis.*

Denoting this distance by ai (so that i denotes the distance when $a = 1$), we have in Fig. 11,

$$ai = CO$$

and, from the triangle ACO,

$$ai = \frac{\rho\sin(\varphi-\varphi')}{\cos\varphi}$$

or, by (88),

$$ai = \frac{ae^2\sin\varphi}{\sqrt{(1-e^2\sin^2\varphi)}} = ae^2\sin\varphi\sec\psi \tag{91}$$

85. *To find the radius of curvature of the terrestrial meridian for a given latitude.*—Denoting this radius by R, we have, from the differential calculus,

$$R = \frac{[1+(D_x y)^2]^{\frac{3}{2}}}{D_x^2 y}$$

where we employ the notation $D_x y$, $D_x^2 y$ to denote the first and second differential coefficients of y relatively to x. We have from the equation of the ellipse

$$D_x y = -\frac{b^2}{a^2}\cdot\frac{x}{y} \qquad D_x^2 y = -\frac{b^4}{a^2y^3}$$

whence

$$R = \frac{(a^4 y^2 + b^4 x^2)^{\frac{3}{2}}}{a^4 b^4}$$

Observing that $b^2 = a^2(1 - e^2)$, we find, by substituting the values of x and y in terms of φ (p. 99),

$$R = \frac{a(1 - e^2)}{(1 - e^2 \sin^2 \varphi)^{\frac{3}{2}}} \tag{92}$$

EXAMPLE.—Find the radius of curvature for the latitude of Greenwich, $\varphi = 51° 28' 38''.2$, taking $a = 6377397$ metres. We find

$$R = 6373850 \text{ metres.}$$

86. *Abnormal deviations of the plumb line.*—Granting the geometrical figure of the earth to be that of an ellipsoid of revolution whose dimensions, taking the mean level of the sea, are as given in Art. 80, it must not be inferred that the direction of the plumb line at any point of the surface always coincides precisely with the normal of the ellipsoid. It would do so, indeed, if the earth were an exact ellipsoid composed of perfectly homogeneous matter, or if, originally homogeneous and plastic, it has assumed its present form solely under the influence of the attraction of gravitation combined with the rotation on its axis. But experience has shown* that the plumb line mostly deviates from the normal to the regular ellipsoid, not only towards the north or south, but also towards the east or west: so that the apparent zenith as indicated by the plumb line differs from the true zenith corresponding to the normal both in declination and right ascension. These deviations are due to local irregularities both in the figure and the density of the earth. Their amount is, however, very small, seldom reaching more than 3″ of arc in any direction.

In order to eliminate the influence of these deviations at a given place, observations are made at a number of places as nearly as possible symmetrically situated around it, and, assuming the dimensions of the general ellipsoid to be as we have given them, the direction of the plumb line at the given place is deduced from its direction at each of the assumed places (by

* U.S. Coast Survey Report for 1853, p. 14*.

the aid of the geodetic measures of its distance and direction from each); or, which is the same thing, the latitude and longitude of the place are deduced from those of each of the assumed places: then the mean of all the resulting latitudes is the *geodetic latitude*, and the mean of all the resulting longitudes is the *geodetic longitude*, of the place. These quantities, then, correspond as nearly as possible to the true normal of the regular ellipsoid; the geodetic latitude being the angle which this normal makes with the plane of the equator, and the geodetic longitude being the angle which the meridian plane containing this normal makes with the plane of the first meridian. The geodetic latitude is identical with the *geographical* latitude as we have defined it in Art. 81.

The *astronomical latitude* of a place is the declination of the apparent zenith indicated by the actual plumb line; but, unless when the contrary is stated, it will be hereafter understood to be identical with the geographical or geodetic latitude.

It has recently been attempted to show that the earth differs sensibly from an ellipsoid of revolution;* but no deduction of this kind can be safely made until the anomalous deviations of the plumb line above noticed have been eliminated from the discussion.

CHAPTER IV.

REDUCTION OF OBSERVATIONS TO THE CENTRE OF THE EARTH.

87. THE places of stars given in the Ephemerides are those in which the stars would be seen by an observer at the centre of the earth, and are called *geocentric*, or *true*, places. Those observed from the surface of the earth are called *observed*, or *apparent*, places.

It must be remarked, however, that the geocentric places of the Ephemeris are also called apparent places when it is intended

* See *Astr. Nach.* No. 1303.

to distinguish them from *mean* places, a distinction which will be considered hereafter (Chap. XI.).

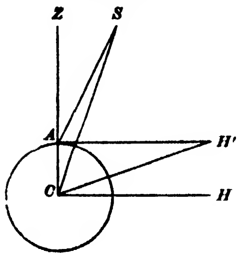
It will also be noticed that we frequently use the terms *true* and *apparent* as *relative* terms only; as, for example, in treating of the effect of parallax, the place of a star as seen from the centre of the earth may be called true, and that in which it would be seen from the surface of the earth were there no atmosphere, may in relation to the former be called apparent; but in considering the effect of refraction, the star's place as it would be seen from the surface of the earth were there no atmosphere may be called true, and the place as affected by the refraction may in relation to the former be called apparent; and similarly in other cases.

PARALLAX.

88. The *parallax* of a star is, in general, the difference of the directions of the straight lines drawn to the star from two different points. The difference of direction of two straight lines being simply the angle contained between them, we may also define parallax as *the angle at the star* contained by the lines drawn to the two points from which it is supposed to be viewed.

In astronomy we frequently use the term parallax to express the difference of altitude or of zenith distance of a star seen from the surface and the centre of the earth respectively; and, in order to express parallax in respect to other co-ordinates, proper qualifying terms are added, as "parallax in declination," &c.

Fig. 12.



Assuming (at first) the earth to be a sphere, let A, Fig. 12, be the position of the observer on its surface, C the centre, CAZ the vertical line, and S a star within a measurable distance CS from the centre. AH', a tangent to the surface at A, and CH, parallel to it, drawn through the centre, may each be regarded as lying in the plane of the celestial horizon (note, p. 19). The true or geocentric altitude of the star above the celestial horizon is then the angle SCH, and the apparent altitude is the angle SAH'. In this case the directions of the star from C and from A are compared with each other by referring them to two

fines which have a common direction, *i.e.* parallel lines. But a still more direct method of comparison is obtained by referring them to one and the same straight line, as CAZ, Z being the zenith. We then call ZCS the true and ZAS the apparent zenith distance, and these are evidently the complements of the true and apparent altitudes respectively.

In the figure we have at once

$$ZAS - ZCS = ASC$$

that is, the parallax in zenith distance or altitude is the angle at the star subtended by the radius of the earth. When the star is in the horizon, as at H', the radius, being at right angles to AH', subtends the greatest possible angle at the star for the same distance, and this maximum angle is called the *horizontal parallax.* The *equatorial horizontal parallax* of a star is the maximum angle subtended at the star by the equatorial radius of the earth.

89. *To find the equatorial horizontal parallax of a star at a given distance from the centre of the earth.*

Let

π = the equatorial horizontal parallax,
Δ = the given distance of the star from the earth's centre,
a = the equatorial radius of the earth,

we have from the triangle CAH' in Fig. 12, if CA is the equatorial radius,

$$\sin \pi = \frac{a}{\Delta} \tag{93}$$

The value of π given in the Ephemeris is always that which is given by this formula when for Δ we employ the distance of the star at the instant for which the parallax is given.

90. *To find the parallax in altitude or zenith distance, the earth being regarded as a sphere.*

Let

ζ = the true zenith distance = ZCS (Fig. 12),
ζ' = the apparent zenith distance = ZAS,

The triangle SAC gives, observing that the angle $SAC = 180^\circ - \zeta'$,

$$\frac{\sin p}{\sin \zeta'} = \frac{a}{\Delta} = \sin \pi$$

or,

$$\sin p = \sin (\zeta' - \zeta) = \sin \pi \sin \zeta' \tag{94}$$

If we put

$$h = \text{the true altitude,}$$
$$h' = \text{the apparent altitude,}$$

then it follows also that

$$\sin p = \sin (h - h') = \sin \pi \cos h' \tag{95}$$

Except in the case of the moon, the parallax is so small that we may consider π and p to be proportional to their sines [Pl. Trig. Art. 55]; and then we have

$$p = \pi \sin \zeta' = \pi \cos h' \tag{96}$$

Since when $\zeta' = 90^\circ$ we have $\sin \zeta' = 1$, and when $\zeta' = 0$, $\sin \zeta' = 0$, it follows that the parallax is a maximum when the star is in the horizon, and zero when the star is in the zenith.

Example.—Given the apparent zenith distance of Venus, $\zeta' = 64^\circ\ 43'$, and the horizontal parallax $\pi = 20''.0$; find the geocentric zenith distance.

		$\log \pi$	1.3010
$\zeta' =$	$64^\circ\ 43'\ 0''0$	$\log \sin \zeta'$	9.9563
$p =$	18.1	$\log p$	1.2573
$\zeta =$	64 42 41.9		

When the true zenith distance is given, to compute the parallax, we may first use this true zenith distance as the apparent, and find an approximate value of p by the formula $p = \pi \sin \zeta$; then, taking the approximate value of $\zeta' = \zeta + p$, we compute a more exact value of p by the formula (94) or (96). This second approximation is unnecessary in all cases except that of the moon, and the parallax of the moon is so great that it becomes necessary to take into account the true figure of the earth, as in the following more general investigation of the subject.

91. In consequence of the spheroidal figure of the earth, the vertical line of the observer does not pass through the centre, and therefore the geocentric zenith distance cannot be directly

referred to this line. If, however, we refer it to the radius drawn from the place of observation (or CAZ', Fig. 11), the zenith distance is that measured from the geocentric zenith of the place; whereas it is desirable to use the geographical zenith. Hence we shall here consider the geocentric zenith distance to be the angle which the straight line drawn from the centre of the earth to the star makes with the straight line drawn through the centre of the earth *parallel to the vertical line of the observer*. These two vertical lines are conceived to meet the celestial sphere in the same point, namely, the geographical zenith, which is the common vanishing point of all lines perpendicular to the plane of the horizon. Thus both the true and the apparent zenith distances will be measured upon the celestial sphere from the pole of the horizon.

The azimuth of a star is, in general, the angle which a vertical plane passing through the star makes with the plane of the meridian. When such a vertical plane is drawn through the centre of the earth, it does not coincide with that drawn at the place of observation, since, by definition (Art. 3), the vertical plane passes through the vertical line, and the vertical lines are not coincident. Hence we shall have to consider a parallax in *azimuth* as well as in *zenith distance*.

92. *To find the parallax of a star in zenith distance and azimuth when the geocentric zenith distance and azimuth are given, and the earth is regarded as a spheroid.**

Let the star be referred to three co-ordinate planes at right angles to each other: the first, the plane of the horizon of the observer; the second, the plane of the meridian; the third, the plane of the prime vertical. Let the axis of x be the meridian line, or intersection of the plane of the meridian and the plane of the horizon; the axis of y, the east and west line; the axis of z, the vertical line. Let the positive axis of x be towards the south; the positive axis of y, towards the west; the positive axis of z, towards the zenith. Let

Δ' = the distance of the star from the origin, which is the place of observation,
ζ' = the apparent zenith distance of the star,
A' = the apparent azimuth " " "

* The investigation which follows is nearly the same as that of OLBERS, to whom the method itself is due.

then, x' y', z' denoting the co-ordinates of the star in this system, we have, by (39),

$$\begin{aligned} x' &= \Delta' \sin \zeta' \cos A' \\ y' &= \Delta' \sin \zeta' \sin A' \\ z' &= \Delta' \cos \zeta' \end{aligned}$$

Again, let the star be referred by rectangular co-ordinates to another system of planes parallel to the former, the origin now being the centre of the earth. In the celestial sphere these planes still represent the horizon, the meridian, and the prime vertical. If then in this system we put

Δ = the distance of the star from the origin,
ζ = the true zenith distance of the star,
A = the true azimuth " "

and denote the co-ordinates of the star in this system by x, y, and z, we have, as before,

$$\begin{aligned} x &= \Delta \sin \zeta \cos A \\ y &= \Delta \sin \zeta \sin A \\ z &= \Delta \cos \zeta \end{aligned}$$

Now, the co-ordinates of the place of observation in this last system, being denoted by a, b, c, we have

$$a = \rho \sin (\varphi - \varphi') \qquad b = 0 \qquad c = \rho \cos (\varphi - \varphi')$$

in which ρ = the earth's radius for the latitude φ of the place of observation, and φ' is the geocentric latitude, $\varphi - \varphi'$ being the reduction of the latitude, Art. 81; and the formulæ of transformation from this second system to the first are (Art. 33)

$$x = x' + a \qquad y = y' + b \qquad z = z' + c$$

or,

$$x' = x - a \qquad y' = y - b \qquad z' = z - c$$

whence, substituting the above values of the co-ordinates,

$$\left.\begin{aligned} \Delta' \sin \zeta' \cos A' &= \Delta \sin \zeta \cos A - \rho \sin (\varphi - \varphi') \\ \Delta' \sin \zeta' \sin A' &= \Delta \sin \zeta \sin A \\ \Delta' \cos \zeta' &= \Delta \cos \zeta - \rho \cos (\varphi - \varphi') \end{aligned}\right\} \tag{97}$$

which are the general relations between the true and apparent zenith distances and azimuths. All the quantities in the second members being given, the first two equations determine $\Delta' \sin \zeta'$, and A'; and then from this value of $\Delta' \sin \zeta'$, and that of $\Delta' \cos \zeta'$ given by the third equation, Δ' and ζ' are determined.

But it is convenient to introduce the horizontal parallax instead of Δ. For, if we put the equatorial radius of the earth $= 1$, we have

$$\sin \pi = \frac{1}{\Delta}$$

and hence, if we divide the equations (97) by Δ, and put

$$f = \frac{\Delta'}{\Delta}$$

we have

$$\left.\begin{aligned} f \sin \zeta' \cos A' &= \sin \zeta \cos A - \rho \sin \pi \sin (\varphi - \varphi') \\ f \sin \zeta' \sin A' &= \sin \zeta \sin A \\ f \cos \zeta' &= \cos \zeta - \rho \sin \pi \cos (\varphi - \varphi') \end{aligned}\right\} \quad (98)$$

To obtain expressions for the difference between ζ and ζ' and between A and A', that is, for the parallax in zenith distance and azimuth, multiply the first equation of (98) by $\sin A$, the second by $\cos A$, and subtract the first product from the second; again, multiply the first by $\cos A$, the second by $\sin A$, and add the products: we find

$$\left.\begin{aligned} f \sin \zeta' \sin (A' - A) &= \rho \sin \pi \sin (\varphi - \varphi') \sin A \\ f \sin \zeta' \cos (A' - A) &= \sin \zeta - \rho \sin \pi \sin (\varphi - \varphi') \cos A \end{aligned}\right\} \quad (99)$$

Multiplying the first of these by $\sin \frac{1}{2} (A' - A)$, the second by $\cos \frac{1}{2} (A' - A)$, and adding the products, we find, after dividing the sum by $\cos \frac{1}{2} (A' - A)$,

$$f \sin \zeta' = \sin \zeta - \rho \sin \pi \sin (\varphi - \varphi') \frac{\cos \frac{1}{2} (A' + A)}{\cos \frac{1}{2} (A' - A)}$$

which with the third equation of (98) will determine ζ'.

If we assume γ such that

$$\tan \gamma = \tan (\varphi - \varphi') \frac{\cos \frac{1}{2} (A' + A)}{\cos \frac{1}{2} (A' - A)} \quad (100)$$

we have the following equations for determining ζ':

$$\left.\begin{aligned} f \sin \zeta' &= \sin \zeta - \rho \sin \pi \cos (\varphi - \varphi') \tan \gamma \\ f \cos \zeta' &= \cos \zeta - \rho \sin \pi \cos (\varphi - \varphi') \end{aligned}\right\} \quad (101)$$

which, by the process employed in deducing (99), give

$$\left.\begin{aligned} f \sin (\zeta' - \zeta) &= \qquad \rho \sin \pi \cos (\varphi - \varphi') \frac{\sin (\zeta - \gamma)}{\cos \gamma} \\ f \cos (\zeta' - \zeta) &= 1 - \rho \sin \pi \cos (\varphi - \varphi') \frac{\cos (\zeta - \gamma)}{\cos \gamma} \end{aligned}\right\} \quad (102)$$

By multiplying the first of these by $\sin \frac{1}{2}(\zeta' - \zeta)$, the second by $\cos \frac{1}{2}(\zeta' - \zeta)$, and adding the products, we find, after dividing by $\cos \frac{1}{2}(\zeta' - \zeta)$,

$$f = 1 - \frac{\rho \sin \pi \cos (\varphi - \varphi') \cos [\frac{1}{2}(\zeta' + \zeta) - \gamma]}{\cos \gamma \cos \frac{1}{2}(\zeta' - \zeta)}$$

or multiplying by Δ,

$$\Delta' = \Delta - \frac{\rho \cos (\varphi - \varphi') \cos [\frac{1}{2}(\zeta' + \zeta) - \gamma]}{\cos \gamma \cos \frac{1}{2}(\zeta' - \zeta)} \tag{103}$$

The equations (99) determine rigorously the parallax in azimuth; then (100) and (102) determine the parallax in zenith distance, and (103) the distance of the star from the observer.

The relation between Δ and Δ' may be expressed under a more simple form. By multiplying the first of the equations (101) by $\cos \gamma$, the second by $\sin \gamma$, the difference of the products gives

$$\Delta' = \Delta \frac{\sin (\zeta - \gamma)}{\sin (\zeta' - \gamma)} \tag{104}$$

93. The preceding formulæ may be developed in series. Put

$$m = \frac{\rho \sin \pi \sin (\varphi - \varphi')}{\sin \zeta}$$

then (99) become

$$f \sin \zeta' \sin (A' - A) = m \sin \zeta \sin A$$
$$f \sin \zeta' \cos (A' - A) = \sin \zeta (1 - m \cos A)$$

whence

$$\tan (A' - A) = \frac{m \sin A}{1 - m \cos A} \tag{105}$$

and therefore [Pl. Trig. Art. 258], $A' - A$ being in seconds,

$$A' - A = \frac{m \sin A}{\sin 1''} + \frac{m^2 \sin 2A}{2 \sin 1''} + \frac{m^3 \sin 3A}{3 \sin 1''} + \&c. \tag{106}$$

To develop γ in series, we take

$$\tan \gamma = \tan (\varphi - \varphi') \frac{\cos [A + \frac{1}{2}(A' - A)]}{\cos \frac{1}{2}(A' - A)}$$
$$= \tan (\varphi - \varphi') [\cos A - \sin A \tan \tfrac{1}{2}(A' - A)]$$

whence, by interchanging arcs and tangents according to the

formulæ $\tan^{-1} y = y - \frac{1}{3} y^3 + \&c.$, $\tan x = x + \frac{1}{3} x^3 + \&c.$ [Pl. Trig. Arts. 209, 213],

$$\gamma = (\varphi - \varphi') \cos A - \frac{(\varphi - \varphi')^2 \rho \sin \pi \sin^2 A \sin 1''}{2 \sin \zeta} + \&c. \quad (107)$$

where the second term of the series is multiplied by $\sin 1''$ because γ and $\varphi - \varphi'$ are supposed to be expressed in seconds.

Again, if we put

$$n = \frac{\rho \sin \pi \cos (\varphi - \varphi')}{\cos \gamma}$$

we find from (102)

$$\tan (\zeta' - \zeta) = \frac{n \sin (\zeta - \gamma)}{1 - n \cos (\zeta - \gamma)} \quad (108)$$

whence, $\zeta' - \zeta$ being in seconds,

$$\zeta' - \zeta = \frac{n \sin (\zeta - \gamma)}{\sin 1''} + \frac{n^2 \sin 2 (\zeta - \gamma)}{2 \sin 1''} + \frac{n^3 \sin 3 (\zeta - \gamma)}{3 \sin 1''} + \&c. \quad (109)$$

Adding the squares of the equations (102), we have

$$f^2 = \left(\frac{\Delta'}{\Delta}\right)^2 = 1 - 2 n \cos (\zeta - \gamma) + n^2$$

whence [equations (A) and (B), Art. 82]

$$\log \Delta' = \log \Delta - M \left(n \cos (\zeta - \gamma) + \frac{n^2 \cos 2 (\zeta - \gamma)}{2} + \&c.\right) \quad (110)$$

where M = the modulus of common logarithms.

94. The second term of the series (107) is of wholly inappreciable effect; so that we may consider as exact the formula

$$\gamma = (\varphi - \varphi') \cos A \quad (111)$$

and the rigorous formulæ (105) and (108) may be readily computed under the following form:

Put

$$\left.\begin{aligned} \sin \vartheta &= m \cos A = \frac{\rho \sin \pi \sin (\varphi - \varphi') \cos A}{\sin \zeta} \\ &\text{then} \\ \tan (A' - A) &= \frac{\sin \vartheta \tan A}{1 - \sin \vartheta} = \tan \vartheta \tan (45° + \tfrac{1}{2} \vartheta) \tan A \end{aligned}\right\} \quad (112)$$

Put

$$\left.\begin{aligned} \sin\vartheta' &= n\cos(\zeta-\gamma) = \frac{\rho\sin\pi\cos(\varphi-\varphi')\cos(\zeta-\gamma)}{\cos\gamma} \\ \text{then} \qquad \tan(\zeta'-\zeta) &= \frac{\sin\vartheta'\tan(\zeta-\gamma)}{1-\sin\vartheta'} \\ &= \tan\vartheta'\tan(45^\circ+\tfrac{1}{2}\vartheta')\tan(\zeta-\gamma) \end{aligned}\right\} \quad (113)$$

EXAMPLE.—In latitude $\varphi = 38^\circ\ 59'$, given for the moon, $A = 320^\circ\ 18'$, $\zeta = 29^\circ\ 30'$, and $\pi = 58'\ 37''.2$, to find the parallax in azimuth and zenith distance.

We have (Table III.) for $\varphi = 38^\circ\ 59'$, $\varphi - \varphi' = 11'\ 15''$, $\log\rho = 9.999428$: hence by (111) $\gamma = 8'\ 39''.3$ and $\zeta - \gamma = 29^\circ\ 21'\ 20''.7$; with which we proceed by (112) and (113) as follows:

	$\log\rho\sin\pi$	8.23118		$\log\rho\sin\pi$	8.231179
	$\log\sin(\phi-\phi')$	7.51488		$\log\cos(\phi-\phi')$	9.999998
	$\log\operatorname{cosec}\zeta$	0.30766		$\log\sec\gamma$	0.000001
	$\log\cos A$	9.88615		$\log\cos(\zeta-\gamma)$	9.940313
$\vartheta = 18''$,	$\log\sin\vartheta$	5.93987	$\vartheta' = 51'\ 1''.5$,	$\log\sin\vartheta'$	8.171491
	$\log\tan\vartheta$	5.93987		$\log\tan\vartheta'$	8.171539
	$\log\tan(45^\circ+\frac{1}{2}\vartheta)$	0.00004		$\log\tan(45^\circ+\frac{1}{2}\vartheta')$	0.006446
	$\log\tan A$	n9.91919		$\log\tan(\zeta-\gamma)$	9.750087
	$\log\tan(A'-A)$	n5.85910		$\log\tan(\zeta'-\zeta)$	7.928072
	$A'-A$	$=-14''.91$		$\zeta'-\zeta$	$= 29'\ 7''.79$
	A'	$= 320^\circ 17'\ 45''.09$		ζ'	$= 29^\circ\ 59'\ 7''.79$

It is evident that we may, without a sacrifice of accuracy, omit the factors $\cos(\varphi-\varphi')$ and $\cos\gamma$ in the computation of $\sin\vartheta'$.

If we neglect the compression of the earth in this example, we find by (94) $\zeta'-\zeta = 29'\ 17''.9$, which is $10''$ in error.

95. *To find the parallax of a star in zenith distance and azimuth when the apparent zenith distance and azimuth are given, the earth being regarded as a spheroid.*

If we multiply the first of the equations (101) by $\cos\zeta'$ and the second by $\sin\zeta'$, the difference of the products gives

$$\sin(\zeta'-\zeta) = \frac{\rho\sin\pi\cos(\varphi-\varphi')\sin(\zeta'-\gamma)}{\cos\gamma}$$

for which, since $\cos(\varphi-\varphi')$ and $\cos\gamma$ are each nearly equal to unity, we may take, without sensible error,

$$\sin(\zeta'-\zeta) = \rho\sin\pi\sin(\zeta'-\gamma) \qquad (114)$$

in which γ has the value found by (111), or, with sufficient accuracy by the formula

$$\gamma = (\varphi - \varphi') \cos A' \tag{115}$$

Again, if we multiply the first of the equations (98) by sin A' and the second by cos A', the difference of the products gives

$$\sin (A' - A) = \frac{\rho \sin \pi \sin (\varphi - \varphi') \sin A'}{\sin \zeta} \tag{116}$$

to compute which, ζ must first be found by subtracting the value of the parallax $\zeta' - \zeta$, found by (114), from the given value of ζ'.

EXAMPLE.—In latitude $\varphi = 38° 59'$, given for the moon $A' = 320° 17' 45''.09$, $\zeta' = 29° 59' 7''.79$, $\pi = 58' 37''.2$, to find the parallax in zenith distance and azimuth.

We have, as in the example Art. 94, $\varphi - \varphi' = 11' 15''$, $\log \rho = 9.999428$, $\gamma = (\varphi - \varphi') \cos A' = 8' 39''.3$, $\zeta' - \gamma = 29° 50' 28''.5$; and hence, by (114) and (116),

$\log \rho \sin \pi$	8.231179	$\log \rho \sin \pi$	8.23118
$\log \sin (\zeta' - \gamma)$	9.696879	$\log \sin (\varphi - \varphi')$	7.51488
$\log \sin (\zeta' - \zeta)$	7.928058	$\log \sin A'$	n9.80538
$\zeta' - \zeta =$	29′ 7″.79	$\log \operatorname{cosec} \zeta$	0.30766
$\zeta =$	29° 30′ 0″	$\log \sin (A' - A)$	n5.85910
		$A' - A =$	$-14''.91$
		$A =$	320° 18′ 0″

agreeing with the given values of Art. 94.

96. *For the planets or the sun*, the following formulæ are always sufficiently precise:

$$\left.\begin{aligned} \zeta' - \zeta &= \rho\pi \sin (\zeta' - \gamma) \\ A' - A &= \rho\pi \sin (\varphi - \varphi') \sin A' \operatorname{cosec} \zeta' \end{aligned}\right\} \tag{117}$$

and in most cases we may take $\zeta' - \zeta = \pi \sin \zeta'$, and $A' - A = 0$.

The quantity $\rho\pi$ is frequently called *the reduced parallax*, and $\pi - \rho\pi = (1 - \rho)\pi$ the *reduction* of the equatorial parallax for the given latitude; and a table for this reduction is given in some collections. This reduction is, indeed, sensibly the same as the correction given in our Table XIII, which will be explained more particularly hereafter. Calling the tabular correction $\Delta\pi$, we shall have, with sufficient accuracy for most purposes,

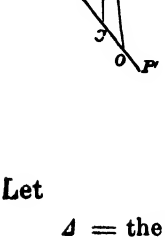
$$\rho\pi = \pi - \Delta\pi$$

97. The preceding methods of computing the parallax enable us to pass *directly* from the geocentric to the apparent azimuth and zenith distance. There is, however, an indirect method which is sometimes more convenient. This consists in reducing both the geocentric and the apparent co-ordinates to *the point in which the vertical line of the observer intersects the axis of the earth.* I shall briefly designate this point as *the point O* (Fig. 11).

We may suppose the point O to be assumed as the centre of the celestial sphere and at the same time as the centre of an imaginary terrestrial sphere described with a radius equal to the normal OA (Fig. 11). Since the point O is in the vertical line of the observer, the azimuth at this point is the same as the apparent azimuth. If, therefore, the geocentric co-ordinates are first reduced to the point O, we shall then avoid the parallax in azimuth, and the parallax in zenith distance will be found by the simple formula for the earth regarded as a sphere, taking the normal as radius.

Since the point O is in the axis of the celestial sphere, the straight line drawn from it to the star lies in the plane of the declination circle of the star; the place of the star, therefore, as seen from the point O, differs from its geocentric place only in declination, and not in right ascension. We have then only to find the reduction of the declination and of the zenith distance to the point O.

Fig. 13.



1st. *To reduce the declination to the point O.*—Let PP', Fig. 13, be the earth's axis; C the centre; O the point in which the vertical line or normal of an observer in the given latitude φ meets the axis; S the star. We have found for CO the expression (Art. 84)

$$CO = ai$$

in which a is the equatorial radius of the earth, and

$$i = \frac{e^2 \sin \varphi}{\sqrt{(1 - e^2 \sin^2 \varphi)}}$$

Let

Δ	= the star's geocentric distance	=	SC,
Δ_1	= the star's distance from the point O	=	SO,
δ	= the geocentric declination	=	$90° - PCS$,
δ_1	= the declination reduced to the point O	=	$90° - POS$

then, drawing SB perpendicular to the axis, the right triangles SCB and SOB give

$$\left.\begin{aligned} \Delta_1 \sin \delta_1 &= \Delta \sin \delta + ai \\ \Delta_1 \cos \delta_1 &= \Delta \cos \delta \end{aligned}\right\} \quad (118)$$

which determine Δ_1 and δ_1. From these we deduce

$$\left.\begin{aligned} \Delta_1 \sin (\delta_1 - \delta) &= ai \cos \delta \\ \Delta_1 \cos (\delta_1 - \delta) &= \Delta + ai \sin \delta \end{aligned}\right\} \quad (119)$$

which determine Δ_1 and the *reduction* of the declination. If we divide these by Δ, and put

$$f_1 = \frac{\Delta_1}{\Delta} \qquad\qquad \sin \pi = \frac{a}{\Delta}$$

in which π denotes, as before, the equatorial horizontal parallax, they become

$$\begin{aligned} f_1 \sin (\delta_1 - \delta) &= i \sin \pi \cos \delta \\ f_1 \cos (\delta_1 - \delta) &= 1 + i \sin \pi \sin \delta \end{aligned}$$

whence

$$\tan (\delta_1 - \delta) = \frac{i \sin \pi \cos \delta}{1 + i \sin \pi \sin \delta}$$

or in series [Pl. Trig. Art. 257],

$$\delta_1 - \delta = \frac{i \sin \pi \cos \delta}{\sin 1''} - \frac{(i \sin \pi)^2 \sin 2\delta}{2 \sin 1''} + \&c$$

But since the second term of the series involves i^2 and consequently e^4, and this is further multiplied by the small factor $\sin^2 \pi$, the term is wholly inappreciable even for the moon; and, as the first term cannot exceed 25″ in any case, we shall obtain extreme accuracy by the simple formula

$$\delta_1 - \delta = i \pi \cos \delta \quad (120)$$

The value of Δ_1 is found from (119), by the same process as was used in finding Δ' in (103), to be

$$\Delta_1 = \Delta \left\{ 1 + i \sin \pi \frac{\sin \frac{1}{2} (\delta_1 + \delta)}{\cos \frac{1}{2} (\delta_1 - \delta)} \right\}$$

or, on account of the small difference between δ_1 and δ,

$$\Delta_1 = \Delta (1 + i \sin \pi \sin \delta) \quad (121)$$

As $\delta_1 - \delta$ is so small, it may be accurately computed with logarithms of four decimal places, and it will be convenient to substitute for i the form

$$i = A \sin \varphi$$

in which

$$A = \frac{e^2}{\sqrt{(1 - e^2 \sin^2 \varphi)}}$$

The value of log A may then be taken from the following table with the argument φ = the geographical latitude

φ	log A
0°	7.8244
10	7.8245
20	7.8246
30	7.8248
40	7.8250
50	7.8253
60	7.8255
70	7.8257
80	7.8258
90	7.8259

We shall then compute $\delta_1 - \delta$ and Δ_1 under the following forms:

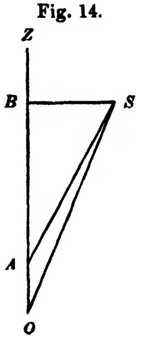
$$\left.\begin{aligned} \delta_1 - \delta &= A\,\pi \sin \varphi \cos \delta \\ \Delta_1 &= \Delta\,(1 + A \sin \pi \sin \varphi \sin \delta) \end{aligned}\right\} \quad (122)$$

If the value of π_1 has been found as below, we may take

$$\delta_1 - \delta = e^2\,\pi_1 \sin \varphi \cos \delta$$

2d. *To find the parallax in zenith distance for the point O.*—Let ZAO, Fig. 14, be the vertical line of the observer at A. The normal AO terminating in the axis being denoted by N, we have, by (90),

Fig. 14.



$$N = \frac{a}{\sqrt{(1 - e^2 \sin^2 \varphi)}}$$

But if in (84) we write $e^4 \sin^4 \varphi$ for $e^4 \sin^2 \varphi$, we have

$$\rho = a \sqrt{(1 - e^2 \sin^2 \varphi)}$$

and this value is sufficiently accurate for the computation of the parallax in all cases. If then we put $a = 1$, we have

$$AO = N = \frac{1}{\rho}$$

If now in the vertical plane passing through the line ZO and the star S we draw SB perpendicular to OZ, and put

$$\begin{aligned} \zeta_1 &= \text{the zenith distance at } O = SOZ \\ \zeta' &= \text{the apparent zenith dist.} = SAZ \end{aligned}$$

the triangles OSB, ASB give

$$\left.\begin{aligned} \Delta' \cos \zeta' &= \Delta_1 \cos \zeta_1 - \frac{1}{\rho} \\ \Delta' \sin \zeta' &= \Delta_1 \sin \zeta_1 \end{aligned}\right\} \tag{123}$$

Dividing these equations by Δ_1, and putting

$$\frac{\Delta'}{\Delta_1} = f_1 \qquad \sin \pi_1 = \frac{1}{\rho \Delta_1}$$

they become

$$\begin{aligned} f_1 \cos \zeta' &= \cos \zeta_1 - \sin \pi_1 \\ f_1 \sin \zeta' &= \sin \zeta_1 \end{aligned}$$

from which we deduce

$$\begin{aligned} f_1 \sin (\zeta' - \zeta_1) &= \sin \pi_1 \sin \zeta_1 \\ f_1 \cos (\zeta' - \zeta_1) &= 1 - \sin \pi_1 \cos \zeta_1 \end{aligned}$$

$$\tan (\zeta' - \zeta_1) = \frac{\sin \pi_1 \sin \zeta_1}{1 - \sin \pi_1 \cos \zeta_1} \tag{124}$$

and in series,

$$\zeta' - \zeta_1 = \frac{\sin \pi_1 \sin \zeta_1}{\sin 1''} + \frac{\sin^2 \pi_1 \sin 2 \zeta_1}{2 \sin 1''} + \text{\&c.} \tag{125}$$

Or, rigorously,

$$\left.\begin{aligned} \sin \vartheta &= \sin \pi_1 \cos \zeta_1 \\ \tan (\zeta' - \zeta_1) &= \tan \vartheta \tan (45° + \tfrac{1}{2} \vartheta) \tan \zeta_1 \end{aligned}\right\} \tag{126}$$

To find π_1 we have

$$\sin \pi_1 = \frac{1}{\rho \Delta_1} = \frac{1}{\rho \Delta (1 + A \sin \pi \sin \varphi \sin \delta)}$$

or

$$\sin \pi_1 = \frac{\sin \pi}{\rho (1 + A \sin \pi \sin \varphi \sin \delta)} \tag{127}$$

But this very precise expression of π_1 will seldom be required: it will generally suffice to take

$$\sin \pi_1 = \frac{\sin \pi}{\rho}$$

or

$$\pi_1 = \frac{\pi}{\rho}$$

which will be found to give the correct value of π_1, even for the moon, within $0''.2$ in every case. Where this degree of accuracy suffices, we may employ a table containing the correction for reducing π to π_1, computed by the formula

$$\Delta\pi = \pi_1 - \pi = \pi\left(\frac{1}{\rho} - 1\right)$$

Table XIII., Vol. II., gives this correction with the arguments π and the geographical latitude φ. Taking the correction from this table, therefore, we have

$$\pi_1 = \pi + \Delta\pi \tag{128}$$

3d. *To compute the parallax in zenith distance for the point O when the apparent zenith distance is given.*

Multiplying the first equation of (123) by $\sin \zeta'$, the second by $\cos \zeta'$, and subtracting, we find

$$\sin(\zeta' - \zeta_1) = \frac{1}{\rho \Delta_1} \sin \zeta'$$

or

$$\sin(\zeta' - \zeta_1) = \sin \pi_1 \sin \zeta' \tag{129}$$

If we denote the apparent altitude by h' and the altitude reduced to the point O by h_1, this equation becomes

$$\sin(h_1 - h') = \sin \pi_1 \cos h' \tag{130}$$

EXAMPLE.—In Latitude $\varphi = 38° 59'$, given the moon's hour angle $t = 341° 1' 36''.85$, geocentric declination $\delta = +14° 39' 24''.54$, and the equatorial horizontal parallax $\pi = 58' 37''.2$, to find the apparent zenith distance and azimuth.

The geocentric zenith distance and azimuth, computed from these data by Art. 14, are $\zeta = 29° 30'$, $A = 320° 18'$, which are the values employed in our example in Art. 94. To compute

by the method of the present article, we first reduce the declination to the point O by (122), as follows:

For $\varphi = 38° \ 59'$		$\log A$	7.8250
$\pi = 3517''.2$		$\log \pi$	3.5462
		$\log \sin \varphi$	9.7987
$\delta = 14° \ 39' \ 24''.54$		$\log \cos \delta$	9.9856
$\delta_1 - \delta =$	$14 \ .31$	$\log(\delta_1 - \delta)$	1.1555
$\delta_1 = 14° \ 39' \ 38''.85$			

With this value of δ_1 and $t = 341° \ 1' \ 36''.85$, the computation of the zenith distance and azimuth by Art. 14 gives for the point O

$$\zeta_1 = 29° \ 29' \ 47''.67 \qquad A_1 = 320° \ 17' \ 45''.09$$

and this value of A_1 is precisely the same as A' found in Art. 94, as it should be, since the azimuth at the point O and at the observer are identical.

We find from Table XIII. $\Delta\pi = 4''.6$, and hence $\pi_1 = 58' \ 37''.2 + 4''.6 = 58' \ 41''.8$; and then, by (126),

		$\log \sin \pi_1$	8.23232
		$\log \cos \zeta_1$	9.93971
$\vartheta = 51' \ 5''$		$\log \sin \vartheta$	8.17203
		$\log \tan \vartheta$	8.17208
		$\log \tan (45° + \frac{1}{2} \vartheta)$	0.00645
$\zeta_1 = 29° \ 29' \ 47''.67$		$\log \tan \zeta_1$	9.75258
$\zeta' - \zeta_1 =$	$29 \ \ 20 \ .03$	$\log \tan (\zeta' - \zeta_1)$	7.93111
$\zeta' = 29° \ 59' \ \ 7''.70$			

agreeing with the value found in Art. 94 within $0''.09$. If we had computed π_1 by (127), the agreement would have been exact.

98. *To find the parallax of a star in right ascension and declination when its geocentric right ascension and declination are given.*

The investigation of this problem is similar to that of Art. 92. Let the star be referred by rectangular co-ordinates to three planes passing through the centre of the earth: the first, the plane of the equator; the second, that of the equinoctial colure; the third, that of the solstitial colure. Let the axis of x be the straight line drawn through the equinoctial points, positive towards the vernal equinox; the axis of y, the intersection of

the plane of the solstitial colure and that of the equator, positive towards that point of the equator whose right ascension is 90°; the axis of z, the axis of the heavens, positive towards the north. Let

$\alpha =$ the star's geocentric right ascension,
$\delta =$ " " declination,
$\Delta =$ " " distance,

then the co-ordinates of the star are

$$x = \Delta \cos \delta \cos \alpha$$
$$y = \Delta \cos \delta \sin \alpha$$
$$z = \Delta \sin \delta$$

Again, let the star be referred to another system of planes parallel to the first, the origin being the observer. The vanishing circles of these planes in the celestial sphere are still the equator, the equinoctial colure, and the solstitial colure. Let

$\alpha' =$ the star's observed right ascension,
$\delta' =$ " " declination,
$\Delta' =$ " distance from the observer,

where by *observed* right ascension and declination we now mean the values which differ from the geocentric values by the parallax depending on the position of the observer on the surface of the earth. The co-ordinates of the star in this system will be

$$x' = \Delta' \cos \delta' \cos \alpha'$$
$$y' = \Delta' \cos \delta' \sin \alpha'$$
$$z' = \Delta' \sin \delta'$$

Now, if

$\Theta =$ the sidereal time $=$ the right ascension of the observer's meridian at the instant of observation,
$\varphi' =$ the reduced latitude of the place of observation,
$\rho =$ the radius of the earth for this latitude,

then Θ, φ', and ρ are the polar co-ordinates of the observer, entirely analogous to α, δ, and Δ of the star, so that the rectangular co-ordinates of the observer, taken in the first system, are

$$a = \rho \cos \varphi' \cos \Theta$$
$$b = \rho \cos \varphi' \sin \Theta$$
$$c = \rho \sin \varphi'$$

and for transformation from one system to the other we have

$$x' = x - a, \qquad y' = y - b, \qquad z' = z - c.$$

Hence

$$\left.\begin{aligned}\Delta' \cos \delta' \cos \alpha' &= \Delta \cos \delta \cos \alpha - \rho \cos \varphi' \cos \Theta \\ \Delta' \cos \delta' \sin \alpha' &= \Delta \cos \delta \sin \alpha - \rho \cos \varphi' \sin \Theta \\ \Delta' \sin \delta' &= \Delta \sin \delta - \rho \sin \varphi'\end{aligned}\right\} \quad (131)$$

or, dividing by Δ, and putting as before

$$f = \frac{\Delta'}{\Delta} \qquad \sin \pi = \frac{1}{\Delta}$$

$$\left.\begin{aligned}f \cos \delta' \cos \alpha' &= \cos \delta \cos \alpha - \rho \sin \pi \cos \varphi' \cos \Theta \\ f \cos \delta' \sin \alpha' &= \cos \delta \sin \alpha - \rho \sin \pi \cos \varphi' \sin \Theta \\ f \sin \delta' &= \sin \delta - \rho \sin \pi \sin \varphi'\end{aligned}\right\} \quad (132)$$

From the first two of these equations we deduce

$$\left.\begin{aligned}f \cos \delta' \sin (\alpha' - \alpha) &= \rho \sin \pi \cos \varphi' \sin (\alpha - \Theta) \\ f \cos \delta' \cos (\alpha' - \alpha) &= \cos \delta - \rho \sin \pi \cos \varphi' \cos (\alpha - \Theta)\end{aligned}\right\} \quad (133)$$

Multiplying the first of these by $\sin \frac{1}{2} (\alpha' \quad \alpha)$, the second by $\cos \frac{1}{2} (\alpha' - \alpha)$, and adding the products, we find, after dividing by $\cos \frac{1}{2} (\alpha' - \alpha)$,

$$f \cos \delta' = \cos \delta - \frac{\rho \sin \pi \cos \varphi' \cos [\frac{1}{2} (\alpha' + \alpha) - \Theta]}{\cos \frac{1}{2} (\alpha' - \alpha)}$$

Put

$$\tan \gamma = \frac{\tan \varphi' \cos \frac{1}{2} (\alpha' - \alpha)}{\cos [\frac{1}{2} (\alpha' + \alpha) - \Theta]} \quad (134)$$

then we have, for determining δ',

$$\left.\begin{aligned}f \sin \delta' &= \sin \delta - \rho \sin \pi \sin \varphi' \\ f \cos \delta' &= \cos \delta - \rho \sin \pi \sin \varphi' \cot \gamma\end{aligned}\right\} \quad (135)$$

whence

$$\left.\begin{aligned}f \sin (\delta' - \delta) &= \rho \sin \pi \sin \varphi' \frac{\sin (\delta - \gamma)}{\sin \gamma} \\ f \cos (\delta' - \delta) &= 1 - \rho \sin \pi \sin \varphi' \frac{\cos (\delta - \gamma)}{\sin \gamma}\end{aligned}\right\} \quad (136)$$

$$f = \frac{\Delta'}{\Delta} = \frac{\sin (\delta - \gamma)}{\sin (\delta' - \gamma)} \quad (137)$$

The equations (133) determine, rigorously, the parallax in right

ascension, or $\alpha' - \alpha$; (136) the parallax in declination, or $\delta' - \delta$; and (137) determines Δ'.

99. To obtain the developments in series, put

$$m = \frac{\rho \sin \pi \cos \varphi'}{\cos \delta}$$

then from (133) we have

$$\tan (\alpha' - \alpha) = \frac{m \sin (\alpha - \Theta)}{1 - m \cos (\alpha - \Theta)} \tag{138}$$

whence

$$\alpha' - \alpha = \frac{m \sin (\alpha - \Theta)}{\sin 1''} + \frac{m^2 \sin 2 (\alpha - \Theta)}{2 \sin 1''} + \&c. \tag{139}$$

Putting

$$n = \frac{\rho \sin \pi \sin \varphi'}{\sin \gamma}$$

we have from (136)

$$\tan (\delta' - \delta) = \frac{n \sin (\delta - \gamma)}{1 - n \cos (\delta - \gamma)} \tag{140}$$

whence

$$\delta' - \delta = \frac{n \sin (\delta - \gamma)}{\sin 1''} + \frac{n^2 \sin 2 (\delta - \gamma)}{2 \sin 1''} + \&c. \tag{141}$$

100. The quantity $\alpha - \Theta$ is the hour angle of the star east of the meridian. According to the usual practice, we shall reckon the hour angle towards the west, and denote it by t, or put

$$t = \Theta - \alpha$$

and then we shall write (138) and (140) as follows:

$$\tan (\alpha - \alpha') = \frac{m \sin t}{1 - m \cos t}$$

$$\tan (\delta - \delta') = \frac{n \sin (\gamma - \delta)}{1 - n \cos (\gamma - \delta)}$$

The rigorous computation will be conveniently performed by the following formulæ:

$$\left.\begin{aligned}
\sin\vartheta &= m\cos t = \frac{\rho\sin\pi\cos\varphi'\cos t}{\cos\delta}\\
\tan(\alpha-\alpha') &= \tan\vartheta\tan(45^\circ+\tfrac{1}{2}\vartheta)\tan t\\
\tan\gamma &= \frac{\tan\varphi'\cos\frac{1}{2}(\alpha-\alpha')}{\cos[t+\frac{1}{2}(\alpha-\alpha')]}\\
\sin\vartheta' &= n\cos(\gamma-\delta) = \frac{\rho\sin\pi\sin\varphi'\cos(\gamma-\delta)}{\sin\gamma}\\
\tan(\delta-\delta') &= \tan\vartheta'\tan(45^\circ+\tfrac{1}{2}\vartheta')\tan(\gamma-\delta)
\end{aligned}\right\}\quad(142)$$

101. Except for the moon, the first terms of the series (139) and (141) will suffice, and we may use the following approximations:

$$\left.\begin{aligned}
\alpha-\alpha' &= \frac{\rho\pi\cos\varphi'\sin t}{\cos\delta}\\
\tan\gamma &= \frac{\tan\varphi'}{\cos t}\\
\delta-\delta' &= \frac{\rho\pi\sin\varphi'\sin(\gamma-\delta)}{\sin\gamma}
\end{aligned}\right\}\quad(143)$$

If the star is on the meridian, we have $t=0$, and hence $\gamma=\varphi'$, and

$$\delta-\delta'=\rho\pi\sin(\varphi'-\delta)$$

Since in the meridian we have $\zeta=\varphi-\delta$, it is easily seen that $\zeta'-\zeta$ found by (108) and $\delta'-\delta$ found by (140) will then be numerically equal, or *the parallax in zenith distance is numerically equal to the parallax in declination when the star is on the meridian.*

102. *To find the parallax of a star in right ascension and declination, when its observed right ascension and declination are given.*

Multiplying the first equation of (132) by $\sin\alpha'$, the second by $\cos\alpha'$, and subtracting one product from the other, we find

$$\sin(\alpha-\alpha')=\frac{\rho\sin\pi\cos\varphi'\sin(\Theta-\alpha')}{\cos\delta}$$

In like manner, from (135) we deduce

$$\sin(\delta - \delta') = \frac{\rho \sin \pi \sin \varphi' \sin(\gamma - \delta')}{\sin \gamma}$$

We have here $\Theta - \alpha'$ equal to the apparent or observed hour angle; and hence, putting

$$t' = \Theta - \alpha'$$

the computation may be made under the following form:

$$\left.\begin{aligned}
\sin(\alpha - \alpha') &= \frac{\rho \sin \pi \cos \varphi' \sin t'}{\cos \delta} \\
\tan \gamma &= \frac{\tan \varphi' \cos \frac{1}{2}(\alpha - \alpha')}{\cos [t' - \frac{1}{2}(\alpha - \alpha')]} \\
\sin(\delta - \delta') &= \frac{\rho \sin \pi \sin \varphi' \sin(\gamma - \delta')}{\sin \gamma}
\end{aligned}\right\} \quad (144)$$

In the first computation of $\alpha - \alpha'$ we employ δ' for δ. The value of $\alpha - \alpha'$ thus found is sufficiently exact for the computation of γ and $\delta - \delta'$. With the computed value of $\delta - \delta'$ we then find δ and correct the computation of $\alpha - \alpha'$.

Example.—Suppose that on a certain day at the Greenwich Observatory the right ascension and declination of the moon were observed to be

$$\alpha' = 7^h\ 41^m\ 20^s.436$$
$$\delta' = 15°\ 50'\ 27''.66$$

when the sidereal time was

$$\Theta = 11^h\ 17^m\ 0^s.02$$

and the moon's equatorial horizontal parallax was

$$\pi = 56'\ 57''.5$$

Required the geocentric right ascension and declination.

We have for Greenwich $\varphi = 51°\ 28'\ 38''.2$, and hence (Table III.) $\varphi - \varphi' = 11'\ 13''.6$, $\varphi' = 51°\ 17'\ 24''.6$, $\log \rho = 9.9991134$. The computation by (144) is then as follows:

α' (in arc) $=$	$115°\ 20'\ 6''.54$
Θ " $=$	$169\ \ 15\ \ 0\ .30$
$t' =$	$53\ \ 54\ \ 53\ .76$
$\frac{1}{2}(\alpha - \alpha') =$	$14\ \ 55\ .8$
$t' - \frac{1}{2}(\alpha - \alpha') =$	$53\ \ 39\ \ 58$
$\log \sec [t' - \frac{1}{2}(\alpha - \alpha')]$	0.227319
$\log \cos \frac{1}{2}(\alpha - \alpha')$	9.999996
$\log \tan \phi'$	0.096133
$\log \tan \gamma$	0.323448
$\gamma =$	$64°\ 35'\ 58''$
$\gamma - \delta' =$	$48\ \ 45\ \ 30$
$\log \rho \sin \pi$	8.218377
$\log \sin \phi'$	9.892275
$\log \sin (\gamma - \delta')$	9.876181
$\log \operatorname{cosec} \gamma$	0.044153
$\log \sin (\delta - \delta')$	8.030986
$\delta - \delta' =$	$+\ 36'\ 55''.24$
$\delta =$	$16°\ 27'\ 22''.90$

$\log \rho \sin \pi$	8.218377
$\log \cos \phi'$	9.796142
$\log \sin t'$	9.907489
(1)	7.922008
$\log \cos \delta'$	9.983185
App. $\log \sin (\alpha - \alpha')$	7.938823
Approx. $\alpha - \alpha' =$	$29'\ 51''.6$
(1)	7.922008
$\log \cos \delta$	9.981835
$\log \sin (\alpha - \alpha')$	7.940173
$\alpha - \alpha' =$	$+\ 29'\ 57''.23$
$\alpha =$	$115°\ 50'\ 3''.77$
$=$	$7^h\ 43^m\ 20^s.251$

103. For all bodies except the moon, the second computation will never affect the result in a sensible degree, and we may use the following approximations:

$$\left.\begin{aligned} \alpha - \alpha' &= \frac{\rho\pi \cos \varphi' \sin t'}{\cos \delta'} \\ \tan \gamma &= \frac{\tan \varphi'}{\cos t'} \\ \delta - \delta' &= \frac{\rho\pi \sin \varphi' \sin (\gamma - \delta')}{\sin \gamma} \end{aligned}\right\} \qquad (145)$$

For the sun, planets, and comets, it is frequently more convenient to use the geocentric distance of the body instead of the parallax, or, at least, to deduce the parallax from the distance, the latter being given. This distance is always expressed in parts of the sun's mean distance as unity. If we put

$\pi_0 =$ the sun's mean equatorial horizontal parallax.
$\Delta_0 =$ the sun's mean distance from the earth,

we have, whatever unit is employed in expressing Δ, Δ_0, and a,

$$\sin \pi = \frac{a}{\Delta} \qquad \sin \pi_0 = \frac{a}{\Delta_0}$$

whence

$$\sin \pi = \frac{\Delta_0}{\Delta} \sin \pi_0$$

and when we take $\Delta_0 = 1$,

$$\sin \pi = \frac{\sin \pi_0}{\Delta} \qquad \text{or } \pi = \frac{\pi_0}{\Delta} \tag{146}$$

According to ENCKE's determination

$$\pi_0 = 8''.57116 \qquad \log \pi_0 = 0.93304$$

EXAMPLE.—DONATI's comet was observed by Mr. JAMES FERGUSON at Washington, 1858 Oct. 13, $6^h\ 26^m\ 21^s.1$ mean time, and its observed right ascension and declination when corrected for refraction were

$$\alpha' = 236^\circ\ 48'\ \ 0''.5$$
$$\delta' = -7^\circ\ 36'\ 52''.8$$

The logarithm of the comet's distance from the earth was $\log \Delta = 9.7444$. Required the geocentric place.

We have for Washington $\varphi = 38^\circ\ 53'\ 39''.3$, whence, by Table III., $\log \rho \cos \varphi' = 9.8917$, $\log \rho \sin \varphi' = 9.7955$. Converting the mean into sidereal time (Art. 50), we find $\Theta = 19^h\ 55^m\ 16^s.98$. Hence, by (145) and (146),

$\Theta =$	$298^\circ\ 49'.2$	$\log \tan \varphi'$	9.9038
$\alpha' =$	$236\ \ 48.0$	$\log \cos t'$	9.6713
$t' =$	$62\ \ 1.2$	$\log \tan \gamma$	0.2325
$\log \pi_0$	0.9330	$\gamma =$	$59^\circ\ 39'.2$
$\log \Delta$	9.7444	$\gamma - \delta' =$	$67\ \ 16.1$
$\log \pi$	1.1886		
$\log \rho\pi \cos \varphi'$	1.0803	$\log \rho\pi \sin \varphi'$	0.9841
$\log \sin t'$	9.9460	$\log \sin (\gamma - \delta')$	9.9649
$\log \sec \delta'$	0.0038	$\log \operatorname{cosec} \gamma$	0.0640
$\log (\alpha - \alpha')$	1.0301	$\log (\delta - \delta')$	1.0130
$\alpha - \alpha' =$	$+10''.7$	$\delta - \delta' =$	$+10''.3$

Hence, for the geocentric place of the comet,

$$\alpha = 236^\circ\ 48'\ 11''.2 \qquad \delta = -7^\circ\ 36'\ 42''.5$$

104. *Parallax in latitude and longitude.*—Formulæ similar to the above obtain for the parallax in latitude and longitude. We

have only to substitute for Θ and φ' (which are the right ascension and declination of the geocentric zenith) the corresponding longitude and latitude of the geocentric zenith (which will be found by Art. 23), and put λ and β in the place of α and δ. Thus, if l and b are the longitude and latitude of the geocentric zenith, the equations (143) give for all objects except the moon.

$$\left.\begin{aligned}
\lambda - \lambda' &= \frac{\rho\pi \cos b \sin (l - \lambda)}{\cos \beta} \\
\tan \gamma &= \frac{\tan b}{\cos (l - \lambda)} \\
\beta - \beta' &= \frac{\rho\pi \sin b \sin (\gamma - \beta)}{\sin \gamma}
\end{aligned}\right\} \qquad (147)$$

In the same manner, the equations (131) may be made to express the general relations between the geocentric and the apparent longitude and latitude, and for the moon we can employ (142), observing to substitute respectively

$$\begin{array}{rcccccc}
\text{for} & \alpha, & \alpha', & \delta, & \delta', & \Theta, & \varphi' \\
\text{the quantities} & \lambda, & \lambda', & \beta, & \beta', & l, & b
\end{array}$$

In all the formulæ, when we choose to neglect the compression of the earth, we have only to put $\varphi = \varphi'$ and $\rho = 1$.

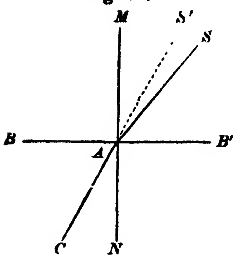
REFRACTION.

105. *General laws of refraction.*—The path of a ray of light is a straight line so long as the ray is passing through a medium of uniform density, or through a vacuum. But when a ray passes obliquely from one medium into another of different density, it is bent or *refracted.* The ray before it enters the second medium is called the *incident* ray; after it enters the second medium it is called the *refracted* ray; and the difference between the directions of the incident and refracted rays is called the *refraction.*

If a normal is drawn to the surface of the refracting medium at the point where the incident ray meets it, the angle which the incident ray makes with this normal is called the *angle of incidence*, the angle which the refracted ray makes with the normal is the *angle of refraction*, and the refraction is the difference of these two angles.

Thus, if SA, Fig. 15, is an incident ray upon the surface BB' of a refracting medium, AC the refracted ray, MN the normal to the surface at A, SAM is the angle of incidence, CAN is the angle of refraction; and if CA be produced backwards in the direction AS', SAS' is the refraction. An observer whose eye is at any point of the line AC will receive the ray as if it had come directly to his eye without refraction in the direction $S'AC$, which is therefore called the *apparent* direction of the ray.

Fig. 15.



Now, it is shown in Optics that this refraction takes place according to the following general laws:

1st. When a ray of light falls upon a surface (of any form). which separates two media of different densities, the plane which contains the incident ray and the normal drawn to the surface at the point of incidence contains the refracted ray also.

2d. When the ray passes from a rarer to a denser medium, it is in general refracted *towards* the normal, so that the angle of refraction is less than the angle of incidence; and when the ray passes from a denser to a rarer medium, it is refracted *from* the normal, so that the angle of refraction is greater than the angle of incidence.

3d. Whatever may be the angle of incidence, the sine of this angle bears a constant ratio to the sine of the corresponding angle of refraction, so long as the densities of the two media are constant. If a ray passes out of a vacuum into a given medium, the number expressing this constant ratio is called the *index of refraction* for that medium. This index is always an improper fraction, being equal to the sine of the angle of incidence divided by the sine of the angle of refraction.

4th. When the ray passes from one medium into another, the sines of the angles of incidence and refraction are reciprocally proportional to the indices of refraction of the two media.

106. *Astronomical refraction.*—The rays of light from a star in coming to the observer must pass through the atmosphere which surrounds the earth. If the space between the star and the upper limit of the atmosphere be regarded as a vacuum, or as filled with a medium which exerts no sensible effect upon the

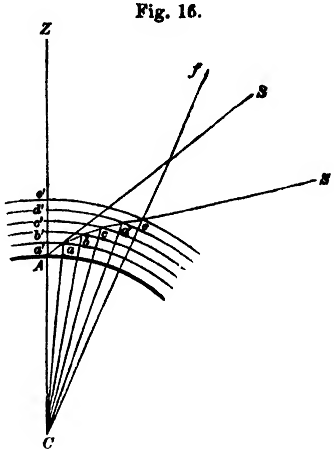
direction of a ray of light, the path of the ray will be at first a straight line; but upon entering the atmosphere its direction will be changed. According to the second law above stated, the new medium being the denser, the ray will be bent towards the normal, which in this case is a line drawn from the centre of the earth to the surface of the atmosphere at the point of incidence.

The atmosphere, however, is not of uniform density, but is most dense near the surface of the earth, and gradually decreases in density to its upper limit, where it is supposed to be of such extreme tenuity that its first effect upon a ray of light may be considered as infinitesimal. The ray is therefore *continually* passing from a rarer into a denser medium, and hence its direction is continually changed, so that its path becomes a curve which is concave towards the earth.

The last direction of the ray, or that which it has when it reaches the eye, is that of a tangent to its curved path at this point; and the difference of the direction of the ray before entering the atmosphere and this last direction is called the *astronomical refraction*, or simply the refraction.

Thus, Fig. 16, the ray *Se* from a star, entering the atmosphere at *e*, is bent into the curve *ecA* which reaches the observer at *A* in the direction of the tangent *S'A* drawn to the curve at *A*. If *CAZ* is the vertical line of the observer, or normal at *A*, by the first law of the preceding article, the vertical plane of the observer which contains the tangent *AS'* must also contain the whole curve *Ae* and the incident ray *Se*. Hence refraction increases the apparent altitude of a star, but does not affect its azimuth.

Fig. 16.



The angle *S'AZ* is the *apparent zenith distance* of the star. The *true* zenith distance* is strictly the angle which a straight line drawn from the star to the point *A*

* By *true* zenith distance we here (and so long as we are considering only the effect of refraction) mean that which differs from the apparent zenith distance only by the refraction.

makes with the vertical line. Such a line would not coincide with the ray Se; but in consequence of the small amount of the refraction, if the line Se be produced it will meet the vertical line AZ at a point so little elevated above A that the angle which this produced line will make with the vertical will differ very little from the true zenith distance. Thus, if the produced line Se be supposed to meet the vertical in b', the difference between the zenith distances measured at b' and at A is the *parallax* of the star for the height Ab', and this difference can be appreciable only in the case of the moon. It is therefore usual to assume Se as identical with the ray that would come to the observer directly from the star if there were no atmosphere.

The only case in which the error of this assumption is appreciable will be considered in the Chapter on Eclipses.

107. *Tables of Refraction.*—For the convenience of the reader who may wish to avail himself of the refraction tables without regard to the theory by which they are computed, I shall first explain the arrangement and use of those which are given at the end of this work.

Since the amount of the refraction depends upon the density of the atmosphere, and this density varies with the pressure and the temperature, which are indicated by the barometer and the thermometer, the tables give the refraction for a *mean* state of the atmosphere; and when the true refraction is required, supplementary tables are employed which give the correction of the mean refraction depending upon the observed height of the barometer and thermometer.

TABLE I. gives the refraction when the barometer stands at 30 inches and the thermometer (Fahrenheit's) at 50°. If we put

$$r = \text{the refraction,}$$
$$z = \text{the apparent zenith distance,}$$
$$\zeta = \text{the true zenith distance,}$$

then

$$\zeta = z + r$$

Where great accuracy is not required, it suffices to take r directly from TABLE I. and to add it to z. (The resulting ζ is that zenith distance which we have heretofore denoted by ζ' in the discussion of parallax.) The argument of this table is the apparent zenith distance z.

TABLE II. is BESSEL'S Refraction Table,* which is generally regarded as the most reliable of all the tables heretofore constructed. In Column A of this table the refraction is regarded as a function of the *apparent* zenith distance z, and the adopted form of this function is

$$r = \alpha \beta^A \gamma^\lambda \tan z$$

in which α varies slowly with the zenith distance, and its logarithm is therefore readily taken from the table with the argument z. The exponents A and λ differ sensibly from unity only for great zenith distances, and also vary slowly; their values are therefore readily found from the table.

The factor β depends upon the barometer. The actual pressure indicated by the barometer depends not only upon the height of the column, but also upon its temperature. It is, therefore, put under the form

$$\beta = BT$$

and log B and log T are given in the supplementary tables with the arguments "height of the barometer," and "height of the attached thermometer," respectively; so that we have

$$\log \beta = \log B + \log T$$

Finally, log γ is given directly in the supplementary table with the argument "external thermometer." This thermometer must be so exposed as to indicate truly the temperature of the atmosphere at the place of observation.

In Column B of the table the refraction is regarded as a function of the *true* zenith distance ζ expressed under the form

$$r = \alpha' \beta^{A'} \gamma^{\lambda'} \tan \zeta$$

and log α', A', and λ' are given in the table with the argument ζ; β and γ being found as before.

Column A will be used when z is given to find ζ; and Column B, when ζ is given to find z.

Column C is intended for the computation of differential refraction, or the difference of refraction corresponding to small

* From his *Astronomische Untersuchungen*, Vol. I.

differences of zenith distance, and will be explained hereafter (Micrometric Observations, Vol. II.).

These tables extend only to 85° of zenith distance, beyond which no refraction table can be relied upon. There occur at times anomalous deviations of the refraction from the tabular value at all zenith distances; and these are most sensible at great zenith distances. Fortunately, almost all valuable astronomical observations can be made at zenith distances less than 85°, and indeed less than 80°; and within this last limit we are justified by experience in placing the greatest reliance in BESSEL's Table. In an extreme case, where an observation is made within 5° of the horizon, we can compute an approximate value of the refraction by the aid of the following supplementary table, which is based upon actual observations made by ARGELANDER.*

App. zen. distance	log Refract.	A	λ
85° 0′	2.76687	1.0127	1.1229
30	2.80590	1.0147	1.1408
86 0	2.84444	1.0172	1.1624
30	2.88555	1.0204	1.1888
87 0	2.93174	1.0244	1.2215
30	2.98269	1.0298	1.2624
88 0	3.03686	1.0368	1.3141
30	3.09723	1.0465	1.3797
89 .0	3.16572	1.0593	1.4653
30	3.24142	1.0780	1.5789

If we call R the refraction whose logarithm is given in this table, the refraction for a given state of the air will be found by the formula

$$r = R\beta^{A}\gamma^{\lambda}$$

EXAMPLE 1.—Given the apparent zenith distance $z = 78°$ 30′ 0″, Barom. 29.770 inches, Attached Therm. − 0°.4 F., External Therm. − 2°.0 F.

We find from Table II., Col. A, for 78° 30′,

$$\log \alpha - 1.74981 \qquad A = 1.0032 \qquad \lambda = 1.0328$$

and from the tables for barometer and thermometer,

* *Tabulæ Regiomontanæ*, p. 539.

$$\begin{aligned}\log B &= +\ 0.00253 \qquad \log \gamma = +\ 0.04545\\ \log T &= +\ \underline{0.00127}\\ \log \beta &= +\ 0.00380\end{aligned}$$

Hence the refraction is computed as follows:

$$\begin{aligned} \log \alpha &= 1.74981\\ A \log \beta = \log \beta^A &= +\ 0.00381\\ \lambda \log \gamma = \log \gamma^\lambda &= +\ 0.04694\\ \log \tan z &= \underline{0.69154}\\ r = 310''.53 = 5'\ 10''.53 \quad \log r &= 2.49210\end{aligned}$$

The true zenith distance is, therefore, 78° 30′ 0″ + 5′ 10″.53 = 78° 35′ 10″.53.

Example 2.—Given the true zenith distance ζ = 78° 35′ 10″.53, Barom. 29.770 inches, Attached Therm. − 0°.4 F., External Therm. − 2°.0 F.

We find from Table II., Col. B, for 78° 35′ 10″,

$$\log \alpha' = 1.74680 \qquad A' = 0.9967 \qquad \lambda' = 1.0261$$

and from the tables for barometer and thermometer, as before,

$$\begin{aligned}\log B &= +\ 0.00253 \qquad \log \gamma = +\ 0.04545\\ \log T &= +\ \underline{0.00127}\\ \log \beta &= +\ 0.00380\end{aligned}$$

The refraction is then computed as follows:

$$\begin{aligned} \log \alpha' &= 1.74680\\ A' \log \beta = \log \beta^{A'} &= +\ 0.00379\\ \lambda' \log \gamma = \log \gamma^{\lambda'} &= +\ 0.04663\\ \log \tan \zeta &= \underline{0.69489}\\ r = 310''.53'' = 5'\ 10''.53 \quad \log r &= 2.49211\end{aligned}$$

and the apparent zenith distance is therefore 78° 30′.

Example 3.—Given z = 87° 30′, barometer and thermometer as in the preceding examples.

By the supplementary table above given,

$$\begin{aligned} & & & & \log R &= 2.98269\\ A &= 1.0298 & \log \beta &= +\ 0.00380 & \log \beta^A &= +\ 0.00391\\ \lambda &= 1.2624 & \log \gamma &= +\ 0.04545 & \log \gamma^\lambda &= +\ \underline{0.05738}\\ & & r &= 18'\ 26''.6 & \log r &= 3.04398\end{aligned}$$

It is important in all cases where great precision is required that the barometer and thermometer be carefully verified, to see that they give true indications. The zero points of thermometers are liable to change after a certain time, and inequalities in the bore of the tube are not uncommon. A special investigation of every thermometer is, therefore, necessary before it is applied in any delicate research. If the capillarity of the barometer has not been allowed for in adjusting the scale, it must be taken into account by the observer in each reading.

We may obtain the true refraction for any state of the air within 1″ or 2″, very expeditiously, by taking the mean refraction from Table I. and correcting it by Table XIV. A, and Table XIV. B. The mode of using this table is obvious from its arrangement. Thus, in Example 1 we find

from Table I.,		Mean refr. =	4′ 38″.9
" XIV. A, for Barom.	29.77,	Corr. =	− 2 .
" XIV. B, " Therm.	− 2°.	" =	+ 32 .
		True refr. =	5′ 9″.

which agrees with BESSEL'S value within 1″.5. For greater accuracy, the height of the barometer should be reduced to the temperature 32° F., which is the standard assumed in these tables. The corrected height of the barometer in this example is 29.85, and the corresponding correction of the refraction would then be − 1″; consequently the true refraction would be 5′ 10″, which is only 0″.5 in error.

These tables furnish good approximations even at great zenith distances. Thus, we find by them, in Example 3, $r =$ 18′ 24″.

108. INVESTIGATION OF THE REFRACTION FORMULA.—In this investigation we may, without sensible error, consider the earth as a sphere, and the atmosphere as composed of an infinite number of concentric spherical strata, whose common centre is the centre of the earth, each of which is of uniform density, and within which the path of a ray of light is a straight line. Let C, Fig. 16, be the centre of the earth, A a point of observation on the surface; CAZ the vertical line; Aa', $a'b'$, $b'c'$, &c. the vertical thicknesses of the concentric strata; Se a ray of light from a star S, meeting the atmosphere at the point e, and successively re-

fracted in the directions ed, dc, &c. to the point A. The last direction of the ray is aA, which, when the number of strata is supposed to be infinite, becomes a tangent to the curve ecA at A, and consequently AaS' is the apparent direction of the star. Let the normals Ce, Cd, &c. be drawn to the successive strata. The angle Sef is the first angle of incidence, the angle Ced the first angle of refraction. At any intermediate point between e and A, as c, we have Ccd, the supplement of the angle of incidence, and Ccb, the angle of refraction.

If now for any point, as c, in the path of the ray, we put

i = the angle of incidence,
f = the angle of refraction,
μ = the index of refraction for the stratum above c,
μ' = " " " below c,

then, Art. 105,

$$\frac{\sin i}{\sin f} = \frac{\mu'}{\mu} \tag{148}$$

If we put

q = the normal Cc to the upper of the two strata,
q' = " Cb " lower " "
i' = the angle of incidence in the lower stratum,
= $180° - Cbc$,

the rectilinear triangle Cbc gives

$$\frac{\sin i'}{\sin f} = \frac{q}{q'}$$

which, with the above proportion, gives

$$q\,\mu \sin i = q'\mu' \sin i'$$

an equation which shows that the product of the normal to any stratum by its index of refraction and the sine of the angle of incidence is the same for any two consecutive strata; that is, it is a constant product for all the strata. If then we put

z = the apparent zenith distance,
a = the normal at the observer, or radius of the earth,
μ_0 = the index of refraction of the air at the observer,

we have, since z is the angle of incidence at the observer,

$$q\mu \sin i = a\mu_0 \sin z \tag{149}$$

in which the second member is constant for the same values of z and μ_0.

Now, we have from (148)

$$\tan \tfrac{1}{2} (i - f) = \frac{\mu' - \mu}{\mu' + \mu} \tan \tfrac{1}{2} (i + f)$$

But $i - f$ is the refraction of the ray in passing from one stratum into the next; and supposing, as we do, that the densities of the strata vary by infinitesimal increments, $i - f$ is the differential of the refraction; and we may, therefore, write $\frac{1}{2}\, dr$ for $\tan \frac{1}{2} (i - f)$ and $d\mu$ for $\mu' - \mu$; consequently, also, 2μ for $\mu' + \mu$, and $\tan i$ for $\tan \frac{1}{2} (i + f)$: hence we have

$$dr = \frac{d\mu}{\mu} \tan i \tag{150}$$

which is the *differential equation of the refraction.*

But, as both μ and i are variable, we cannot integrate this equation unless we can express i as a function of μ. This we could do by means of (149) if the relation between q and μ were given, that is, if the law of the decrease of density of the air for increasing heights above the surface of the earth were known. This, however, is unknown, and we are obliged to make an hypothesis respecting this law, and ultimately to test the validity of the hypothesis by comparing the refractions computed by the resulting formula with those obtained by direct observation. I shall first consider the hypothesis of BOUGUER, both on account of the simplicity of the resulting formula and of its historical interest.*

109. *First hypothesis.*—Let it be assumed that the law of decrease of density is such that some constant power of the refraction index μ is reciprocally proportional to the normal q, an hypothesis expressed by the equation

---

* I shall consider but two hypotheses: the first, because it leads to the simple formula of BRADLEY, which, though imperfect, is often useful as an approximate expression of the refraction; the second, because the tables formed from it by BESSEL have thus far appeared to be the most correct and in greatest accordance with observation, although on theoretical grounds even the hypothesis of BESSEL is open to objection. For a review of the labors of astronomers and physicists upon this difficult subject, from the earliest times to the present, see *Die Astronomische Strahlenbrechung in ihrer historischen Entwickelung dargestellt, von* DR. C. BRUHNS. Leipzig. 1861.

$$\left(\frac{\mu}{\mu_0}\right)^{n+1} = \frac{a}{q} \tag{151}$$

which with (149) gives

$$\sin i = \left(\frac{\mu}{\mu_0}\right)^n \sin z \tag{152}$$

or, logarithmically,

$$\log \sin i = n \log \mu + \log\left(\frac{\sin z}{\mu_0{}^n}\right)$$

where the last term is constant. By differentiation, therefore,

$$\frac{di}{\tan i} = n\,\frac{d\mu}{\mu}$$

which with (150) gives

$$dr = \frac{di}{n}$$

and, integrating,

$$r = \frac{i}{n} + C$$

To determine the constant C, the integral is to be taken from the upper limit of the atmosphere to the surface of the earth. At the upper limit $r = 0$; and if we put $\vartheta =$ the value of i at that limit, we have

$$0 = \frac{\vartheta}{n} + C$$

At the lower limit the value of r is the whole atmospheric refraction, and $i = z$: hence

$$r = \frac{z}{n} + C$$

Eliminating the constant, we have

$$r = \frac{z - \vartheta}{n} \tag{153}$$

To find ϑ, we have, by putting $\mu = 1$ in (152), since the density of the air at the upper limit is to be taken as zero,

$$\sin \vartheta = \frac{\sin z}{\mu_0{}^n} \tag{154}$$

Having then found μ_0 at the surface of the earth and suitably

determined n, we find ϑ by (154), and then r by (153). The two equations may be expressed in a single formula thus:

$$r = \frac{1}{n}\left[z - \sin^{-1}\left(\frac{\sin z}{\mu_0^n}\right)\right] \tag{155}$$

which is known as SIMPSON'S formula, but is in fact equivalent to the formula first given by BOUGUER in 1729 in a memoir on refraction which gained the prize of the French Academy.

From (154) we find

$$\frac{\sin z - \sin\vartheta}{\sin z + \sin\vartheta} = \frac{\mu_0^n - 1}{\mu_0^n + 1}$$

whence

$$\tan\tfrac{1}{2}(z - \vartheta) = \frac{\mu_0^n - 1}{\mu_0^n + 1}\tan\tfrac{1}{2}(z + \vartheta)$$

and, reducing by (153),

$$\tan\frac{n}{2}r = \frac{\mu_0^n - 1}{\mu_0^n + 1}\tan\left(z - \frac{n}{2}r\right) \tag{156}$$

which is equivalent to BRADLEY'S formula. If we are content to represent the refraction approximately by our formula, we can write this in the form

$$r = g\tan(z - fr)$$

and we shall find, with BRADLEY, that for a mean state of the air corresponding to the barometer 29.6 and thermometer 50° Fahr. we can express the observed refractions, very nearly, by taking

$$g = 57''.036, \qquad f = 3.$$

110. But, as we wish our formula to represent, if possible, the actual constitution of the atmosphere, let us endeavor to test the hypothesis upon which it rests. In order to correspond with the real state of nature, *it is necessary that the constitution of the atmosphere which the hypothesis involves should not only agree with the observed refraction, but also with the height of the barometer, and with the observed diminution of heat as the altitude of the observer above the earth's surface increases.*

The discussion of the formula will be more simple if we substitute the density of the air in the place of the index of refraction. Put

δ_0 = the density of the air at the surface of the earth,
δ = the density of the air at any point above the surface.

The relation between δ and μ, according to Optics, is expressed by

$$\mu^2 - 1 = 4k\delta \tag{157}$$

in which $4k$ is a constant determined by experiment. According to the experiments of BIOT,

$$4k = 0.000588768$$

Since k is so small that its square will be inappreciable, we may take

$$\mu = (1 + 4k\delta)^{\frac{1}{2}} = 1 + 2k\delta \tag{158}$$

and, consequently,

$$\mu_0 = 1 + 2k\delta_0$$
$$\mu_0^n = 1 + 2nk\delta_0$$

and (156) becomes, still neglecting k^2,

$$\tan \frac{n}{2} r = nk\delta_0 \tan \left(z - \frac{n}{2} r \right) \tag{159}$$

If we denote the horizontal refraction, or that for $z = 90°$, by r_0, this formula gives

$$\tan \frac{n}{2} r_0 = nk\delta_0 \cot \frac{n}{2} r_0$$

or

$$\tan \frac{n}{2} r_0 = \sqrt{nk\delta_0}$$

and, putting the small arc $\frac{n}{2} r_0$ for its tangent,

$$r_0 = \sqrt{\frac{4k\delta_0}{n}} \tag{160}$$

We can find δ_0 from the observed state of the barometer and thermometer at the surface of the earth, so that in order to compute the horizontal refraction by this formula, for the purpose of comparing it with the observed horizontal refraction, we have only to determine the value of n.

Let

x = the height of any assumed point in the atmosphere above the surface of the earth,

δ, p, g = the density and pressure of the air, and the force of gravity, respectively, at that point,

δ_0, p_0, g_0 = the same quantities at the earth's surface.

At an elevation greater than x by an infinitesimal distance dx, the pressure p is diminished by dp. The weight of a column of air whose height is dx, density δ, and gravity g, is expressed by $g\delta dx$, and this is equal to the decrement of the pressure: hence the equation

$$dp = - g\delta dx$$

By the law of gravity, we have

$$g = g_0 \frac{a^2}{(a + x)^2}$$

and hence

$$\begin{aligned} dp &= - g_0 a^2 \delta \frac{dx}{(a + x)^2} \\ &= g_0 a \delta \, d\left(\frac{a}{a + x}\right) \end{aligned} \tag{161}$$

Now, in the hypothesis under consideration, we have

$$\frac{a}{a + x} = \left(\frac{\mu}{\mu_0}\right)^{n+1} = \left(\frac{1 + 4k\delta}{1 + 4k\delta_0}\right)^{\frac{n+1}{2}}$$

or, neglecting the square of k,

$$\frac{a}{a + x} = 1 - 2(n + 1)k(\delta_0 - \delta).$$

which gives

$$d\left(\frac{a}{a + x}\right) = 2(n + 1)k\, d\delta$$

Hence

$$dp = 2\, g_0 a (n + 1) k\delta\, d\delta$$

Integrating,

$$p = g_0 a (n + 1) k\delta^2 \tag{162}$$

no constant being necessary, since p and δ vanish together.

To compare this with the observed pressure p_0, let

$l =$ the height of a column of air of the density δ_0 which acted upon by the gravity g_0 will be in equilibrium with the pressure p_0;

in other words, let l be the height of a homogeneous atmosphere of the density δ_0 which would exert the pressure p_0. Then, by this definition,

$$p_0 = g_0 \delta_0 l \tag{163}$$

which with (162) gives

$$\frac{p}{p_0} = (n+1)\frac{a}{l}\cdot\frac{k\delta^n}{\delta_0} \tag{164}$$

At the surface of the earth, where p becomes p_0 and δ becomes δ_0, this equation gives

$$1 = (n+1)\frac{a}{l}\cdot k\delta_0 \tag{165}$$

whence

$$n = \frac{\frac{l}{a}}{k\delta_0} - 1$$

and this reduces the expression of the horizontal refraction (160) to

$$r_0 = \frac{2k\delta_0}{\sqrt{\left[\frac{l}{a} - k\delta_0\right]}} \tag{166}$$

Taking as the unit of density the value of δ_0 which corresponds to the barometer 0.76 metres and thermometer 0° C., we have, according to BIOT,

$$4k\delta_0 = 0.000588768$$

The constant l for this state of the air is the height of a homogeneous atmosphere which would produce the pressure $0^m.76$ of the barometer when the temperature is 0° C.; and this height is to that of the barometric column as the density of mercury is to that of the air. According to REGNAULT, for Barom. $0^m.76$ and Therm. 0° C., mercury is 10517.3 times as heavy as air: hence we have

$$l = 0^m.76 \times 10517.3 = 7993^m.15$$

For a we shall here use the mean radius of the earth, since we have supposed the earth to be spherical, or

$$a = 6366738 \text{ metres}$$

which gives

$$\frac{l}{a} = 0.00125545 \tag{167}$$

Substituting these values in (166), we find, after dividing by $\sin 1''$ to reduce to seconds,

$$r_0 = 1824'' = 30'\,24''$$

But, according to ARGELANDER'S observations, we should have

for Barom. $0^m.76$ and Therm. $0°$ C., $r_0 = 37' 31''$; and the hypothesis therefore gives the horizontal refraction too small by more than $7'$.

111. The hypothesis can be tested further by examining whether it represents the law of decreasing temperatures for increasing heights in the atmosphere. In the first place, we observe that *in this hypothesis the densities of the strata of the atmosphere decrease in arithmetical progression when the altitudes increase in arithmetical progression.* For, since x is very small in comparison with a, we have very nearly

$$\frac{a}{a+x} = 1 - \frac{x}{a}$$

and hence

$$\frac{x}{a} = 2\,(n+1)\,k\delta_0\left(1 - \frac{\delta}{\delta_0}\right)$$

or, by (165),

$$x = 2l\left(1 - \frac{\delta}{\delta_0}\right) \tag{168}$$

which shows that equal increments of x correspond to equal decrements of δ.

This last equation also gives for the upper limit of the atmosphere, where $\delta = 0$, $x = 2\,l$; that is, *in this hypothesis the height of the atmosphere is double that of a homogeneous atmosphere of the same pressure.*

Again, we have, by (164), (165), and (168),

$$\frac{p\delta_0}{p_0\delta} = \frac{\delta}{\delta_0} = 1 - \frac{x}{2l} \tag{169}$$

The function $\dfrac{p\delta_0}{p_0\delta}$ expresses the law of heat of the strata of the atmosphere. For let τ_0 be the temperature at the surface of the earth, τ the temperature at the height x. If the temperature were τ_0 in both cases, we should have

$$\frac{p}{p_0} = \frac{\delta}{\delta_0} \tag{170}$$

but when the temperature is changed from τ_0 to τ the density is diminished in the ratio $1 + \varepsilon\,(\tau - \tau_0) : 1$, ε being a constant which

is known from experiment; so that the true relation between the pressures and densities at different temperatures is expressed by the known formula

$$\frac{p}{p_0} = \frac{\delta}{\delta_0}[1 + \varepsilon(\tau - \tau_0)]$$

whence

$$\frac{p\delta_0}{p_0\delta} = 1 + \varepsilon(\tau - \tau_0) \tag{171}$$

which combined with (169) gives

$$x = 2l\varepsilon(\tau_0 - \tau)$$

and hence equal increments of x correspond to equal decrements of τ. Hence, *in this hypothesis, the heat of the strata of the atmosphere decreases as their density in arithmetical progression.* The value of ε, according to RUDBERG and REGNAULT, is very nearly $\frac{1}{273}$. Hence we must ascend to a height $\frac{2l}{273} = 58.6$ metres, in order to experience a decrease of temperature of 1° C. But, according to the observations of GAY LUSSAC in his celebrated balloon ascension at Paris (in the year 1804), the decrease of temperature was 40°.25 C. for a height of 6980 metres, or 1° C. for 173 metres, so that in the hypothesis under consideration the height is altogether too small, or the decrease of temperature is too rapid. This hypothesis, therefore, is not sustained either by the observed refraction or by the observed law of the decrease of temperature.

112. *Second hypothesis.*—Before proposing a new hypothesis, let us determine the relation between the height and the density of the air at that height, when the atmosphere is assumed to be throughout of the same temperature, in which case we should have the condition (170). Resuming the differential equation (161),

$$dp = g_0 a\delta\, d\left(\frac{a}{a+x}\right)$$

put

$$\frac{a}{a+x} = 1 - s$$

in which s is a new variable very nearly proportional to x. We then have

$$dp = -g_0 a\delta ds$$

which with the supposition (170) gives

$$\frac{dp}{p} = -\frac{g_0\delta_0 a\, ds}{p_0}$$

Integrating,

$$\log p = -\frac{g_0\delta_0}{p_0} as + C$$

in which the logarithm is Napierian. The constant being determined so that p becomes p_0 when $s = 0$, we have

$$\log p_0 = C$$

and therefore

$$\log \frac{p}{p_0} = -\frac{g_0\delta_0}{p_0} as = -\frac{as}{l}$$

where l has the value (163). Hence, e being the Napierian base,

$$\frac{p}{p_0} = \frac{\delta}{\delta_0} = e^{-\frac{as}{l}} \tag{172}$$

which is the expression of the law of decreasing densities upon the supposition of a uniform temperature. In our first hypothesis the temperatures decrease, but nevertheless too rapidly. *We must, then, frame an hypothesis between that and the hypothesis of a uniform temperature.*

Now, in our first hypothesis we have by (169), within terms involving the second and higher powers of s,

$$\frac{p\delta_0}{p_0\delta} = 1 - \frac{as}{2l}$$

and in the hypothesis of a uniform temperature,

$$\frac{p\delta_0}{p_0\delta} = 1$$

The arithmetical mean between these would be

$$\frac{p\delta_0}{p_0\delta} = 1 - \frac{as}{4l}$$

but, as we have no reason for assuming exactly the arithmetical mean, BESSEL proposes to take

$$\frac{p\delta_0}{p_0\delta} = e^{-\frac{as}{h}} = 1 - \frac{as}{h} + \frac{1}{2}\left(\frac{as}{h}\right)^2 - \&c. \tag{173}$$

h being a new constant *to be determined so as to satisfy the observed refractions.* This equation, which we shall adopt as our second hypothesis, expresses the assumed law of decreasing temperatures, since, by (171), it amounts to assuming

$$1 + \varepsilon\,(\tau - \tau_0) = e^{-\frac{as}{h}} \tag{174}$$

and it follows that in this hypothesis the temperatures will not decrease in arithmetical progression with increasing heights, though they will do so very nearly for the smaller values of s, that is, near the earth's surface.

Now, combining the supposition (173) with the equation

$$dp = -\,g_0 a\delta\, ds$$

we have

$$\frac{dp}{p} = -\frac{g_0\delta_0 a}{p_0}\, e^{\frac{as}{h}}\, ds = -\frac{a}{l}\, e^{\frac{as}{h}}\, ds$$

Integrating and determining the constant so that for $s = 0$, p becomes p_0, we have

$$\frac{p}{p_0} = e^{-\frac{h}{l}\left(e^{\frac{as}{h}} - 1\right)}$$

which with (173) gives*

$$\delta = \delta_0\, e^{-\frac{h}{l}\left(e^{\frac{as}{h}} - 1\right) + \frac{as}{h}}$$

Inasmuch as the law of the densities thus expressed is still hypothetical, we may simplify the exponent of e. For if h is much greater than l (as is afterwards shown), we may in this exponent put $e^{\frac{as}{h}} - 1 = \frac{as}{h}$ and we shall have as the expression of our hypothesis

$$\delta = \delta_0\, e^{-\frac{as}{l} + \frac{as}{h}} = \delta_0\, e^{-\frac{h-l}{h}\cdot\frac{as}{l}} \tag{175}$$

---

* BESSEL. Fundamenta Astronomiæ, p. 28.

By comparing this with (172), we see that this new hypothesis differs from that of a uniform temperature by the correction $\frac{as}{h}$ applied to the exponent of e.

Putting, for brevity,

$$\beta = \frac{h - l}{h} \cdot \frac{a}{l} \tag{176}$$

we have

$$\delta = \delta_0 \, e^{-\beta s} \tag{177}$$

in which β is constant. This expression of the density is to be introduced into the differential equation of the refraction (150).

Now, by (149), in which $q = a + x$, we have

$$\sin i = \frac{a\mu_0 \sin z}{(a + x)\mu} = \frac{(1 - s)\,\mu_0 \sin z}{\mu}$$

whence

$$\tan i = \frac{\sin i}{\sqrt{(1 - \sin^2 i)}} = \frac{(1 - s) \sin z}{\sqrt{\left[\frac{\mu^2}{\mu_0^2} - (1 - s)^2 \sin^2 z\right]}}$$

$$= \frac{(1 - s) \sin z}{\sqrt{\left[\cos^2 z - \left(1 - \frac{\mu^2}{\mu_0^2}\right) + (2s - s^2) \sin^2 z\right]}}$$

From the equation $\mu^2 = 1 + 4k\delta$ we deduce

$$\frac{d\mu}{\mu} = \frac{2k\,d\delta}{1 + 4k\delta}$$

and if we introduce as a constant the quantity

$$\alpha = \frac{2k\delta_0}{1 + 4k\delta_0} \tag{178}$$

(which for Barom. $0^m.76$ and Therm. 0° C. is $\alpha = 0.000294211$)

$$\frac{d\mu}{\mu} = \frac{\alpha \frac{d\delta}{\delta_0}}{1 - 2\alpha\left(1 - \frac{\delta}{\delta_0}\right)}$$

We might neglect the square of k, and consequently, also, that of

α, with hardly appreciable error, and then this expression would become simply $\alpha \frac{d\delta}{\delta_0}$, but for greater accuracy we can retain the denominator, employing its mean value, as it varies within very narrow limits. For its greatest value, when $\delta = \delta_0$, is $= 1$, and its least value, when $\delta = 0$, is $= 1 - 2\alpha$, and the mean between these values is $1 - \alpha$. Hence we shall take

$$\frac{d\mu}{\mu} = \frac{\alpha}{1-\alpha} \cdot \frac{d\delta}{\delta_0}$$

In the denominator of the value of tan i we have also to substitute

$$1 - \frac{\mu^2}{\mu_0^2} = 1 - \frac{1+4k\delta}{1+4k\delta_0} = 2\alpha\left(1 - \frac{\delta}{\delta_0}\right)$$

Therefore, substituting in (150), we have

$$dr = \frac{\alpha \sin z\,(1-s)\,\frac{d\delta}{\delta_0}}{(1-\alpha)\left[\cos^2 z - 2\alpha\left(1-\frac{\delta}{\delta_0}\right) + (2s - s^2)\sin^2 z\right]^{\frac{1}{2}}}$$

or, by (177),

$$dr = \frac{-\alpha\beta \sin z\,(1-s)\,e^{-\beta s}ds}{(1-\alpha)\left[\cos^2 z - 2\alpha\,(1-e^{-\beta s}) + (2s - s^2)\sin^2 z\right]^{\frac{1}{2}}}$$

In the integration of this equation we may change the sign of the second member, since our object is only to obtain the numerical value of r. It is apparent that if we put 1 for $1 - s$ in the numerator of this expression, and also neglect the term $s^2 \sin^2 z$ in the denominator, the error will be almost or quite insensible; but, not to reject terms without examination, let us develop the expression into series. For this purpose, put the radical in the denominator under the form $\sqrt{y^2 - s^2 \sin^2 z}$, in which

$$y = \left[\cos^2 z - 2\alpha\,(1 - e^{-\beta s}) + 2s \sin^2 z\right]^{\frac{1}{2}}$$

Then

$$\frac{1-s}{(y^2 - s^2\sin^2 z)^{\frac{1}{2}}} = \frac{1-s}{y}\left(1 - \frac{s^2 \sin^2 z}{y^2}\right)^{-\frac{1}{2}}$$

$$= \frac{1-s}{y}\left(1 + \frac{s^2\sin^2 z}{2y^2} + \&c.\right)$$

$$= \frac{1}{y} - \frac{2sy^2 - s^2 \sin^2 z}{2y^3} - \&c.$$

Hence, restoring the value of y, we have

$$dr = \frac{\alpha\beta \sin z \, e^{-\beta s} ds}{(1-\alpha)\left[\cos^2 z - 2\alpha(1 - e^{-\beta s}) + 2s \sin^2 z\right]^{\frac{1}{2}}}$$

$$- \frac{\alpha\beta \sin z \, e^{-\beta s} s ds \left[\cos^2 z - 2\alpha(1 - e^{-\beta s}) + \frac{3}{2} s \sin^2 z\right]}{(1-\alpha)\left[\cos^2 z - 2\alpha(1 - e^{-\beta s}) + 2s \sin^2 z\right]^{\frac{3}{2}}}$$

$$- \&c. \quad \ldots \quad (179)$$

We shall hereafter show that the second term of this development is insensible. Confining ourselves for the present to the first term, let us, after the method of LAPLACE, introduce the new variable s' such that

$$s = s' + \frac{\alpha(1 - e^{-\beta s})}{\sin^2 z} \tag{180}$$

then this term takes the form

$$dr = \frac{\alpha\beta \sin z \, e^{-\beta s} ds}{(1-\alpha)\left[\cos^2 z + 2s' \sin^2 z\right]^{\frac{1}{2}}} \tag{181}$$

in which we have yet to reduce the numerator to a function of the new variable s'. Now, by *Lagrange's Theorem*,* when

---

* See PEIRCE's Curves and Functions, Vol. I. Art. 181. For the convenience of the reader, however, I add the following demonstration of this theorem. It is proposed to develop the function $u = fy$ in a series of ascending powers of x, x and y being connected by the equation

$$y = t + x\phi y$$

and the functions f and ϕ being given. If from this equation y could be found as an explicit function of x and substituted in the equation $u = fy$, the development could be effected at once by Maclaurin's Theorem, according to which we should have

$$u = u_0 + D_x u_0 x + D_x^2 u_0 \frac{x^2}{1.2} + \ldots + D_x^n u_0 \frac{x^n}{1.2\ldots n} + \&c.$$

where u_0, $D_x u_0$, &c. denote the values of u and its successive derivatives when $x = 0$. It is proposed to find the values of the derivatives without recourse to the elimination of y, as this elimination is often impracticable. For brevity, put $Y = \phi y$; then the derivatives of

$$y = t + xY$$

relatively to x and t are

$$s = s' + a\varphi s$$

we have

$$fs = fs' + \frac{a}{1}\left[\varphi s'.\, Dfs'\right] + \frac{a^2}{1.2} D\left[(\varphi s')^2.Dfs'\right]$$

$$+ \frac{a^3}{1.2.3} D^2\left[(\varphi s')^3.\,Dfs'\right] + \&c.$$

in which f and φ denote any functions whatever, and D, D^2, &c. the successive derivatives of the functions to which they are prefixed. Hence, by putting

$$fs = e^{-\beta s} \qquad \varphi s = \frac{1 - e^{-\beta s}}{\sin^2 z}$$

this theorem gives

$$D_x y = Y + xD_y YD_x y \qquad D_t y = 1 + x\, D_y YD_t\, y$$

whence, eliminating x,

$$D_x y = YD_t\, y$$

Multiplying this by $D_y u$, it gives

$$D_x u = YD_t u \tag{a}$$

The derivative of this equation relatively to t is

$$D_x[D_t u] = D_t[YD_t u]$$

This is a general theorem, whatever function u is of y, and consequently, also, whatever function $D_t u$ is of y. We may then substitute in it the function $Y^n D_t u$ for $D_t u$, and we shall have

$$D_x[Y^n D_t u] = D_t[Y^{n+1} D_t u] \tag{b}$$

Now, the successive derivatives of (a) relatively to x are, by the successive application of (b), making $n = 1, 2, 3,$ &c.,

$$\begin{aligned} D_x^2\, u &= D_x[YD_t u] &&= D_t[Y^2 D_t u] \\ D_x^3\, u &= D_t{}^2{}_x[Y^2 D_t u] &&= D_t{}^2[Y^3 D_t u] \\ \ldots & \quad \ldots\ldots && \quad \ldots\ldots \\ D_x^n\, u &= D_t{}^{n-1}[Y^n D_t u] \end{aligned}$$

But when $x = 0$, we have $y = t$, $Y = \phi t$, and hence

$$u_0 = ft, \qquad D_x u_0 = \phi t\,.\,Dft, \quad \ldots \quad D_x^n\, u_0 = D^{n-1}[(\phi t)^n Dft]$$

where the subscript letter of the D is omitted in the second members as unnecessary, since t is now the only variable. These values substituted in Maclaurin's Theorem give Lagrange's Theorem:

$$fy = ft + \phi t\,.\,Dft\;\frac{x}{1} + D[(\phi t)^2\,.\,Dft]\frac{x^2}{1.2} + \ldots + D^{n-1}[(\phi t)^n\,.\,Dft]\frac{x^n}{1.2\ldots.n} + \&c.$$

$$e^{-\beta s} = e^{-\beta s'} - \frac{a\beta}{\sin^2 z}\left[1 - e^{-\beta s'}\right] e^{-\beta s'}$$

$$- \frac{a^2\beta}{1.2\sin^4 z} D\left[(1 - e^{-\beta s'})^2 e^{-\beta s'}\right]$$

$$- \frac{a^3\beta}{1.2.3\sin^6 z} D^2\left[(1 - e^{-\beta s'})^3 e^{-\beta s'}\right]$$

. . . .

$$- \frac{a^n\beta}{1.2.3\ldots n\sin^{2n} z} D^{n-1}\left[(1 - e^{-\beta s'})^n e^{-\beta s'}\right]$$

$$- \text{\&c.} \tag{182}$$

But we have in the numerator of (181)

$$\beta e^{-\beta s}\, ds = - de^{-\beta s}$$

and hence, differentiating (182) and substituting the result in (181), we find

$$dr = \frac{a\beta\sin z\, ds'}{(1-a)\left[\cos^2 z + 2s'\sin^2 z\right]^{\frac{3}{2}}}\left\{e^{-\beta s'} + \frac{a}{\sin^2 z} D\left[(1 - e^{-\beta s'})e^{-\beta s'}\right]\right.$$

$$+ \frac{a^2}{1.2\sin^4 z} D^2\left[(1 - e^{-\beta s'})^2 e^{-\beta s'}\right]$$

. . . . .

$$+ \frac{a^n}{1.2.3\ldots n\sin^{2n} z} D^n\left[(1 - e^{-\beta s'})^n e^{-\beta s'}\right]$$

$$\left. + \text{\&c.}\right\} \tag{183}$$

To effect the differentiations expressed in the several terms of this series, we take the general expression

$$(1 - e^{-\beta s'})^n e^{-\beta s'} = (- e^{-\beta s'} + 1)^n e^{-\beta s'}$$

$$= \pm\left(e^{-(n+1)\beta s'} - ne^{-n\beta s'} + \frac{n(n-1)}{1.2} e^{-(n-1)\beta s'} - \text{\&c.}\right)$$

where the upper sign is to be used when n is even, and the lower sign when n is odd. Differentiating this n times successively, we have

$$D^n\left[(1 - e^{-\beta s'})^n e^{-\beta s'}\right] = + \beta^n\left[(n+1)^n e^{-(n+1)\beta s'} - n^n . ne^{-n\beta s'} + \text{\&c.}\right]$$

by means of which, making $n = 1 . 2 . 3 \ldots$ successively, we reduce (183) to the following form:

$$dr = \frac{\alpha\beta \sin z\, ds'}{(1-\alpha)\left[\cos^2 z + 2\, s' \sin^2 z\right]^{\frac{1}{2}}} \left\{ e^{-\beta s'} + \frac{\alpha\beta}{\sin^2 z}\left(2\, e^{-2\beta s'} - e^{-\beta s'}\right) \right.$$

$$+ \frac{\alpha^2\beta^2}{1\,.\,2 \sin^4 z}\left(3^2\, e^{-3\beta s'} - 2^2 . 2 e^{-2\beta s'} + e^{-\beta s'}\right)$$

$$+ \frac{\alpha^3\beta^3}{1.2.3 \sin^6 z}\left(4^3 e^{-4\beta s'} - 3^3.3 e^{-3\beta s'} + 2^3.3\, e^{-2\beta s'} - e^{-\beta s'}\right)$$

$$\left. + \,\&c. \right\} \tag{184}$$

We have now to integrate the terms of this series, after having multiplied each by the factor without the brackets. The integrals are to be taken from the surface of the earth, where $s = 0$, to the upper limit of the atmosphere; that is, q being the normal to any stratum (Art. 108), they are to be taken between the limits $q = a$ and $q = a + H$, H being the height of the atmosphere. Now, this height is not known; but since at the upper limit the density is zero and beyond this limit the ray suffers no refraction to infinity, we can without error take the integrals between the limits $q = a$ and $q = \infty$, *i.e.* between $s = 0$ and $s = 1$. But we may make the upper limit of s also equal to infinity. For, by (176), β will not differ greatly from $\frac{a}{l}$, and consequently will be a very large number, nearly equal to 800, as we find from (167); hence for $s = 1$ we have in (172) $\delta = \frac{\delta_0}{(2.718..)^{800}}$ which will be sensibly equal to zero, and consequently the same as we should find by taking $s = \infty$. Hence the integrals may be taken between the limits $s = 0$ and $s = \infty$; consequently, also, according to (180), between the limits $s' = 0$ and $s' = \infty$.

Now, as every term of the series will be of the form

$$\frac{\beta \sin z\, ds' e^{-n\beta s'}}{\left[\cos^2 z + 2\, s' \sin^2 z\right]^{\frac{1}{2}}} = \frac{\beta\, ds' e^{-n\beta s'}}{\left[\cot^2 z + 2 s'\right]^{\frac{1}{2}}} \tag{185}$$

multiplied by constants, we have only to integrate this general form. Let t be a new variable, such that

$$\cot^2 z + 2 s' = \frac{2 t^2}{n\beta} \tag{186}$$

then (185) becomes

$$\sqrt{\frac{2\beta}{n}}\cdot dt\, e^{\frac{n\beta}{2}\cot^2 z - t^2}$$

the integral of which is to be taken from $t =$

$$\sqrt{\frac{n\beta}{2}}\cot z = T \tag{187}$$

to $t = \infty$, which are the limits given by (186) for $s' = 0$ and $s' = \infty$. If, therefore, we denote by $\psi(n)$ a function such that

$$\int_T^\infty dt\, e^{-t^2} = e^{-\frac{n\beta}{2}\cot^2 z}\,\psi(n)$$

or

$$\psi(n) = e^{TT}\int_T^\infty dt\, e^{-tt} \tag{188}$$

the integral of (185) will become

$$\int_0^\infty \frac{\beta ds' \sin z e^{-n\beta s'}}{[\cos^2 z + 2s' \sin^2 z]^{\frac{1}{2}}} = \sqrt{2\beta}\cdot\frac{\psi(n)}{\sqrt{n}} \tag{189}$$

Substituting this value in (184), making successively $n = 1, 2, 3,$ &c., we find the following expression of the refraction:

$$\begin{aligned} r = \frac{\alpha\sqrt{2\beta}}{1-\alpha}\Big\{ & \psi(1) \\ & + \frac{\alpha\beta}{\sin^2 z}\left[2^{\frac{1}{2}}\psi(2) - \psi(1)\right] \\ & + \frac{\alpha^2\beta^2}{1\,.\,2\,.\,\sin^4 z}\left[3^{\frac{3}{2}}\psi(3) - 2^{\frac{3}{2}}.\,2\psi(2) + \psi(1)\right] \\ & + \frac{\alpha^3\beta^3}{1\,.\,2\,.\,3\sin^6 z}\left[4^{\frac{5}{2}}\psi(4) - 3^{\frac{5}{2}}.\,3\psi(3) + 2^{\frac{5}{2}}.3\psi(2) - \psi(1)\right] \\ & + \&c. \Big\} \end{aligned} \tag{190}$$

which, since we have in general

$$1 - \frac{x}{1} + \frac{x^2}{1\,.\,2} - \frac{x^3}{1\,.\,2\,.\,3} + \ldots . = e^{-x}$$

can also be written as follows:*

$$r = \frac{\alpha\sqrt{2\beta}}{1-\alpha}\left\{\begin{array}{l} \quad e^{-\frac{\alpha\beta}{\sin^2 z}}\,\psi(1) \\ + \quad 2^{\frac{1}{2}} \cdot \frac{\alpha\beta}{\sin^2 z}\, e^{-\frac{2\alpha\beta}{\sin^2 z}}\,\psi(2) \\ + \frac{3^{\frac{3}{2}}}{1.2} \cdot \frac{\alpha^2\beta^2}{\sin^4 z}\, e^{-\frac{3\alpha\beta}{\sin^2 z}}\,\psi(3) \\ + \frac{4^{\frac{5}{2}}}{1.2.3} \cdot \frac{\alpha^3\beta^3}{\sin^6 z}\, e^{-\frac{4\alpha\beta}{\sin^2 z}}\,\psi(4) \\ + \&c. \end{array}\right\} \tag{191}$$

113. The only remaining difficulty is to determine the function $\psi(n)$, (188). In the case of the horizontal refraction, where $\cot z = 0$ and therefore also $T = 0$, this function becomes independent of (n), and reduces to the well-known integral†

$$\int_0^\infty dt\, e^{-tt} = \frac{\sqrt{\pi}}{2} \tag{192}$$

---

* LAPLACE, *Mécanique Céleste*, Vol. IV. p. 186 (BOWDITCH's Translation); where, however, $\frac{a}{l}$ stands in the place of the more general symbol β here employed. This form of the refraction is due to KRAMP, *Analyse des réfractions astronomiques et terrestres*, Strasbourg, 1799.

† This useful definite integral may be readily obtained as follows. Put $k = \int_0^\infty dt\, e^{-tt}$. Then, since the definite integral is independent of the variable, we also have $k = \int_0^\infty dv\, e^{-vv}$, and, multiplying these expressions together,

$$k^2 = \int_0^\infty dt\, e^{-tt} \int_0^\infty dv\, e^{-vv} = \int_0^\infty \int_0^\infty dt\, dv\, e^{-(tt+vv)}$$

the order of integration being arbitrary Let

$$v = tu;\ \text{whence } dv = t\, du$$

(for in integrating, regarding v as variable, t is regarded as constant): then we have

$$k^2 = \int_0^\infty \int_0^\infty du\,.\,dt\,.\,te^{-tt(1+uu)} = \int_0^\infty du \int_0^\infty dt\,.\,te^{-tt(1+uu)}$$

$$= \int_0^\infty du \cdot \frac{1}{2(1+u^2)} = \tfrac{1}{2}(\tan^{-1}\infty - \tan^{-1}0) = \frac{\pi}{4}$$

whence

$$k = \int_0^\infty dt\, e^{-tt} = \frac{\sqrt{\pi}}{2}$$

where $\pi = 3.1415926\ldots.$ The expression for the horizontal refraction is therefore found at once by putting $\frac{1}{2}\sqrt{\pi}$ for $\psi\,(n)$ in every term of (191), and $\sin z = 1$, namely:

$$r_0 = \frac{\alpha}{1-\alpha}\sqrt{\frac{\beta\pi}{2}}\left\{\begin{array}{l} e^{-\alpha\beta} \\ + 2^{\frac{1}{2}}\,\alpha\beta\, e^{-2\alpha\beta} \\ + \dfrac{3^{\frac{3}{2}}}{1\,.\,2}\,\alpha^2\beta^2\, e^{-3\alpha\beta} \\ + \dfrac{4^{\frac{5}{2}}}{1\,.\,2\,.\,3}\,\alpha^3\beta^3\, e^{-4\alpha\beta} \\ + \&c. \end{array}\right\} \tag{193}$$

For small values of T, that is, for great zenith distances, we may obtain the value of the integral in (188) by a series of ascending powers of T. We have

$$\int_T^{\infty} dt\, e^{-tt} = \int_0^{\infty} dt\, e^{-tt} - \int_0^{T} dt\, e^{-tt} \tag{194}$$

The first integral of the second member is given by (192). The second is

$$\int_0^T dt\, e^{-tt} = \int_0^T dt\left(1 - t^2 + \frac{t^4}{1\,.\,2} - \frac{t^6}{1\,.\,2\,.\,3} + \&c.\right)$$

$$= T - \frac{T^3}{3} + \frac{1}{1\,.\,2}\cdot\frac{T^5}{5} - \frac{1}{1\,.\,2\,.\,3}\cdot\frac{T^7}{7} + \&c. \tag{195}$$

Another development for the same case is obtained by the successive application of the method of integration by parts, as follows:*

$$\int dt\, e^{-tt} = t\, e^{-tt} + 2\int t^2\, dt\, e^{-tt}$$

$$= t\, e^{-tt} + \frac{2}{3}\, t^3 e^{-tt} + \frac{2^2}{3}\int t^4\, dt\, e^{-tt}$$

---

* By the formula $\int x\, dy = xy - \int y\, dx$, making always $x = e^{-tt}$, and dy successively $= dt,\ t^2 dt,\ t^4 dt,$ &c.

$$= e^{-tt}\, t\left(1 + \frac{2t^2}{3} + \frac{(2t^2)^2}{3\,.\,5} + \frac{(2t^2)^3}{3\,.\,5\,.\,7} + \&c.\right)$$

whence, by introducing the limits,

$$\int_0^T dt\, e^{-tt} = e^{-TT}\, T\left(1 + \frac{2\,T^2}{3} + \frac{(2\,T^2)^2}{3\,.\,5} + \frac{(2\,T^2)^3}{3\,.\,5\,.\,7} + \&c.\right) \quad (196)$$

As the denominators increase, these series finally become convergent for all values of T; but they are convenient only for small values.

For the greater values of T, a development according to the descending powers may be obtained, also by the method of integration by parts, as follows:* We have

$$\int dt\, e^{-tt} = -\frac{1}{2t}\, e^{-tt} - \tfrac{1}{2}\int \frac{dt}{t^2}\, e^{-tt}$$

$$= -\frac{1}{2t}\, e^{-tt} + \frac{1}{2^2 t^3}\, e^{-tt} + \frac{1\,.\,3}{2^2}\int \frac{dt}{t^4}\, e^{-tt}$$

Hence

$$\int_T^\infty dt\, e^{-tt} = \frac{e^{-TT}}{2\,T}\left\{1 - \frac{1}{2\,T^2} + \frac{1\,.\,3}{(2\,T^2)^2} - \frac{1\,.\,3\,.\,5}{(2\,T^2)^3} + \cdots\right.$$

$$\left.\pm \frac{1\,.\,3\,.\,5\ldots(2n-1)}{(2\,T^2)^n}\right\} \mp \frac{1\,.\,3\,.\,5\ldots(2n+1)}{2^{n+1}}\int_T^\infty \frac{dt}{t^{2n+2}}\, e^{-tt} \quad (197)$$

The sum of a number of consecutive terms of this series is alternately greater and less than the value of the integral. But since the factors of the numerators increase, the series will at last become divergent for any value of T. Nevertheless, if we stop at any term, *the sum of all the remaining terms will be less than this term;* for if we take the sum of all the terms in the brackets, the sum of the remaining terms is

$$\mp \frac{1\,.\,3\,.\,5\ldots(2n+1)}{2^{n+1}}\int_T^\infty \frac{dt}{t^{2n+2}}\, e^{-tt}$$

---

* By the formula $\int x\, dy = xy - \int y\, dx$, making always $dy = t\, dt\, e^{-tt}$, and x successively $= \frac{1}{t}, \frac{1}{t^3}, \frac{1}{t^5}$, &c.

The integral in this expression is evidently less than the product of the integral

$$\int_T^\infty \frac{dt}{t^{2n+2}} = \frac{1}{(2n+1)\,T^{2n+1}}$$

multiplied by the greatest value of e^{-tt} between the limits T and ∞, and this greatest value is e^{-TT}. Hence the above remainder is always numerically less than

$$\mp \frac{1 \,.\, 3 \,.\, 5 \,\ldots\ldots (2n-1)}{2^{n+1}\; T^{2n+1}}\, e^{-TT}$$

which expression is nothing more than the last term of the series (when multiplied by the factor without the brackets), taken with a contrary sign. Hence, if we do not continue the summation until the terms begin to increase, but stop at some sufficiently small term, the error of the result will always be less than this term.

Finally, the integral may be developed in the form of a continued fraction, as was shown by LAPLACE. Putting for brevity

$$\psi(n) = u_0 = \frac{1}{2T}\left(1 - \frac{1}{2T^2} + \frac{1\,.\,3}{(2T^2)^2} - \frac{1\,.\,3\,.\,5}{(2T^2)^3} + \&c.\right) \tag{198}$$

and denoting the successive derivatives of u_0 relatively to T by u_1, u_2, &c., we have first

$$u_1 = -\frac{1}{2T^2} + \frac{1\,.\,3}{(2T^2)^2} - \frac{1\,.\,3\,.\,5}{(2T^2)^3} + \&c. \tag{199}$$

or

$$u_1 = 2Tu_0 - 1 \tag{200}$$

Differentiating this equation, successively, we have

$$\begin{aligned} u_2 &= 2Tu_1 + 2u_0 \\ u_3 &= 2Tu_2 + 4u_1 \\ u_4 &= 2Tu_3 + 6u_2 \\ &\&c. \end{aligned}$$

or, in general,

$$u_{n+1} = 2Tu_n + 2nu_{n-1}$$

n having any value in the series 1 . 2 . 3 . 4 . . . &c.

Hence we derive

$$\frac{u_n}{u_{n-1}} = -\frac{2n}{2T - \frac{u_{n+1}}{u_n}}$$

or, putting

$$k = \frac{1}{2T^2} \tag{201}$$

$$\frac{u_n}{u_{n-1}} = -\frac{2n\left(\frac{k}{2}\right)^{\frac{1}{2}}}{1 - \left(\frac{k}{2}\right)^{\frac{1}{2}}\frac{u_{n+1}}{u_n}} \tag{202}$$

By (200) we have

$$u_0 = \frac{1}{2T - \frac{u_1}{u_0}}$$

or

$$2Tu_0 = \frac{1}{1 - \left(\frac{k}{2}\right)^{\frac{1}{2}}\frac{u_1}{u_0}} \tag{203}$$

But from (202), by making n successively 1, 2, 3, &c., we have

$$\frac{u_1}{u_0} = -\frac{2\left(\frac{k}{2}\right)^{\frac{1}{2}}}{1 - \left(\frac{k}{2}\right)^{\frac{1}{2}}\frac{u_2}{u_1}}, \qquad \frac{u_2}{u_1} = -\frac{2 \,.\, 2\left(\frac{k}{2}\right)^{\frac{1}{2}}}{1 - \left(\frac{k}{2}\right)^{\frac{1}{2}}\frac{u_3}{u_2}}, \quad \text{\&c.},$$

which successively substituted in (203) give

$$2Tu_0 = \cfrac{1}{1 + \cfrac{k}{1 + \cfrac{2k}{1 + \cfrac{3k}{1 + \cfrac{4k}{1 + \text{\&c.}}}}}} \tag{204}$$

This can be employed for all values of T, but when k exceeds $\frac{1}{4}$ it will be more convenient to employ (195) or (196).

The successive approximating fractions of (204) are

$$\frac{1}{1}, \quad \frac{1}{1+k}, \quad \frac{1+2k}{1+3k}, \quad \frac{1+5k}{1+6k+3k^2}, \quad \frac{1+\ 9k+\ 8k^2}{1+10k+15k^2}, \text{ \&c.}$$

and, in general, denoting the n^{th} approximating fraction by $\frac{a_n}{b_n}$,

$$\frac{a_n}{b_n} = \frac{a_{n-1} + (n-1)\,k a_{n-2}}{b_{n-1} + (n-1)\,k b_{n-2}}$$

By the preceding methods, then, the function $\psi(n)$ can be computed for any value of T. A table containing the logarithm of this function for all values of T from 0 to 10, is given by Bessel (*Fundamenta Astronomiæ*, pp. 36, 37), being an extension of that first constructed by Kramp. By the aid of this table the computation of the refraction is greatly facilitated.

114. Let us now examine the second term of (179.) This term will have its greatest value in the horizontal refraction, when $z = 90°$, in which case it reduces to

$$-\frac{\alpha\beta\, e^{-\beta s}\, s ds\,\left[\frac{3}{2} s - 2\alpha\,(1 - e^{-\beta s})\right]}{(1-\alpha)\left[2s - 2\alpha\,(1 - e^{-\beta s})\right]^{\frac{3}{2}}}$$

Moreover, the most sensible part of the integral corresponds to small values of s, and therefore, since α is also very small, we may put $2\alpha\,(1 - e^{-\beta s}) = 2\alpha\beta s$. The integral thus becomes

$$-\frac{\alpha\beta\,(3 - 4\alpha\beta)}{2^{\frac{5}{2}}\,(1-\alpha)\,(1-\alpha\beta)^{\frac{3}{2}}}\int_0^\infty s^{\frac{1}{2}}\, ds\, e^{-\beta s}$$

Now we have, by integrating by parts,

$$\int s^{\frac{1}{2}}\, ds\, e^{-\beta s} = -\frac{s^{\frac{1}{2}}\, e^{-\beta s}}{\beta} + \frac{1}{2\beta}\int s^{-\frac{1}{2}}\, ds\, e^{-\beta s}$$

and hence

$$\int_0^\infty s^{\frac{1}{2}}\, ds\, e^{-\beta s} = \frac{1}{2\beta}\int_0^\infty s^{-\frac{1}{2}}\, ds\, e^{-\beta s}$$

Putting $\beta s = x^2$, this becomes, by (192),

$$\frac{1}{\beta^{\frac{3}{2}}}\int_0^\infty dx\, e^{-xx} = \frac{1}{2\beta}\sqrt{\frac{\pi}{\beta}}$$

Hence the term becomes

$$-\frac{\alpha\,(3 - 4\alpha\beta)}{8\,(1-\alpha)\,(1-\alpha\beta)^{\frac{3}{2}}}\sqrt{\frac{\pi}{2\beta}}$$

Taking BESSEL's value of $h = 116865.8$ toises* $= 227775.7$ metres, and the value of $l = 7993.15$ metres (p. 141), we find by (176) $\beta = 768.57$. Substituting this and $\alpha = 0.000294211$ (p. 146), the value of the above expression, reduced to seconds of arc by dividing by sin 1″, is found to be only 0″.72, which in the horizontal refraction is insignificant. This term, therefore, can be neglected (and consequently also all the subsequent terms), and the formula (191) may be regarded as the rigorous expression of the refraction.

115. In order to compute the refraction by (191), it only remains to determine the constants α and β. The constant α might be found from (178) by employing the value of k determined by BIOT by direct experiment upon the refractive power of atmospheric air, but in order that the formula may represent as nearly as possible the observed refractions, BESSEL preferred to determine both α and β from observations.†

Now, α depends upon the density of the air at the place of observation, and is, therefore, a function of the pressure and temperature; and β, which involves l, also depends upon the thermometer, since by the definition of l it must vary with the temperature. The constants must, then, be determined for some assumed normal state of the air, and we must have the means of deducing their values for any other given state. Let

$p_0 =$ the assumed normal pressure,
$\tau_0 =$ " " temperature,
$p =$ the observed pressure,
$\tau =$ " " temperature,
$\delta_0 =$ the normal density corresponding to p_0 and τ_0,
$\delta =$ the density corresponding to p and τ;

* *Fundamenta Astronomiæ*, p. 40.

† It should be observed that the assumed expression of the density (177) may represent various hypotheses, according to the form given to β. Thus, if we put $\beta = \frac{a}{l}$, we have the form (172) which expresses the hypothesis of a uniform temperature. We may therefore readily examine how far that hypothesis is in error in the horizontal refraction; for by taking the reciprocal of (167) we have in this case $\beta = 796.53$, and hence with $\alpha = 0.000294211$ we find, by taking fifteen terms of the series (193), $r_0 = 39' \, 54''.5$, which corresponds to Barom. 0^m. 76, and Therm. 0° C. This is 2′ 23″.5 *greater* than the value given by ARGELANDER's Observations (p. 141). Our first hypothesis gave a result too small by more than 7′, and hence a true hypothesis must be intermediate between these, as we have already shown from a con

then we have by (171)

$$\delta = \frac{\delta_0}{1 + \varepsilon(\tau - \tau_0)} \cdot \frac{p}{p_0}$$

in which ε is the coefficient of expansion of atmospheric air, or the expansion for 1° of the thermometer. *If the thermometer is Centigrade*, we have, according to BESSEL,*

$$\varepsilon = 0.0036438$$

From (178) it follows that α is sensibly proportional to the density, and hence if we put

α_0 = the value of α for the normal density δ_0,

we have, for any given state of the air,

$$\alpha = \frac{\alpha_0}{1 + \varepsilon(\tau - \tau_0)} \cdot \frac{p}{p_0} \tag{205}$$

in which for p and p_0 we may use the heights of the barometric column, provided these heights are reduced to the same temperature of the mercury and of the scales.

Again, if

l_0 = the height of a homogeneous atmosphere of the temperature τ_0, and any given pressure,

then the height l for the same pressure, when the temperature is τ, is

$$l = l_0 [1 + \varepsilon(\tau - \tau_0)] \tag{206}$$

The normal state of the air adopted by BESSEL in the determination of the constants, so as to represent BRADLEY'S observations, made at the Greenwich Observatory in the years 1750–1762, was a mean state corresponding to the barometer 29.6 inches, and thermometer 50° Fahrenheit = 10° Centigrade; and for this state he found

$$\alpha_0 = 0.000278953$$

---

sideration of the law of temperatures. At the same time, we see that the hypothesis of a uniform temperature is nearer to the truth than the first hypothesis, and we are so far justified in adhering to the form $\delta = \delta_0 e^{-\beta s}$ with the modification of substituting a corrected value of β.

* This value, determined by BESSEL, from the observations of stars, differs slightly from the value $\frac{1}{273}$ more recently determined by RUDBERG and REGNAULT by direct experiments upon the refractive power of the air.

or, dividing by sin 1″,

$$\alpha_0 = 57''.538$$

and

$$h = 116865.8 \text{ toises} = 227775.7 \text{ metres.}$$

For the constant l_0 at the normal temperature 50° F., BESSEL employed

$$l_0 = 4226.05 \text{ toises} = 8236.73 \text{ metres.*}$$

Since the strata of the atmosphere are supposed to be parallel to the earth's surface, BESSEL employed for a the radius of curvature of the meridian for the latitude of Greenwich (the observations of Bradley being taken in the meridian), and, in accordance with the compression of the earth assumed at the time when this investigation was made, he took

$$a = 6372970 \text{ metres.}$$

Hence we have

$$\beta_0 = \frac{h - l_0}{h} \cdot \frac{a}{l_0} = 745.747$$

These values of α_0 and β_0 being substituted for α and β in (193), the horizontal refraction is found to be only about 1′ too great, which is hardly greater than the probable error of the observed horizontal refraction. At zenith distances less than 85°, however, BESSEL afterwards found that the refraction computed with these values of the constants required to be multiplied by the factor 1.003282 in order to represent the Königsberg observations.

116. By the preceding formulæ, then, the values of the constants α and β can be found for any state of the air, as given by the barometer and thermometer at the place of observation, and then the true refraction might be directly computed by (191). But, as this computation would be too troublesome in practice, the *mean refraction* is computed for the assumed normal values of α and β, and given in the refraction tables. From this mean

---

* According to the later determination of REGNAULT which we have used on p. 143, we should have $l_0 = 8286.1$ metres. The difference does not affect the value of BESSEL's tables, which are constructed to represent actual observations.

refraction we must deduce the true refraction in any case by applying proper corrections depending upon the observed state of the barometer and thermometer. For facility of logarithmic computation, BESSEL adopted the form

$$r = r_0 \left(\frac{p}{p_0}\right)^A \left\{ \frac{1}{1 + \varepsilon\,(\tau - \tau_0)} \right\}^\lambda \tag{207}$$

in which r_0 is the tabular refraction corresponding to p_0 and τ_0, and r is the refraction corresponding to the observed p and τ. Let us see what interpretation must be given to the exponents A and λ. If the pressure remained p_0, the refraction corresponding to the temperature τ would be

$$r_0 + \frac{dr}{d\tau}\,(\tau - \tau_0) + \frac{d^2r}{d\tau^2}\,\frac{(\tau - \tau_0)^2}{1\,.\,2} + \&c.$$

or, with sufficient precision,

$$r_0 \left\{ 1 + \frac{1}{r_0} \cdot \frac{dr}{d\tau}\,(\tau - \tau_0) \right\}$$

In like manner, if the temperature were constant, and the pressure is increased by the quantity $p - p_0$, the refraction would become nearly

$$r_0 \left\{ 1 + \frac{1}{r_0} \cdot \frac{dr}{dp}\,(p - p_0) \right\}$$

Hence, when both pressure and temperature vary, we shall have, very nearly,

$$r = r_0 \left\{ 1 + \frac{1}{r_0} \cdot \frac{dr}{dp}\,(p - p_0) \right\} \times \left\{ 1 + \frac{1}{r_0} \cdot \frac{dr}{d\tau}\,(\tau - \tau_0) \right\} \tag{208}$$

Now, putting $\frac{p}{p_0}$ in (207) under the form $1 + \frac{p - p_0}{p_0}$, and developing by the binomial theorem, we have

$$r = r_0 \left\{ 1 + \frac{A}{p_0}\,(p - p_0) + \&c. \right\} \times \left\{ 1 - \lambda\varepsilon\,(\tau - \tau_0) + \&c. \right\}$$

Therefore, neglecting the smaller terms, we must have

$$A = \frac{p_0}{r_0} \cdot \frac{dr}{dp} \qquad \lambda = -\frac{1}{\varepsilon\, r_0} \cdot \frac{dr}{d\tau} \tag{209}$$

to determine which we are now to find the derivatives of (191) relatively to p and τ. Put

$$x = \frac{\alpha\beta}{\sin^2 z} \tag{210}$$

and $q_1 = \psi(1)$, $q_2 = 2^{\frac{3}{2}}\psi(2)$, $q_3 = 3^{\frac{5}{2}}\psi(3)$, &c., or in general

$$q_n = n^{\frac{2n-1}{2}}\psi(n) \tag{211}$$

then, if we also put

$$Q = x\,e^{-x}\,q_1 + \frac{x^2}{1\,.\,2}\,e^{-2x}\,q_2 \ldots\ldots + \frac{x^n}{1\,.\,2\,\ldots\ldots\,n}\,e^{-nx}\,q_n + \text{\&c.} \tag{212}$$

the formula (191) becomes

$$(1-\alpha)\,r = \sin^2 z\sqrt{\frac{2}{\beta}}\,.\,Q \tag{213}$$

in which, since the variations of $\frac{\alpha}{1-\alpha}$ in (191) are sensibly the same as those of α, we may regard $1-\alpha$ as constant. Differentiating this, observing that Q varies with both p and τ, while β varies only with τ, we have

$$\left.\begin{aligned}
(1-\alpha)\frac{dr}{dp} &= \sin^2 z\sqrt{\frac{2}{\beta}}\,.\,\frac{dQ}{dp} \\
(1-\alpha)\frac{dr}{d\tau} &= \sin^2 z\sqrt{\frac{2}{\beta}}\,.\,\frac{dQ}{d\tau} - (1-\alpha)\frac{r}{2\beta}\,.\,\frac{d\beta}{d\tau}
\end{aligned}\right\} \tag{214}$$

In differentiating Q, it will be convenient to regard it as a function of the two variables x and β, the quantities q_1, q_2, &c. varying only with β. We have, since β does not vary with p,

$$\frac{dQ}{dp} = \frac{dQ}{dx}\,.\,\frac{dx}{dp} \tag{215}$$

and since both x and β vary with τ,

$$\frac{dQ}{d\tau} = \frac{dQ}{dx}\,.\,\frac{dx}{d\tau} + \frac{dQ}{d\beta}\,.\,\frac{d\beta}{d\tau} \tag{216}$$

From (212) we find

$$\frac{dQ}{dx} = \frac{1-x}{x} Q' \tag{217}$$

in which

$$Q' = xe^{-x} q_1 + \frac{x^2}{1.2} e^{-2x} 2q_2 + \frac{x^3}{1.2.3} e^{-3x} 3q_3 + \&c. \tag{218}$$

Also,

$$\frac{dQ}{d\beta} = xe^{-x} \frac{dq_1}{d\beta} + \frac{x^2}{1.2} e^{-2x} \frac{dq_2}{d\beta} + \&c. \tag{219}$$

In which we have generally, by (211),

$$\frac{dq_n}{d\beta} = n^{\frac{2n-1}{2}} \frac{d\psi(n)}{dT} \cdot \frac{dT}{d\beta}$$

But by (200), in which $u_0 = \psi(n)$, we have

$$n^{\frac{2n-1}{2}} \cdot \frac{d\psi(n)}{dT} = 2Tq_n - n^{\frac{2n-1}{2}}$$

and by (187)

$$\frac{dT}{d\beta} = \frac{T}{2\beta}$$

whence

$$\frac{dq_n}{d\beta} = \frac{T^2}{\beta} q_n - \frac{T}{2\beta} n^{\frac{2n-1}{2}}$$

$$= \frac{\cot^2 z}{2} nq_n - \frac{\cot z}{2\sqrt{2\beta}} n^n$$

Substituting the values of this expression for $n = 1, 2, 3$, &c. in (219), we have

$$\frac{dQ}{d\beta} = \frac{\cot^2 z}{2}\left(xe^{-x} q_1 + \frac{x^2}{1.2} e^{-2x} 2q_2 + \frac{x^3}{1.2.3} e^{-3x} 3q_3 + \&c.\right)$$

$$-\frac{\cot z}{2\sqrt{2\beta}}\left(xe^{-x} + \frac{x^2}{1.2} e^{-2x} 2^2 + \frac{x^3}{1.2.3} e^{-3x} 3^3 + \&c.\right)$$

The first series in this expression $= Q'$. The second, when e^{-x}, e^{-2x}, &c. are developed in series, becomes

$$x + x^2 + x^3 + \&c. = \frac{x}{1-x}$$

and hence

$$\frac{dQ}{d\beta} = \frac{\cot^2 z}{2} Q' - \frac{\cot z}{2\sqrt{2\beta}} \cdot \frac{x}{1-x} \tag{220}$$

We have, further, from (210) and the values of α, l, and β in the preceding article,

$$\frac{dx}{dp} = \frac{x}{\alpha} \cdot \frac{d\alpha}{dp} = \frac{x}{\alpha} \cdot \frac{\alpha}{p} = \frac{x}{p}$$

$$\frac{d\beta}{d\tau} = \frac{d\beta}{dl} \cdot \frac{dl}{d\tau} = -\frac{a}{l^2} \cdot \varepsilon\, l = -\varepsilon \beta \cdot \frac{h}{h-l}$$

$$\frac{d\alpha}{d\tau} = -\varepsilon \alpha$$

$$\frac{dx}{d\tau} = \frac{x}{\alpha} \cdot \frac{d\alpha}{d\tau} + \frac{x}{\beta} \cdot \frac{d\beta}{d\tau} = -\varepsilon x \cdot \frac{2h-l}{h-l}$$

Substituting these values in (215) and (216), and then substituting in (214), we find*

$$(1-\alpha)\frac{dr}{dp} = \sin^2 z \sqrt{\frac{2}{\beta}} \cdot Q' \cdot \frac{1-x}{p}$$

$$(1-\alpha)\frac{dr}{d\tau} = -\varepsilon \sin^2 z \sqrt{\frac{2}{\beta}} \cdot Q' \left\{ \frac{2h-l}{h-l}(1-x) + \frac{h}{h-l} \cdot \frac{\beta}{2} \cdot \cot^2 z \right\}$$
$$+ \varepsilon \left\{ (1-\alpha)\, r + \frac{\alpha\beta \cot z}{1-x} \right\} \frac{\frac{1}{2}h}{h-l} \tag{221}$$

These formulæ are to be computed with the normal values of α, β, r, l, and p, and for the different zenith distances, after which A and λ are computed by (209). The values of A and λ thus found are given in Table II.

117. Finally, in tabulating the formula (207), Bessel puts

$$r_0 = \alpha \tan z \tag{222}$$

$$\frac{p}{p_0} = \beta, \qquad \gamma = \frac{1}{1 + \varepsilon(\tau - \tau_0)}$$

(where α and β no longer have the same signification as in the preceding articles).

---

* Bessel, *Fundamenta Astronomiæ*, p. 34.

The true refraction then takes the form

$$r = \alpha \beta^A \gamma^\lambda \tan z \tag{223}$$

The quantity here denoted by β is the ratio of the observed and normal heights of the barometer, both being reduced to the same temperature of the mercury and of their scales. First, to correct for the temperature of the scale, let $b^{(l)}$, $b^{(e)}$, or $b^{(m)}$ denote the observed reading of the barometer scale according as it is graduated in Paris lines, English inches, or French metres. The standard temperatures of the Paris line is 13° Réaumur, of the English inch 62° Fahrenheit, and of the French metre 0° Centigrade; that is, the graduations of the several scales indicate true heights only when the attached thermometers indicate these temperatures respectively. The expansion of brass from the freezing point to the boiling point is .0018782 of its length at the freezing point. If then the reading of the attached thermometer is denoted either by r', f', or c', according as it is Réaumur's, Fahrenheit's, or the Centigrade, the true height observed will be (putting $s =$ 0.0018782)

$$b^{(l)} \cdot \frac{1 + \frac{s}{80} r'}{1 + \frac{s}{80} \cdot 13}, \qquad b^{(e)} \cdot \frac{1 + \frac{s}{180}(f' - 32)}{1 + \frac{s}{180} \cdot 30}, \qquad b^{(m)} \cdot \frac{1 + \frac{s}{100} \cdot c'}{1}$$

or

$$b^{(l)} \cdot \frac{80 + r's}{80 + 13s}, \qquad b^{(e)} \cdot \frac{180 + (f' - 32)s}{180 + 30s}, \qquad b^{(m)} \cdot \frac{100 + c's}{100} \tag{224}$$

where the multipliers $1 + \frac{s}{80} r'$, &c. evidently reduce the reading to what it would have been if the observed temperature had been that of freezing, and the divisors $1 + \frac{s}{80} \cdot 13$, &c. further reduce these to the respective temperatures of graduation, and consequently give the true heights.

This true height of the mercury will be proportional to the pressure only when the temperature of the mercury is constant. We must, therefore, reduce the height to what it would be if the temperature were equal to the adopted normal temperature, which is in our table 8° Réaumur = 50° F. = 10° C. Now, mercury expands $\frac{1}{55.5}$ of its volume at the freezing point of water, when

its temperature is raised from that point to the boiling point of water. Hence, putting $q = \frac{1}{55.5}$, the above heights will be reduced to the normal temperature by multiplying them respectively by the factors

$$\frac{80 + 8q}{80 + r'q}, \qquad \frac{180 + 18q}{180 + (f' - 32)q}, \qquad \frac{100 + 10q}{100 + c'q} \tag{225}$$

The normal height of the barometer adopted by Bessel was 29.6 inches of Bradley's instrument, or 333.28 Paris lines; but it afterwards appeared that this instrument gave the heights too small by $\frac{1}{2}$ a Paris line, so that the normal height in the tables is 333.78 Paris lines, at the adopted normal temperature of 8° R. Reducing this to the standard temperature of the Paris line = 13° R., we have

$$p_0 = 333.78\,\frac{80 + 8s}{80 + 13s} \tag{226}$$

In comparing this with the observed heights, the $b^{(e)}$ and $b^{(m)}$ must be reduced to lines by observing that one English inch = 11.2595 Paris lines, and one metre = 443.296 Paris lines. Making this reduction, the value of $\beta = \frac{p}{p_0}$ is found by dividing the product of (224) and (225) by (226). The result may then be separated into two factors, one of which depends upon the observed height of the barometric column, and the other upon the attached thermometer; so that if we put

$$\left.\begin{aligned}
B &= \frac{b^{(l)}}{333.78}\cdot\frac{80 + 8q}{80 + 8s} \\
&= b^{(e)}\cdot\frac{11.2595}{333.78}\cdot\frac{80 + 13s}{80 + 8s}\cdot\frac{180 + 18q}{180 + 30s} \\
&= b^{(m)}\cdot\frac{443.296}{333.78}\cdot\frac{80 + 13s}{80 + 8s}\cdot\frac{100 + 10q}{100} \\
&\text{and} \\
T &= \frac{80 + r's}{80 + r'q} = \frac{180 + (f' - 32)s}{180 + (f' - 32)q} = \frac{100 + c's}{100 + c'q}
\end{aligned}\right\} \tag{227}$$

we shall have $\beta = BT$, or

$$\log\beta = \log B + \log T \tag{228}$$

The quantity γ would be computed directly under the form

$$\gamma = \frac{1}{1 + \varepsilon(\tau - \tau_0)}$$

if τ_0 were at once the freezing point and the normal temperature of the tables; for ε is properly the expansion of the air for each degree of the thermometer above the freezing point, the density of the air at this point being taken as the unit of density. But if the normal temperature is denoted by τ_0, that of the freezing point by τ_1, the observed by τ, we shall have

$$\gamma = \frac{1 + \varepsilon(\tau_0 - \tau_1)}{1 + \varepsilon(\tau - \tau_1)}$$

an expression which, if we neglect the square of ε, will be reduced to the above more simple one by dividing the numerator and denominator by $1 + \varepsilon(\tau_0 - \tau_1)$. BESSEL adopted for τ_0 the value 50° F. by BRADLEY's thermometer; but as this thermometer was found to give 1°.25 too much, the normal value of the tables is $\tau_0 = 48°.75$ F. Hence, if r, f, or c denote the temperature indicated by the external thermometer, according as it is Réaumur, Fahr., or Cent., we have*

$$\left.\begin{aligned} \gamma &= \frac{180 + 16.75 \times 0.36438}{180 + \frac{9}{4}r \times 0.36438} \\ &= \frac{180 + 16.75 \times 0.36438}{180 + (f - 32) \times 0.36438} \\ &= \frac{180 + 16.75 \times 0.36438}{180 + \frac{9}{5}c \times 0.36438} \end{aligned}\right\} \qquad (229)$$

The tables constructed according to these formulæ give the values of log B, log T, and log γ, with the arguments barometer, attached thermometer, and external thermometer respectively, and the computation of the true refraction is rendered extremely simple. An example has already been given in Art. 107.

118. In the preceding discussion we have omitted any consideration of the hygrometric state of the atmosphere. The

* *Tabulæ Regiomontanæ*, p. LXII.

refractive power of aqueous vapor is greater than that of atmospheric air of the same density, but under the same pressure its density is less than that of air; and LAPLACE has shown that "the greater refractive power of vapor is in a great degree compensated by its diminished density."*

119. *Refraction table with the argument true zenith distance.*—When the true zenith distance ζ is given, we may still find the refraction from the usual tables, or Col. A of Table II., where the apparent zenith distance z is the argument, by successive approximations. For, entering the table with ζ instead of z, we shall obtain an approximate value of r, which, subtracted from ζ, will give an approximate value of z; with this a more exact value of r can be found, and a second value of z, and so on, until the computed values of r and z exactly satisfy the equation $z = \zeta - r$. But it is more convenient to obtain the refraction directly with the argument ζ. For this purpose Col. B of Table II. gives the quantities α', A', λ', which are entirely analogous to the α, A, and λ, so that the refraction is computed under the form

$$r = \alpha' \beta^{A'} \gamma^{\lambda'} \tan \zeta \tag{230}$$

where β and γ have the same values as before.

The values of α', A', and λ' are deduced from those of α, A, and λ after the latter have been tabulated. They are to be so determined as to satisfy the equations

$$\alpha \beta^{A} \gamma^{\lambda} \tan z = \alpha' \beta^{A'} \gamma^{\lambda'} \tan \zeta \tag{231}$$
$$z = \zeta - \alpha' \beta^{A'} \gamma^{\lambda'} \tan \zeta \tag{232}$$

and this for any values of β and γ. Let (z) denote the value of z which corresponds to ζ when $\beta = 1$, $\gamma = 1$; that is, when the refraction is at its mean tabular value. The value of (z) may be found by successive approximations from Col. A., as above explained. Let (α), (A), (λ), and (r) denote the corresponding values of α, A, λ, r. We have

$$(r) = (\alpha) \tan (z) = \alpha' \tan \zeta$$
$$(z) = \zeta - \alpha' \tan \zeta$$

whence, by (232),

* *Méc. Cél.* Book X.

$$z = (z) - \alpha' \tan \zeta \, (\beta^{A'} \gamma^{\lambda'} - 1)$$

But, taking Napierian logarithms, we have

$$l\,(\beta^{A'} \gamma^{\lambda'}) = A' \, l\beta + \lambda' \, l\gamma$$

and hence, e being the Napierian base,

$$\beta^{A'} \gamma^{\lambda'} = e^{A' l\beta + \lambda' l\gamma} = 1 + (A' \, l\beta + \lambda' \, l\gamma) + \&c.$$

where, as β and γ differ but little from unity, the higher powers of $A' l\beta + \lambda' l\gamma$ may be omitted. Hence

$$z = (z) - (r) \, [A' \, l\beta + \lambda' \, l\gamma]$$

Now, taking the logarithm of (231), we have

$$l\,(\alpha \tan z) + A \, l\beta + \lambda \, l\gamma = l\,(\alpha' \tan \zeta) + A' \, l\beta + \lambda' \, l\gamma$$

The first member is a function of z, which we may develop as a function of (z); for, denoting this first member by fz, and putting

$$y = -(r) \, [A' l\beta + \lambda' l\gamma]$$

we have $z = (z) + y$, and hence

$$fz = f\,[(z) + y] = f\,(z) + \frac{df(z)}{d\,(z)} \, y + \&c.,$$

where we may also neglect the higher powers of y. But since $f(z)$ is what fz becomes when $z = (z)$, and consequently $A = (A)$, $\lambda = (\lambda)$, we have

$$f(z) = l\,[(\alpha) \tan (z)] + (A) \, l\beta + (\lambda) \, l\gamma$$

$$\frac{df(z)}{d(z)} = \frac{dl[(\alpha) \tan (z)]}{d\,(z)} = \frac{d\,[(\alpha) \tan (z)]}{(\alpha) \tan (z) \, d\,(z)} = \frac{1}{(r)} \cdot \frac{d(r)}{d(z)}$$

Hence we have

$$\begin{aligned} fz &= l\,[(\alpha) \tan (z)] + (A) \, l\beta + (\lambda) \, l\gamma - \frac{d(r)}{d(z)} \, [A' \, l\beta + \lambda' \, l\gamma] \\ &= l\,[\alpha' \tan \zeta] + A' \, l\beta + \lambda' \, l\gamma \end{aligned}$$

or, since $(\alpha) \tan (z) = \alpha' \tan \zeta$,

$$A'\left\{1+\frac{d\,(r)}{d\,(z)}\right\}l\beta+\lambda'\left\{1+\frac{d\,(r)}{d\,(z)}\right\}l\gamma=(A)\,l\beta+(\lambda)l\gamma$$

Since this is to be satisfied for indeterminate values of β and γ, the coefficients of $l\beta$ and $l\gamma$ in the two members must be equal; and therefore

$$\left.\begin{aligned} A' &= \frac{(A)}{1+\dfrac{d(r)}{d(z)}} \\ \lambda' &= \frac{(\lambda)}{1+\dfrac{d(r)}{d(z)}} \\ &\text{and also} \\ \alpha' &= (\alpha)\frac{\tan\,(z)}{\tan\,\zeta} \end{aligned}\right\} \quad (233)$$

All the quantities in the second members of these formulæ may be found from Column A of Table II., and thus Column B may be formed.*

If we put

$$k'=\alpha'\,\beta^{A'}\,\gamma^{\lambda'}$$

we shall now find the refraction under the form

$$r=k'\tan\zeta$$

120. *To find the refraction of a star in right ascension and declination.*

The declination δ and hour angle t of the star being given, together with the latitude φ of the place of observation, we first compute the true zenith distance ζ and the parallactic angle q by (20). The refraction will be expressed under the form

$$r=k'\tan\zeta$$

in which

$$k'=\alpha'\,\beta^{A'}\,\gamma^{\lambda'}$$

The latitude and azimuth being here constant (since refraction acts only in the vertical circle), we have from (50), by put-

* See also Bessel, *Astronomische Untersuchungen*, Vol I p 159

ting $d\varphi = 0$, $dA = 0$, $d\zeta = r \quad k' \tan \zeta$, $dt = -\, d\alpha$, (α = star's right ascension),

$$\left.\begin{aligned} d\delta &= -\, k' \tan \zeta \cos q \\ \cos \delta \, d\alpha &= -\, k' \tan \zeta \sin q \end{aligned}\right\} \quad (234)$$

which are readily computed, since the logarithms of $\tan \zeta \cos q$ and $\tan \zeta \sin q$ will already have been found in computing ζ by (20). The value of $\log k'$ will be found from Table II. Column B, with the argument ζ.

The values of $d\delta$ and $d\alpha$ thus found are those which are to be algebraically added to the *apparent* declination and right ascension to free them from the effect of refraction.

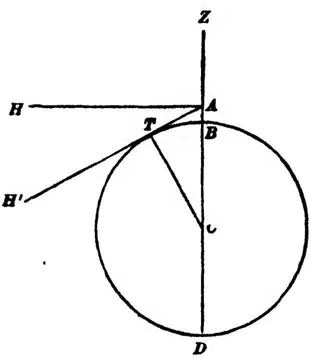
The mean value of k' is about $57''$, which may be employed when a very precise result is not required.

DIP OF THE HORIZON.

121. The *dip of the horizon* is the angle of depression of the visible sea horizon below the true horizon, arising from the elevation of the eye of the observer above the level of the sea.

Let CZ, Fig. 17, be the vertical line of an observer at A, whose height above the level of the sea is AB. The plane of the true horizon of the observer at A is a plane at right angles to the vertical line (Art. 3). Let a vertical plane be passed through CZ, and let BTD be the intersection of this plane with the earth's surface regarded as a sphere, AH its intersection with the horizontal plane. Draw ATH' in this plane, tangent to the circular section of the earth at T. Disregarding for the present the effect of the atmosphere, T will be the most distant point of the surface visible from A. If we now conceive the vertical plane to revolve about CZ as an axis, AH will generate the plane of the celestial horizon, while AH' will generate the surface of a cone touching the earth in the small circle called the visible horizon; and the angle HAH' will be the dip of the horizon.

Fig. 17.



122. *To find the dip of the horizon, neglecting the atmospheric refraction.* Let

$$x = \text{the height of the eye} = AB,$$
$$a = \text{the radius of the earth,}$$
$$D = \text{the dip of the horizon.}$$

We have in the triangle CAT, $ACT = HAH' = D$, and hence

$$\tan D = \frac{AT}{CT}$$

By geometry, we have

$$AT = \sqrt{AB \times AD} = \sqrt{x(2a + x)}$$

whence

$$\tan D = \frac{\sqrt{2ax + x^2}}{a} = \sqrt{\frac{2x}{a} + \left(\frac{x}{a}\right)^2}$$

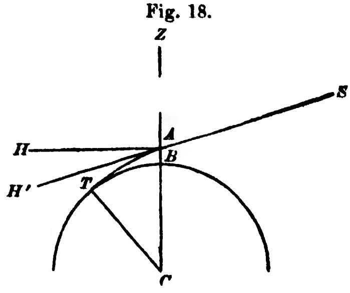
As x is always very small compared with a, the square of the fraction $\frac{x}{a}$ is altogether inappreciable: so that we may take simply

$$\tan D = \sqrt{\frac{2x}{a}} \tag{235}$$

123. *To find the dip of the horizon, having regard to the atmospheric refraction.*

The curved path of a ray of light from the point T, Fig. 18, to the eye at A, is the same as that of a ray from A to T; and this is a portion of the whole path of a ray (as from a star S) which passes through the point A, and is tangent to the earth's surface at T. The direction in which the observer at A sees the point T is that of the tangent to the curved path at A, or AH'; the true dip is therefore the angle HAH', and is less than that found in the preceding article. It is also evident that the most distant visible point of the earth's

Fig. 18.



surface is more remote from the observer than it would be if the earth had no atmosphere.

Now, recurring to the investigation of the refraction in Art. 108, we observe that the angle HAH' is the complement of the angle of incidence of the ray at the point A, there denoted by i; and it was there shown that if q, μ, and i are respectively the normal, the index of refraction, and the angle of incidence for a point elevated above the earth's surface, while a, μ_0, and z are the same quantities at the surface, we have

$$q\,\mu \sin i = a\,\mu_0 \sin z$$

But in the present case we have $z = 90°$; and hence, putting

$$D' = \text{the true dip} = 90° - i$$
$$q \;= a + x$$

we have

$$\sin i = \cos D' = \frac{\mu_0}{\mu} \cdot \frac{a}{a+x} = \frac{\mu_0}{\mu}\left(1 + \frac{x}{a}\right)^{-1}$$

Developing and neglecting the square of $\frac{x}{a}$ as before,

$$\cos D' = \frac{\mu_0}{\mu}\left(1 - \frac{x}{a}\right) \tag{236}$$

which would suffice to determine D' when μ_0 and μ have been obtained from the observed densities of the air at the observer and at the level of the sea. But, as D' is small, it is more convenient to determine it from its sine; and we may also introduce the density of the air directly into the formula by putting (Art. 110),

$$\frac{\mu_0}{\mu} = \sqrt{\frac{1 + 4k\delta_0}{1 + 4k\delta}}$$

Substituting the value of α from (178), namely,

$$\alpha = \frac{2k\delta_0}{1 + 4k\delta_0}$$

we may give this the form

$$\frac{\mu_0}{\mu} = \left\{ \frac{1}{1 - 2\alpha\left(1 - \frac{\delta}{\delta_0}\right)} \right\}^{\frac{1}{2}}$$

$$= \left\{ 1 - 2\alpha\left(1 - \frac{\delta}{\delta_0}\right) \right\}^{-\frac{1}{2}}$$

which, by neglecting the square of the second term, gives,

$$\frac{\mu_0}{\mu} = 1 + \alpha\left(1 - \frac{\delta}{\delta_0}\right)$$

Hence, still neglecting the higher powers of α and $\frac{x}{a}$, as well as their product, we have

$$\sin D' = \sqrt{1 - \cos^2 D'} = \sqrt{\left\{ \frac{2x}{a} - 2\alpha\left(1 - \frac{\delta}{\delta_0}\right) \right\}} \qquad (237)$$

which agrees with the formula given by LAPLACE, *Méc Cél.* Book X.

For an altitude of a few feet, the difference of pressure will not sensibly affect the value of D', and may be disregarded, especially since a very precise determination of the dip is not possible unless we know the density of the air *at the visible horizon*, which cannot usually be observed. We may, however, assume the temperature of the water to be that of the lowest stratum of the air, and, denoting this by τ_0, while τ denotes the temperature of the air at the height of the eye, we have [making $p = p_0$ in (171)], approximately,

$$\frac{\delta}{\delta_0} = \frac{1}{1 + \varepsilon(\tau - \tau_0)} = 1 - \varepsilon(\tau - \tau_0)$$

in which for Fahrenheit's thermometer $\varepsilon = 0.002024$. Hence

$$\sin D' = \sqrt{\frac{2x}{a}} \left\{ 1 - \frac{a}{x}\alpha\varepsilon(\tau - \tau_0) \right\}^{\frac{1}{2}}$$

$$= \sin D \left\{ 1 - \frac{\alpha\varepsilon(\tau - \tau_0)}{\sin^2 D} \right\}$$

where D is the dip, computed by (235), when the refraction is neglected, the sine of so small an angle being put for its tangent. If we substitute the values $\alpha = 0.00027895$, $\sin D = D \sin 1''$, and $\varepsilon = 0.002024$, this formula becomes

$$D' = D - \frac{24021(\tau - \tau_0)}{D}$$

in which D is in seconds. If D is expressed in minutes in the last term, it will be sufficiently accurate to take

$$D' = D - 400 \times \frac{\tau - \tau_0}{D} \tag{238}$$

This will give $D' = D$ when $\tau = \tau_0$, as it should do, since in that case the atmosphere is supposed to be of uniform density from the level of the sea to the height of the observer. If $\tau < \tau_0$, we have $D' > D$. In extreme cases, where τ is much greater than τ_0, we may have $D' < 0$, or negative, and the visible horizon will appear above the level of the eye, a phenomenon occasionally observed. I know of no observations sufficiently precise to determine whether this simple formula, deduced from theoretical considerations, accurately represents the observed dip in every case.

124. If, however, we wish to compute the value of D' for a mean state of the atmosphere without reference to the actually observed temperatures, we may proceed as follows: In the equation above found,

$$\cos D' = \frac{\mu_0}{\mu} \cdot \frac{a}{a + x}$$

we may substitute the value

$$\left(\frac{\mu}{\mu_0}\right)^{n+1} = \frac{a}{a + x}$$

which is our first hypothesis as to the law of decrease of density of the strata of the atmosphere, Art. 109. This hypothesis will serve our present purpose, provided n is so determined as to represent the actually observed mean *horizontal* refraction. We have, then,

$$\cos D' = \left(1 + \frac{x}{a}\right)^{-\frac{n}{n+1}}$$

and developing, neglecting the higher powers of $\frac{x}{a}$,

$$\cos D' = 1 - \frac{n}{n+1} \cdot \frac{x}{a}$$

$$\sin D' = \sqrt{\frac{n}{n+1} \cdot \frac{2x}{a}} = \tan D \sqrt{\frac{n}{n+1}}$$

or

$$D' = D \sqrt{\frac{n}{n+1}}$$

To determine n, we have by (160), reducing r_0 to seconds,

$$n = \frac{4\,k\delta_0}{(r_0 \sin 1'')^2}$$

where, for Barom. $0^m.76$, Therm. 10°C., which nearly represent the mean state of the atmosphere at the surface of the earth, we have $4k\delta_0 = 0.00056795$, and $r_0 = 34'\ 30''$ (which is about the mean of the determinations of the horizontal refraction by different astronomers); and hence we find

$$n = 5.639, \qquad \sqrt{\frac{n}{n+1}} = 0.9216 = 1 - 0.0784$$

$$D' = D - .0784\,D \tag{239}$$

The coefficient .0784 agrees very nearly with DELAMBRE's value .07876, which was derived from a large number of observations upon the terrestrial refraction at different seasons of the year

To compute D' directly, we have

$$D' = \frac{0.9216}{\sin 1''} \sqrt{\frac{2}{a}} \times \sqrt{x}$$

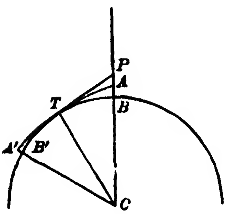
If x is in feet, we must take a in feet. Taking the mean value $a = 20888625$ feet, and reducing the constant coefficient of $\sqrt{x}$, we have

$$D' = 58''.82 \sqrt{x \text{ in feet.}} \tag{240}$$

Table XI., Vol. II., is computed by this formula.

125. *To find the distance of the sea horizon, and the distance of an object of known height just visible in the horizon.*—The small portion TA, Fig. 19, of the curved path of a ray of light, may be regarded as the arc of a circle; and then the refraction elevates A as seen from T as much as it elevates T as seen from A. Drawing the tangent TP, the observer at T would see the point A at P; and if the chord TA were drawn, the angle PTA would be the refraction of A. This refraction, being the same as that of T as seen from A, is, by (239), equal to $.0784 D$. In the triangle TPA, TAP is so nearly a right angle (with the small elevations of the eye here considered) that if we put

Fig. 19.



$$x_1 = AP$$

we may take as a sufficient approximation

$$x_1 = TA \times \tan PTA = a \tan D \times .0784 \tan D$$

But we have $a \tan^2 D = 2x$, and hence

$$x_1 = .1568 x$$

Putting

$$d = \text{the distance of the sea horizon,}$$

we have

$$PT = \sqrt{(2CB + PB) \times PB}$$

or, nearly,

$$d = \sqrt{2a(x + x_1)} = \sqrt{2.3136 ax}$$

If x is given in feet, we shall find d in statute miles by dividing this value by 5280. Taking a as in the preceding article, we find

$$\frac{\sqrt{2.3136 a}}{5280} = 1.317$$

and, therefore,

$$d \text{ (in statute miles)} = 1.317 \sqrt{x \text{ in feet}} \qquad (241)$$

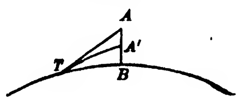
If an observer at A' at the height $A'B' = x'$ sees the object A, whose height is x, in the horizon, he must be in the curve de-

scribed by the ray from A which touches the earth's surface at T. The distance of A' from T will be $= 1.317 \sqrt{x'}$, and hence the whole distance from A to A' will be $= 1.317 (\sqrt{x} + \sqrt{x'})$.

The above is a rather rough approximation, but yet quite as accurate as the nature of the problem requires; for the anomalous variations of the horizontal refraction produce greater errors than those resulting from the formula. By means of this formula the navigator approaching the land may take advantage of the first appearance of a mountain of known height, to determine the position of the ship. For this purpose the formula (241) is tabulated with the argument "height of the object or eye;" and the sum of the two distances given in the table, corresponding to the height of the object and of the eye respectively, is the required distance of the object from the observer.

126. *To find the dip of the sea at a given distance from the observer.*—By the dip of the sea is here understood the apparent depression of any point of the surface of the water nearer than the visible horizon. Let T, Fig. 20, be such a point, and A the position of the observer. Let TA' be a ray of light from T, tangent to the earth's surface at T, meeting the vertical line of the observer in A'. Put

Fig. 20.



$D'' =$ the dip of T as seen from A,
d $=$ the distance of T in statute miles,
x $=$ the height of the observer's eye in feet $= AB$,
$x' = A'B$.

We have, by (241),

$$x' = \left(\frac{d}{1.317}\right)^2$$

and the dip of T, as seen from A', is, therefore, by (240),

$$= 58''.82 \sqrt{x'} = 44''.66\, d.$$

Now, supposing the chords TA, TA' to be drawn, the dip of T at A exceeds that at A' by the angle ATA', very nearly; and we have nearly

$$\text{angle } ATA' = \frac{AA'}{TA'} \times \frac{1}{\sin 1''} = \frac{x - x'}{5280\, d \sin 1''}$$

whence

$$D' = 44''.66\, d + \frac{x - x'}{5280\, d \sin 1''}.$$

Substituting the value of x' in terms of d,

$$D'' = 22''.14\, d + 39''.07 \frac{x}{d} \text{ (}x\text{ being in feet and } d \text{ in statute miles).} \qquad (242)$$

If d is given in sea miles, we find, by exchanging d for $\frac{69\frac{1}{2}}{60} d$,

$$D'' = 25''.65\, d + 33''.73 \frac{x}{d} \text{ (}x\text{ being in feet and } d \text{ in sea miles).} \qquad (243)$$

The value of D'' is given in nautical works in a small table with the arguments x and d. The formula (243) is very nearly the same as that adopted by BOWDITCH in the *Practical Navigator*.

127. At sea the altitude of a star is obtained by measuring its angular distance above the visible horizon, which generally appears as a well-defined line. The observed altitude then exceeds the apparent altitude by the dip, remembering that by apparent altitude we mean the altitude referred to the true horizon, or the complement of the apparent zenith distance. Thus, h' being the observed altitude, h the apparent altitude,

$$h = h' - D'$$

or, when the star has been referred to a point nearer than the visible horizon,

$$h = h' - D''$$

SEMIDIAMETERS OF CELESTIAL BODIES.

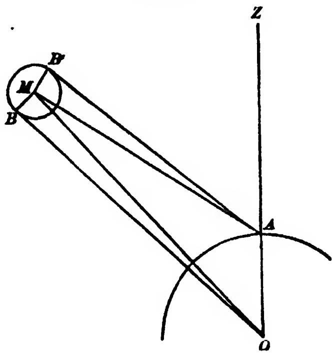
128. In order to obtain by observation the position of the centre of a celestial body which has a well-defined disc, we observe the position of some point of the limb and deduce that of the centre by a suitable application of the angular semidiameter of the body.

I shall here consider only the case of a spherical body. The apparent outline of a planet, whether spherical or spheroidal, and whether fully or partially illuminated by the sun, will be

discussed in connection with the theory of occultations in Chapter X.

The angular semidiameter of a spherical body is the angle subtended at the place of observation by the radius of the disc. I shall here call it simply the semidiameter, and distinguish the linear semidiameter as the *radius*.

Fig. 21.



Let O, Fig. 21, be the centre of the earth, A the position of an observer on its surface, M the centre of the observed body; OB, AB', tangents to its surface, drawn from O and A. The triangle OBM revolved about OM as an axis will describe a cone touching the spherical body in the small circle described by the point B, and this circle is the *disc* whose angular semidiameter at O is MOB. Put

S = the geocentric semidiameter, MOB,
S' = the apparent semidiameter, MAB',
Δ, Δ' = the distances of the centre of the body from the centre of the earth and the place of observation respectively,
a = the equatorial radius of the earth,
a' = the radius of the body,

then the right triangles OMB, AMB' give

$$\sin S = \frac{a'}{\Delta} \qquad \sin S' = \frac{a'}{\Delta'} \tag{244}$$

But if

π = the equatorial horizontal parallax of the body,

we have, Art. 89,

$$\sin \pi = \frac{a}{\Delta}$$

and hence

$$\sin S = \frac{a'}{a} \sin \pi \qquad \sin S' = \frac{\Delta}{\Delta'} \sin S \tag{245}$$

or, with sufficient precision in most cases,

$$S = \frac{a'}{a} \pi \qquad S' = \frac{\Delta}{\Delta'} S \tag{246}$$

The geocentric semidiameter and the horizontal parallax have therefore a constant ratio $= \frac{a'}{a}$. For the moon, we have

$$\frac{a'}{a} = 0.272956 \tag{247}$$

as derived from the Greenwich observations and adopted by HANSEN (*Tables de la Lune*, p. 39).

If the body is in the horizon of the observer, its distance from him is nearly the same as from the centre of the earth, and hence the geocentric is frequently called the horizontal semidiameter; but this designation is not exact, as the latter is somewhat greater than the former. In the case of the moon the difference is between 0″.1 and 0″.2. See Table XII.

If the body is in the zenith, its distance from the observer is less than its geocentric distance by a radius of the earth, and the apparent semidiameter has then its greatest value.

The apparent semidiameter at a given place on the earth's surface is computed by the second equation of (245) or (246), in which the value of $\frac{\Delta}{\Delta'}$ is that found by (104); so that, putting $z =$ the true (geocentric) zenith distance of the body, $\zeta' =$ the apparent zenith distance (affected by parallax), $A =$ its azimuth, $\varphi - \varphi'$ the reduction of the latitude, we have, (by (111) and (104),

$$\left.\begin{aligned} \gamma &= (\varphi - \varphi') \cos A \\ \sin S' &= \sin S \frac{\sin (\zeta' - \gamma)}{\sin (\zeta - \gamma)} \end{aligned}\right\} \tag{248}$$

129. This last formula is rigorous, but an approximate formula for computing the difference $S' - S$ will sometimes be convenient. In (103) we may put

$$\frac{\cos (\varphi - \varphi')}{\cos \gamma \cos \frac{1}{2} (\zeta' - \zeta)} = 1$$

without sensible error in computing the very small difference in question; we thus obtain

$$\frac{\Delta'}{\Delta} = 1 - \rho \sin \pi \cos [\tfrac{1}{2} (\zeta' + \zeta) - \gamma]$$

Putting

$$m = \rho \sin \pi \cos [\tfrac{1}{2}(\zeta' + \zeta) - \gamma] \tag{249}$$

we have

$$\frac{\Delta}{\Delta'} = \frac{1}{1-m} = 1 + m + m^2 + \&c.$$

and hence, since the third power of m is evidently insensible,

$$S' - S = Sm + Sm^2 \tag{250}$$

which is practically as exact as (248). The value of ζ' required in (249) will be found with sufficient accuracy by (114), or

$$\zeta' - \zeta = \rho\,\pi \sin(\zeta' - \gamma)$$

The quantity $S' - S$ is usually called the *augmentation* of the semidiameter. It is appreciable only in the case of the moon.

130. If we neglect the compression of the earth, which will not involve an error of more than 0″.05 even for the moon,* we may develop (250) as follows. Putting $\rho = 1$ and $\gamma = 0$ in (249), we may take

$$\begin{aligned} m &= \sin \pi \cos \tfrac{1}{2}(\zeta' + \zeta) \\ &= \sin \pi \cos [\zeta' - \tfrac{1}{2}(\zeta' - \zeta)] \\ &= \sin \pi \cos \zeta' + \tfrac{1}{2} \sin \pi \sin (\zeta' - \zeta) \sin \zeta \\ &= \sin \pi \cos \zeta' + \tfrac{1}{2} \sin^2 \pi \sin^2 \zeta' \end{aligned}$$

which substituted in (250) gives, by neglecting powers of $\sin \pi$ above the second,

$$\begin{aligned} S' - S &= S \sin \pi \cos \zeta' + \tfrac{1}{2} S \sin^2 \pi \sin^2 \zeta' + S \sin^2 \pi \cos^2 \zeta' \\ &= S \sin \pi \cos \zeta' + \tfrac{1}{2} S \sin^2 \pi + \tfrac{1}{2} S \sin^2 \pi \cos^2 \zeta' \end{aligned}$$

But we have

$$S = \frac{a'}{a}\pi = \frac{a'}{a} \cdot \frac{\sin \pi}{\sin 1''}$$

---

* The greatest declination of the moon being less than 30°, it can reach great altitudes only in low latitudes, where the compression is less sensible. A rigorous investigation of the error produced by neglecting the compression shows that the maximum error is less than 0″.06.

and if we put

$$h = \frac{a}{a'} \sin 1'', \qquad \log h = 5.2495$$

we have $\sin \pi = hS$, which substituted above gives the following formula for computing the augmentation of the moon's semidiameter:

$$S' - S = h\,S^2 \cos \zeta' + \tfrac{1}{2} h^2 S^3 + \tfrac{1}{2} h^2 S^3 \cos^2 \zeta' \tag{251}$$

EXAMPLE.—Find the augmentation for $\zeta' = 40°$, $S = 16' \, 0''$ $= 960''$.

$\log S^2$	5.9645	$\log S^3$	8.947	1st term	$= 12''.54$
$\log h$	5.2495	$\log \frac{1}{2} h^2$	0.198	2d "	$= 0\ .14$
$\log \cos \zeta'$	9.8843	log 2d term	9.145	3d "	$= 0.\ 08$
log 1st term	1.0983	$\log \cos^2 \zeta'$	9.769	$S' - S$	$= 12\ .76$
		log 3d term	8.914		

The value of $S' - S$ may be taken directly from Table XII. with the argument apparent altitude $= 90° - \zeta'$.

131. If the geocentric hour angle (t) and declination (δ) are given, we have, by substituting (137) in (245),

$$\sin S' = \sin S \frac{\sin (\delta' - \gamma)}{\sin (\delta - \gamma)} \tag{252}$$

for which γ and δ' are to be determined by (134) and (136), or with sufficient accuracy for the present purpose by the formulæ

$$\tan \gamma = \frac{\tan \varphi'}{\cos t}$$

$$\delta' - \delta = \frac{\rho \pi \sin \varphi' \sin (\delta - \gamma)}{\sin \gamma}$$

132. *To find the contraction of the vertical semidiameter of the sun or moon produced by atmospheric refraction.*

Since the refraction increases with the zenith distance, the refraction for the centre of the sun or the moon will be greater than that for the upper limb, and that for the lower limb will be greater than that for the centre. The apparent distance of the

limbs is therefore diminished, and the whole disc, instead of being circular, presents an oval figure, the vertical diameter of which is the least, and the horizontal diameter the greatest. The refraction increasing more and more rapidly as the zenith distance increases, the lower half of the disc is somewhat more contracted than the upper half.

The contraction of the vertical semidiameter may be found directly from the refraction table, by taking the difference of the refractions for the centre and the limb.

EXAMPLE.—The true semidiameter of the moon being 16′ 0″, and the apparent zenith distance of the centre 84°, find the contraction of the upper and lower semidiameters in a mean state of the atmosphere (Barom. 30 inches, Therm. 50° F.). We find from Table I.

For apparent zen. dist. of centre,	84° 0′	Refr. =	8′ 28″.0
" approx. " upper limb,	83 44	" =	8 9 .4
" " " lower "	84 16	" =	8 48 .1

Hence,

Approx. contraction upper semid. = 8′ 28″.0 — 8′ 9″.4 = 18″.6
" " lower " = 8 48 .1 — 8 28 .0 = 20 .1

These results are but approximate, since we have supposed the apparent zenith distance of the limb to differ from that of the centre by the true semidiameter, whereas they differ only by the apparent or contracted semidiameter. Hence we must repeat as follows:

App. zen. dist. upper limb =	83° 44′ 18″.6	Refr. =	8′ 9″.7
" " lower " =	84 15 39 .9	" =	8 47 .7

Contraction of upper semid. = 8′ 28″.0 — 8′ 9″.7 = 18″.3
" lower " = 8 47 .7 — 8 28 .0 = 19 .7

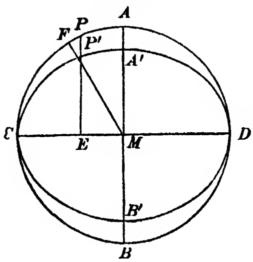
Observations at great zenith distances, where this contraction is most sensible, do not usually admit of great precision, on account of the imperfect definition of the limbs and the uncertainty of the refraction itself. It is, therefore, sufficiently exact to assume the contraction of either the upper or lower semidiameter to be equal to the mean of the two. In the above example, which offers an extreme case, if we take the mean

19″ 0 as the contraction for either semidiameter, the error will be only 0″.7, which is quite within the limit of error of observations at such zenith distances.

133. *To find the contraction of any inclined semidiameter, produced by refraction.*

Let M, Fig. 22, be the apparent place of the sun's or the moon's centre; $ACBD$, a circle described with a radius MA equal to the true semidiameter, will represent the disc as it would appear if the refraction were the same at all points of the limb. The point A, however, being less refracted than M, will appear at A', P at P', &c.; while B, being more refracted than M, appears at B'. The contraction is sensible only at great zenith distances, where we may assume that AM and $PP'E$, small portions of vertical circles drawn through A and P, are sensibly parallel. If then we put

Fig. 22.



$S =$ the true vertical semidiameter $= AM$,
$S_1 =$ the contracted vert. semid. $= A'M$,
$S_q =$ the contracted inclined semid. $= MP'$, which makes an angle q with the vertical circle,
$\Delta S_1 =$ the contraction of the vertical semid. $= S - S_1$
$\Delta S_q =$ the contraction of the inclined semid. $= S - S_q$

we shall have

$S_q \cos q = P'E =$ the difference of the apparent zenith distances of M and P',
$S_1 =$ the difference of the app. zen. dist. of M and A'.

Now, the difference of the refractions at M and A' is AA', and the difference of the refractions at M and P' is PP'; and, since these small differences are nearly proportional to the differences of zenith distance, we have

$$S_1 : S_q \cos q = AA' : PP'$$

$$PP' = \Delta S_1 \frac{S_q \cos q}{S_1}$$

The small triangle PFP' may be regarded as rectilinear and right-angled at F; whence

$$FP' = PP' \times \cos q$$

or

$$\Delta S_q = \Delta S_1 \frac{S_q \cos^2 q}{S_1}$$

If we put S_1 for S_q in the second member, the resulting value of ΔS_q will never be in error $0''.2$ for zenith distances less than 85°, and it suffices to take

$$\Delta S_q = \Delta S_1 \cos^2 q \tag{253}$$

This formula is sufficiently exact for all purposes to which we shall have occasion to apply it.

134. *To find the contraction of the horizontal semidiameter.*—The formula (253) for $q = 90°$ makes the contraction of the horizontal semidiameter $= 0$. This results from our having assumed that the portions of vertical circles drawn through the several points of the limb are parallel, and this assumption departs most from the truth in the case of the two vertical circles drawn through the extremities of the horizontal diameter. To investigate the error in this case, let ZM, Fig. 23, be the vertical circle drawn through the centre of the body, ZM' that drawn through the extremity of the horizontal semidiameter MM'. In consequence of the refraction, the points M and M' appear at N and N'. If we denote the zenith distances of M and N by ζ and z, those of M' and N' by ζ' and z', the refraction MN may be expressed as a function either of z or of ζ, Art. 107, and we shall have

Fig. 23.

$$r = k \tan z = k' \tan \zeta$$

where k and k' are given by the refraction table with the arguments z and ζ. The zenith distance of the point M' differs so little from that of M that the values of k and k' will be sensibly the same for both points, and we shall have for the refraction $M'N'$,

$$r' = k \tan z' = k' \tan \zeta'$$

These two equations give

$$\frac{\tan z}{\tan z'} = \frac{\tan \zeta}{\tan \zeta'}$$

But if the triangle ZNN' is right-angled at N, we have

$$\cos Z = \frac{\tan z}{\tan z'}$$

and hence, also,

$$\cos Z = \frac{\tan \zeta}{\tan \zeta'}$$

Therefore the triangle ZMM' is also right-angled, and it gives

$$\tan Z = \frac{\tan S}{\sin(z + r)} = \frac{\tan S'}{\sin z}$$

in which $S = MM'$ and $S' = NN'$. Hence

$$\frac{\tan S}{\tan S'} = \frac{\sin(z + r)}{\sin z} = \cos r + \sin r \cot z$$

or, very nearly,

$$\frac{S}{S'} = 1 + r \sin 1'' \cot z = 1 + k \sin 1''$$

Hence the contraction of the horizontal semidiameter is expressed by the following formula:

$$S - S' = S' k \sin 1''$$

In the zenith, the mean value of $\log k$ is 1.76156; at the zenith distance 85°, it is 1.71020. For $S' = 16'$, therefore, the contraction found by this formula is $0''.27$ in the zenith, and $0''.24$ for 85°. Thus, *for all zenith distances less than 85° the contraction of the horizontal semidiameter is very nearly constant and equal to one-fourth of a second.*

When the body is in the horizon, we have $k = r \cot z = 0$, and hence $S - S' = 0$, which follows also from the sensible parallelism of the vertical circles at the horizon.

REDUCTION OF OBSERVED ZENITH DISTANCES TO THE CENTRE OF THE EARTH.

135. It is important to observe a proper order in the application of the several corrections which have been treated of in this chapter.

The zenith distance of any point of the heavens observed with any instrument is generally affected with the index error and other instrumental errors. These errors will be treated of in the second volume; here we assume that they have been duly allowed for, and we shall call "observed" zenith distance that which would be obtained with a perfect instrument, and shall denote it by z.

In all cases the first step in the reduction is to find the refraction r $(= a\beta^A\gamma^\lambda \tan z)$ with the argument z, and then $z + r$ is the zenith distance freed from refraction.

1st. In the case of a *fixed star*,

$$\zeta = z + r$$

is at once the required geocentric zen. dist.

2d. In the case of the *moon*, the zenith distance observed is that of the upper or lower limb. If S is the geocentric and S' the augmented semidiameter found by Art. 128, 129, or 130,

$$\zeta' = z + r \pm S'$$

is the apparent zenith distance of the moon's centre freed from refraction, and affected only by parallax, and, consequently, it is that which has been denoted by the same symbol in the discussion of the parallax. With this, therefore, we compute the parallax in zenith distance, $\zeta' - \zeta$, by Art. 95, and then

$$\zeta = \zeta' - (\zeta' - \zeta)$$

is the required geocentric zenith distance of the moon's centre.

To compute S' by (248), (250), or (251), we must first know ζ'; but it will suffice to employ in these formulæ the approximate value $\zeta' = z + r \pm S$.

We can, however, avoid the computation of S', when extreme precision is not required, by computing the parallax for the zenith distance of the limb. Thus, putting $\zeta' = z + r$, and

computing $\zeta' - \zeta$ by Art. 95, the quantity $\zeta = \zeta' - (\zeta' - \zeta)$ is the geocentric zenith distance of the limb; and therefore, applying the geocentric semidiameter, $\zeta \pm S$ is the required geocentric zenith distance of the moon's centre. This process involves the error of assuming the horizontal parallax for the limb to be the same as that for the moon's centre. It can easily be shown, however, that the error in the result will never amount to $0''.2$, which in most cases in practice is unimportant. The exact amount will be investigated in the next article.

3d. In the case of the *sun* or a *planet*, when the limb has been observed, the process of reduction is, theoretically, the same as for the moon; but the parallax is so small that the augmentation of the semidiameter is insensible. We therefore take

$$\zeta' = z + r \pm S$$

and then, computing the parallax by Art. 96, or even by Art. 90, $\zeta = \zeta' - (\zeta' - \zeta)$ is the true geocentric zenith distance.

If a point has been referred to the sea horizon and the measured altitude is H, then, D being the dip of the horizon, $h' = H - D$ is properly the observed altitude, and $z = 90° - h'$ the observed zenith distance, with which we proceed as above.

136. The process above given for reducing the observed zenith distance of the moon's limb to the geocentric zenith distance of the moon's centre, is that which is usually employed; but the whole reduction, exclusive of refraction, may be directly and rigorously computed as follows. Putting

$\zeta' = z + r =$ the apparent zenith distance of the moon's limb corrected for refraction,

$\zeta =$ the geocentric zenith distance of the moon's centre,

then, S' being the augmented semidiameter, we must substitute $\zeta' \pm S'$ for ζ' in the formulæ for parallax, and, by (101), we have

$$f \sin(\zeta' \pm S') = \sin \zeta - \rho \sin \pi \cos(\varphi - \varphi') \tan \gamma$$
$$f \cos(\zeta' \pm S') = \cos \zeta - \rho \sin \pi \cos(\varphi - \varphi')$$

Multiplying the first of these by $\cos \zeta'$, the second by $\sin \zeta'$, and subtracting, we have

$$\pm f \sin S' = -\sin(\zeta' - \zeta) + \frac{\rho \sin \pi \cos(\varphi - \varphi')}{\cos \gamma} \sin(\zeta' - \gamma)$$

in which $f = \frac{\Delta'}{\Delta}$. By (245) we have also

$$f \sin S' = \sin S$$

and hence the rigorous formula

$$\sin(\zeta' - \zeta) = \rho \sin \pi \sin(\zeta' - \gamma) \frac{\cos(\varphi - \varphi')}{\cos \gamma} \mp \sin S$$

for which, however, we may employ with equal accuracy in practice

$$\sin(\zeta' - \zeta) = \rho \sin \pi \sin(\zeta' - \gamma) \mp \sin S \tag{254}$$

in which, A being the moon's azimuth, we have

$$\gamma = (\varphi - \varphi') \cos A$$

If we put (Art. 128)

$$k = \frac{a'}{a} = 0.272956$$

we have $\sin S = k \sin \pi$, and (254) may be written as follows:

$$\sin(\zeta' - \zeta) = [\rho \sin(\zeta' - \gamma) \mp k] \sin \pi \tag{255}$$

For convenience in computation, however, it will be better to make the following transformation. Put

$$\sin p = \rho \sin \pi \sin(\zeta' - \gamma) \tag{256}$$

then (254) becomes

$$\begin{aligned}\sin(\zeta' - \zeta) &= \sin p \mp \sin S \\ &= \sin(p \mp S) + \sin p\,(1 - \cos S) \mp \sin S\,(1 - \cos p) \\ &= \sin(p \mp S) + 2 \sin p \sin^2 \tfrac{1}{2} S \mp 2 \sin S \sin^2 \tfrac{1}{2} p\end{aligned}$$

where the last two terms never amount to $0''.2$, and therefore the formula may be considered exact under the form

$$\sin(\zeta' - \zeta) = \sin(p \mp S) \mp \tfrac{1}{2}(p \mp S) \sin 1'' \sin p \sin S$$

Since $\zeta' - \zeta$ and $p \mp S$ differ by so small a quantity, there will

be no appreciable error in regarding them as proportional to their sines; and hence we have

$$\zeta' - \zeta = p \mp S \mp \tfrac{1}{2}(p \mp S) \sin p \sin S \tag{257}$$

the upper signs being used for the upper limb and the lower signs for the lower limb.

In this formula, p is the parallax computed for the zenith distance of the limb, and the small term $\frac{1}{2}(p \mp S) \sin p \sin S$ may be regarded as the correction for the error of assuming the parallax of the limb to be the same as that of the centre.

Example.—In latitude $\varphi = 38° 59'$ N., given the observed zenith distance of the moon's lower limb, $z = 47° 29' 58''$, the azimuth $A = 33° 0'$, Barom. 30.25 inches, At. Therm. 65° F., Ext. Therm 64° F., Eq. hor. par. $\pi = 59' 10''.20$; find the geocentric zenith distance of the moon's centre:

(Table III.) $(\varphi - \varphi') =$	$11' 15''$	$z =$	$47° 29' 58''.00$
$\log (\varphi - \varphi') =$	2.8293	(Table II.) $r =$	1 2 .27
$\log \cos A$	9.9236	$\zeta' =$	47 31 0 .27
$\log \gamma$	2.7529	$\gamma =$	9 26 .
(Table III.) $\log \rho$	9.999428	$\zeta' - \gamma =$	47 21 34 .
$\log \sin \pi$	8.235806		
$\log \sin (\zeta' - \gamma)$	9.866652		
$\log \sin p$	8.101886	$p =$	$43' 28''.09$
$\log \sin \pi$	8.235806	$S =$	16 9 .00
(Art. 128) $\log (0.272956)$	9.436093	$p + S =$	59 37 .09
$\log \sin S$	7.671899	$\frac{1}{2}(p + S) \sin p \sin S =$	0 .11
$\log \sin p \sin S$	5.7739	$\zeta' - \zeta =$	59 37 .20
$\log (p + S)$	3.5535		
$\log \frac{1}{2}$	9.6990	$\zeta =$	$46° 31' 23''.07$
$\log \frac{1}{2}(p + S) \sin p \sin S$	9.0264		

It is hardly necessary to observe that if the geocentric zenith distance of the centre of the moon or other body is given, the apparent zenith distance of the limb affected by parallax and refraction will be deduced by reversing the order of the steps above explained.

If altitudes are given, we may employ altitudes throughout the computation, putting everywhere $90° - h$, &c. for z, &c., and making the necessary obvious modifications in the formulæ.

CHAPTER V.

FINDING THE TIME BY ASTRONOMICAL OBSERVATIONS.

137. We have seen, Art. 55, that the local time at any place is readily found when the hour angle of any known heavenly body is given. This hour angle is obtained by observation, but, a direct measure of it being in general impracticable, we must have recourse to observations from which it can be deduced.

The observer is supposed to be provided with a clock, chronometer, or watch, which is required to show the time, mean or sidereal, either at his own or at some assumed meridian, such as that of Greenwich.

The *clock correction** is the quantity which must be added algebraically to the time shown by the clock to obtain the correct time at the meridian for which the clock is regulated. If we put

$$T = \text{the clock time,}$$
$$T' = \text{the true time,}$$
$$\Delta T = \text{the clock correction,}$$

we have

$$T' = T + \Delta T$$

or

$$\Delta T = T' - T \qquad (258)$$

and the clock correction will be *positive* or *negative*, according as the clock is *slow* or *fast*. It is generally the immediate object of an observation for time to determine this correction. At the instant of the observation, the time T is noted by the clock, and if this time agrees with the time T' computed from the observation, the clock is correct; otherwise the clock is in error, and its correction is found by the equation $\Delta T = T' - T$.

The *clock rate* is the daily or hourly increase of the clock correction. Thus, if

* For brevity, I shall use *clock* to denote any time-keeper.

$$\begin{aligned} \Delta T_0 &= \text{the clock correction at a time } T_0, \\ \Delta T &= \quad\text{"}\qquad\text{"}\qquad\qquad\text{"}\quad T, \\ \delta T &= \text{the clock rate in a unit of time,} \end{aligned}$$

we have

$$\Delta T = \Delta T_0 + \delta T\,(T - T_0) \tag{259}$$

where $T - T_0$ must be expressed in days, hours, &c., according as δT is the rate in one day, one hour, &c.

When, therefore, the clock correction and rate have been found at a certain instant T_0, we can deduce the true time from the clock indication T (or "clock face," as it is often called) at any other instant, by the equation

$$T' = T + \quad T_0 + \delta T\,(T - T_0) \tag{260}$$

If the clock correction has been determined at two different times T_0 and T, the rate is inferred by the equation

$$\delta T = \frac{\Delta T - \Delta T_0}{T - T_0} \tag{261}$$

But these equations are to be used only so long as we can regard the rate as *constant.*

Since such uniformity of rate cannot be assumed for any great length of time, even with the best clocks (although the performance of some of them is really surprising), it is proper to make the interval between the observations for time so small that the rate may be taken as constant for that interval. The length of the interval will depend upon the character of the clock and the degree of accuracy required.

EXAMPLE.—At noon, May 5, the correction of a mean time clock is $-16^m\,47^s.30$; at noon, May 12, it is $-16^m\,13^s.50$; what is the mean time on May 25, when the clock face is $11^h\,13^m\,12^s.6$, supposing the rate to be uniform?

May 5, corr.	=	$-\,16^m\,47^s.30$
" 12, "	=	$-\,16\ \ 13\,.50$
Rate in 7 days	=	$+\ \ 33\,.80$
δT	=	$+\ \ \ 4\,.829$

Taking, then, as our starting point $T_0 = $ May 12, 0^h, we have

for the interval to $T =$ May 25, $11^h\ 13^m\ 12^s.6$, $T - T_0 = 13^d\ 11^h\ 13^m\ 12^s.6 = 13^d.467$. Hence we have

$$\begin{array}{rl} \Delta T_0 = & -\ 16^m\ 13^s.50 \\ \delta T \cdot (T - T_0) = & +\ \ \ 1\ \ \ 5\ .03 \\ \hline \Delta T = & -\ 15\ \ \ 8\ .47 \\ T = & 11^h\ 13^m\ 12^s.60 \\ \hline T' = & 10\ 58\ \ \ 4\ .13 \end{array}$$

But in this example the rate is obtained for one true mean day, while the unit of the interval $13^d.467$ is a mean day as shown by the clock. The proper interval with which to compute the rate in this case is $13^d\ 10^h\ 58^m\ 4^s.13 = 13^d.457$ with which we find

$$\begin{array}{rl} \Delta T_0 = & -\ 16^m\ 13^s.50 \\ \delta T \times 13.457 = & +\ \ \ 1\ \ \ 4\ .98 \\ \hline \Delta T = & -\ 15\ \ \ 8\ .52 \\ T = & 11^h\ 13^m\ 12^s.60 \\ \hline T' = & 10\ 58\ \ \ 4\ .08 \end{array}$$

This repetition will be rendered unnecessary by always giving the rate in a *unit of the clock*. Thus, suppose that on June 3, at $4^h\ 11^m\ 12^s.35$ by the clock, we have found the correction $+\ 2^m\ 10^s.14$; and on June 4, at $14^h\ 17^m\ 49^s.82$, we have found the correction $+\ 2^m\ 19^s.89$; the rate *in one hour of the clock* will be

$$\delta T = \frac{+\ 9^s.75}{34.1104} = +\ 0^s.2858$$

For practical details respecting the care of clocks and other time-keepers, the methods of comparing their indications, &c., see Vol. II.; see also Chapter VII., "Longitude by Chronometer." I shall here confine myself to the methods of determining their correction by astronomical observation.

Those methods, however, which involve details depending upon the peculiar nature of the instrument with which the observation is made, will be treated very briefly in this chapter, and their full discussion will be reserved for Vol. II.

FIRST METHOD.—BY TRANSITS.

138. At the instant of a star's passage over the meridian, note the time T by the clock. The star's hour angle at that instant is $= 0^h$, whence the local sidereal time T' is (Art. 55)

$$T' = \alpha = \text{the star's right ascension.}$$

If the clock is regulated to the local sidereal time, we have, therefore,

$$\Delta T = \alpha - T$$

But if the clock is regulated to the local mean time, we first convert the sidereal time α into the corresponding mean time T' (Art. 52), and then we have

$$\Delta T = T' - T$$

This, then, is in theory the simplest and most direct method possible. It is also practically the most precise when properly carried out with the transit instrument. But, as the transit instrument is seldom, if ever, precisely adjusted in the meridian, the clock time T of the true meridian transit of a star is itself deduced from the observed time of the transit over the instrument by applying proper corrections, the theory of which will be fully discussed in Vol. II.

It will there be seen, also, that the time may be found from transits over any vertical circle.

SECOND METHOD.—BY EQUAL ALTITUDES.

139. (A.) *Equal altitudes of a fixed star.*—The time of the meridian transit of a fixed star is the mean between the two times when it is at the same altitude east and west of the meridian; so that the observation of these two times is a convenient substitute for that of the meridian passage when a transit instrument is not available. The observation is most frequently made with the sextant and artificial horizon; but any instrument adapted to the measurement of altitudes may be employed. It is, however, not required that the instrument should indicate the true altitude; it is sufficient if the altitude is *the same* at both observa-

tions. If we use the same instrument, and take care not to change any of its adjustments between the two observations, we may generally assume that the same readings of its graduated arc represent the same altitude. Small inequalities, however, may still exist, which will be considered hereafter.*

The clock correction will be found directly by subtracting the mean of the two clock times of observation from the computed time of the star's transit.

EXAMPLE 1.—March 19, 1856; an altitude of *Arcturus* east of the meridian was noted at $11^h\ 4^m\ 51^s.5$ by a sidereal clock, and the same altitude west of the meridian at $17^h\ 21^m\ 30^s.0$; find the clock correction.

East		11^h	4^m	$51^s.5$
West		17	21	30 .0
Merid. transit by clock $= T$	$=$	14	13	10 .75
March 19, *Arcturus* R. A $= \alpha$	$=$	14	9	7 .11
Clock correction $= \Delta T$	$=$	—	4	3 .64

This is the clock correction at the sidereal time $14^h\ 9^m\ 7^s.11$ or at the clock time $14^h\ 13^m\ 10^s.75$.

EXAMPLE 2.—March 15, 1856, at the Cape of Good Hope, Latitude 33° 56′ S., Longitude $1^h\ 13^m\ 56^s$ E.; equal altitudes of *Spica* are observed with the sextant as below, the times being noted by a chronometer regulated to mean Greenwich time. The artificial horizon being employed, the altitudes recorded are double altitudes.

	East.	2 Alt. Spica.	West.
	$10^h\ 20^m\ 0^s.5$	104° 0′	$2^h\ 40^m\ 38^s.$
	" 20 28 .	" 10	" 40 10 .5
	" 20 55 .	" 20	" 39 42 .
Means	10 20 27 .83		2 40 10 .17
			10 20 27 .83
Merid. Transit. by Chronom. $= T =$			12 30 19 .00

The chronometer being regulated to Greenwich time, we must compute the Greenwich mean time of the star's transit at the Cape (Art. 52). We have

* For the method of observing equal altitudes with the sextant, see Vol. II., "Sextant."

Local sidereal time of transit $= \alpha =$		13^h	17^m	$37^s.92$
Longitude	$= -$	1	13	56.
Greenwich sidereal time	$=$	12	3	41.92
March 15, sid. time of mean noon	$=$	23	33	5.37
Sid. interval from mean noon	$=$	12	30	36.55
Reduction to mean time	$=$	$-$	2	2.97
Mean Gr. time of star's local transit	$= T' =$	12	28	33.58
Chronometer time of do.	$= T =$	12	30	19.00
Chronometer correction	$= \Delta T = -$		1	45.42

140. (B). *Equal altitudes of the sun before and after noon.*—If the declination of the sun were the same at both observations, the hour angles reckoned from the meridian east and west would be equal when the altitudes were equal, and the mean of the two clock times of observation would be the time by the clock at the instant of apparent noon, and we should find the clock correction as in the case of a fixed star. To find the correction for the change of declination, let

φ = the latitude of the place of observation,
δ = the sun's declination at apparent (local) noon,
$\Delta\delta$ = the increase of declination from the meridian to the west observation, or the decrease to the east observation,
h = the sun's true altitude at each observation,
T_0 = the mean of the clock times A.M. and P.M.,
ΔT_0 = the correction of this mean to reduce to the clock time of apparent noon,
t = half the elapsed time between the observations.

Then we have

$t + \Delta T_0$ = the hour angle at the A.M. observation reckoned towards the east,
$t - \Delta T_0$ = the hour angle at the P.M. observation,
$\delta - \Delta\delta$ = the declination at the A.M. "
$\delta + \Delta\delta$ = " " P.M "

and, by the first equation of (14) applied to each observation,

$$\sin h = \sin\varphi \sin(\delta - \Delta\delta) + \cos\varphi \cos(\delta - \Delta\delta)\cos(t + \Delta T_0)$$
$$\sin h = \sin\varphi \sin(\delta + \Delta\delta) + \cos\varphi \cos(\delta + \Delta\delta)\cos(t - \Delta T_0)$$

If we substitute

$$\begin{aligned}\sin(\delta \pm \Delta\delta) &= \sin\delta\cos\Delta\delta \pm \cos\delta\sin\Delta\delta \\ \cos(\delta \pm \Delta\delta) &= \cos\delta\cos\Delta\delta \mp \sin\delta\sin\Delta\delta \\ \cos(t \pm \Delta T_0) &= \cos t\cos\Delta T_0 \mp \sin t\sin\Delta T_0\end{aligned}$$

and then subtract the first equation from the second, we shall find

$$\begin{aligned}0 = 2\sin\varphi\cos\delta\sin\Delta\delta &- 2\cos\varphi\sin\delta\sin\Delta\delta\cos t\cos\Delta T_0 \\ &+ 2\cos\varphi\cos\delta\sin t\cos\Delta\delta\sin\Delta T_0\end{aligned}$$

whence, by transposing and dividing by the coefficient of $\sin\Delta T_0$,

$$\sin\Delta T_0 = -\frac{\tan\Delta\delta\,.\tan\varphi}{\sin t} + \frac{\tan\Delta\delta\,.\tan\delta}{\tan t}\cos\Delta T_0$$

This is a rigorous expression of the required correction ΔT_0, but the change of declination is so small that we may put $\Delta\delta$ for its tangent, ΔT_0 for its sine, and unity for $\cos\Delta T_0$, without any appreciable error; and, since $\Delta\delta$ is expressed in seconds of arc, we shall obtain ΔT_0 in seconds of time by dividing the second member by 15. We thus find the formula*

$$\Delta T_0 = -\frac{\Delta\delta\,.\tan\varphi}{15\sin t} + \frac{\Delta\delta\,.\tan\delta}{15\tan t} \tag{262}$$

The Ephemeris gives the hourly change of δ. If we take it for the Greenwich instant corresponding to the local noon, and call it $\Delta'\delta$, and if t is reduced to hours, we have

$$\Delta\delta = \Delta'\delta\,.\,t$$

and our formula becomes

$$\Delta T_0 = -\frac{\Delta'\delta\,.\,t\tan\varphi}{15\sin t} + \frac{\Delta'\delta\,.\,t\tan\delta}{15\tan t} \qquad \begin{bmatrix}\text{Equation}\\ \text{for noon.}\end{bmatrix} \tag{263}$$

To facilitate the computation in practice, we put

$$\left.\begin{aligned} A &= -\frac{t}{15\sin t} & B &= \frac{t}{15\tan t} \\ a &= A\,.\,\Delta'\delta\,.\tan\varphi & b &= B\,.\,\Delta'\delta\,.\tan\delta \end{aligned}\right\} \tag{264}$$

then we have

$$\Delta T_0 = a + b$$

* As first given by Gauss, *Monatliche Correspondenz*, Vol. 23.

The correction ΔT_0 is called *the equation of equal altitudes.* The computation according to the above form is rendered extremely simple by the aid of our Table IV., which gives the values of $\log A$ and $\log B$ with the argument "elapsed time" ($= 2t$). Then a and b are computed as above, the algebraic signs of the several factors being duly observed. When the sun is moving towards the *north*, give $\Delta'\delta$ the *positive* sign; and also when φ and δ are north, give them the positive sign; in the opposite cases they take the negative sign. The signs of A and B are given in the table; A being negative only when $t < 12^h$ and B positive when $t < 6^h$ or $> 18^h$.

When we have applied ΔT_0 to the mean of the clock times (or the "middle time"), we have the time

$$T = T_0 + \Delta T_0$$

as shown by the clock at the instant of the sun's meridian transit. Then, computing the time T', whether mean or sidereal, which the clock is required to show at that instant, we have the clock correction, as before,

$$\Delta T = T' - T$$

EXAMPLE.—March 5, 1856, at the U. S. Naval Academy, Lat. 38° 59′ N., Long. $5^h\ 5^m\ 57^s.5$ W., the sun was observed at the same altitude, A.M. and P.M., by a chronometer regulated to mean Greenwich time; the mean of the A.M. times was $1^h\ 8^m\ 26^s.6$, and of the P.M. times $8^h\ 45^m\ 41^s.7$; find the chronometer correction at noon.

We have first	A.M. Chro. Time	$= 1^h\ 8^m\ 26^s.6$
	P.M. " "	$= 8\ 45\ 41.7$
	Elapsed time $2t$	$= 7\ 37\ 15.1$
	Middle time T_0	$= 4\ 57\ \ 4.15$

From the Ephemeris we find for the local apparent noon of March 5, 1856,

$$\delta = -5°\ 46'\ 22''.5 \qquad \text{Equation of time} = +11^m\ 35^s.11$$
$$\Delta'\delta = +58''.10$$

For the utmost precision, we reduce $\Delta'\delta$ to the instant of local

noon. With these quantities and $\varphi = 38° 59'$, we proceed as follows:

Arg. $7^h 37^m$ Table IV.	$\log A$	$n9.4804$	$\log B$	9.2151
	$\log \Delta'\delta$	1.7642	$\log \Delta'\delta$	1.7642
	$\log \tan \varphi$	9.9081	$\log \tan \delta$	$n9.0047$
	$\log a$	$n1.1527$	$\log b$	$n9.9840$
		$a = -14^s.21$		$b = -0^s.96$

$$\begin{array}{lr}
\text{Middle Chro. time } T_0 & = 4^h\ 57^m\ \ 4^s.15 \\
\Delta T_0 = a + b = & -\ 15.17 \\
\hline
\text{Chro. Time of app. noon } T = & 4\ \ 56\ \ 48.98
\end{array}$$

This quantity is to be compared with the Greenwich time of the local apparent noon, since the chronometer is regulated to Greenwich time. We have

$$\begin{array}{lr}
\text{Mean local time of app. noon} & = 0^h\ 11^m\ 35^s.11 \\
\text{Longitude} & = 5\ \ \ 5\ \ 57\ .50 \\
\hline
\text{Mean Greenwich time} \quad \text{"} = T' & = 5\ \ 17\ \ 32\ .61 \\
\hline
\Delta T = T' - T & = +\ 20^m\ 43^s.63
\end{array}$$

If the correction of the chronometer to mean local time is required, we have only to omit the application of the longitude. Thus, we should have

$$\begin{array}{lr}
\text{Chro. time of app. noon} & = 4^h\ \ 56^m\ 48^s.98 \\
\text{Equation of time} & = -\ 11\ \ \ 35\ .11 \\
\hline
\text{Chro. time of mean noon} & = 4\ \ \ 45\ \ \ 13\ .87
\end{array}$$

and since at mean noon a chronometer regulated to the local time should give $0^h\ 0^m\ 0^s$, it is here *fast*, and its correction to local time is $-\ 4^h\ 45^m\ 13^s.87$.

141. (C.) *Equal altitudes of the sun in the afternoon of one day and the morning of the next following day; i.e. before and after midnight.*— It is evident that when equal zenith distances are observed in the latitude $+\varphi$, their supplement to 180° may be considered as equal zenith distances observed at the antipode in latitude $-\varphi$ on the same meridian. Hence the formula (263) will give the equation for noon at the antipode by substituting $-\varphi$ for $+\varphi$, that is, by changing the sign of the first term; but this noon at

the antipode is the same absolute instant as the midnight of the observer, and hence

$$\Delta T_0 = \frac{\Delta'\delta \, . \, t \tan \varphi}{15 \sin t} + \frac{\Delta'\delta \, . \, t \tan \delta}{15 \tan t} \quad \begin{bmatrix}\text{Equation for} \\ \text{midnight.}\end{bmatrix} \qquad (265)$$

and this is computed with the aid of the logarithms of A and B in Table IV. precisely as in (264), only changing the sign of A. The sign for this case is given in the table.*

142. *To find the correction for small inequalities in the altitudes.*—If from a change in the condition of the atmosphere the refraction is different at the two observations, equal apparent altitudes will not give equal true altitudes. To find the change Δt in the hour angle t produced by a change Δh in the altitude h, we have only to differentiate the equation

$$\sin h = \sin \varphi \sin \delta + \cos \varphi \cos \delta \cos t$$

regarding φ and δ as constant; whence

$$\cos h \, . \, \Delta h = - \cos \varphi \cos \delta \sin t \, . \, 15 \Delta t$$

where Δh is in seconds of arc and Δt in seconds of time.

If the altitude at the *west* observation is the greater by Δh, the hour angle is increased by Δt, and the middle time is increased by $\frac{1}{2} \Delta t$. The *correction* for the difference of altitudes is therefore $-\frac{1}{2} \Delta t$, and, denoting it by $\Delta' T_0$, we have, by the above equation,

$$\Delta' T_0 = \frac{\Delta h . \cos h}{30 \cos \varphi \cos \delta \sin t} \qquad (266)$$

This correction is to be added algebraically to the middle clock time in any of the cases (A), (B), (C) of the preceding articles.

Example.—Suppose that in Example 2, Art. 139, there had been observed at the east observation Barom. 30.30 inches, Therm. 35° F., but at the west observation Barom. 29.55 inches, Therm. 52° F. We have for the altitude 52° 5′ or zenith distance 37° 55′, by Table I., the mean refraction 45″.4. By Table

* For an example and some practical remarks, see my "Improved method of finding the error and rate of a chronometer by equal altitudes," Appendix to the American Ephemeris for 1856 and 1857.

XIV.A and XIV.B, the corrections for the barometer and thermometer are as follows, taking for greater accuracy one-eighth of the corrections for 6′:

East Obs.		West Obs.	
Barom. 30.30	$+ 0''.5$	Barom. 29.55	$- 0''.6$
Therm. 35°.	$+ 1\ .4$	Therm. 52°.	$- 0\ .1$
	$+ 1\ .9$		$- 0\ .7$

The difference of these numbers gives $\Delta h = + 2''.6$ as the excess of the *true* altitude at the west observation. Hence, by the formula (266),

Δh	$= + \ 2''.6$	$\log \Delta h$	0.415
h	$= \ 52^\circ \ 5'$	$\log \cos h$	9.789
φ	$= - 33 \ 56$	$\log \sec \varphi$	0.081
δ	$= - 10 \ 25$	$\log \sec \delta$	0.007
$t = \frac{1}{2}$ elapsed time	$= 2^h \ 9^m \ 51^s.$	$\log \operatorname{cosec} t$	0.270
		$\log \frac{1}{30}$	8.523
$\Delta' T_0$	$= + \ 0^s.12$	$\log \Delta' T_0$	9.085

When, however, several altitudes have been observed, as in this example, we may obtain this correction from the observations themselves; for we see that the double altitude of *Spica* changed $20' = 1200''$ in about 55^s, and hence we have the proportion

$$1200'' : 2''.6 = 55^s : \Delta' T_0$$

which gives $\Delta' T_0 = + 0^s.12$ as before. By taking the change in the double altitude, the fourth term is the value of $\frac{1}{2} \Delta t$, or $\Delta' T_0$.

If this correction be applied, we find the corrected time of transit $= 12^h \ 30^m \ 19^s.12$, and consequently the chronometer correction $\Delta T = - 1^m \ 45^s.54$.

The altitudes may differ from other causes besides a change in the refraction; for instance, the second observation may be interrupted by passing clouds, so that the precisely corresponding altitude cannot be taken; but, rather than lose the whole observation, if we can observe an altitude differing but little from the first, we may use it as an equal altitude, and compute the correction for the difference by the formula (266).

143. *Effect of errors in the latitude, declination, and altitude upon the time found by equal altitudes.*—The time found by equal altitudes of a fixed star is wholly independent of errors in the latitude

and declination, since these quantities do not enter into the computation. In observations of the sun, an error in the latitude affects the term

$$a = A\,\Delta'\delta \tan\varphi$$

by differentiating which we find that an error $d\varphi$ produces in a the error $da = A\,\Delta'\delta \,.\, \sec^2\varphi \,.\, d\varphi$, or, putting $\sin d\varphi$ for $d\varphi$,

$$da = A\,\Delta'\delta \sec^2\varphi \sin d\varphi$$

In the same manner, we find that an error $d\delta$ in the declination produces in b the error

$$db = B\,\Delta'\delta \sec^2\delta \sin d\delta$$

In the example of Art. 140, suppose the latitude and declination were each in error 1′. We have

$\log A\,\Delta'\delta$	$n1.2446$	$\log B\,\Delta'\delta$	0.9793
$\log \sec^2\varphi$	0.2188	$\log \sec^2\delta$	0.0044
$\log \sin 1'$	6.4637	$\log \sin 1'$	6.4637
$\log da$	$n7.9271$	$\log db$	7.4474
$da =$	$-0^s.008$	$db =$	$+0^s.003$

If $d\varphi$ and $d\delta$ had opposite signs, the whole error in this case would be $0^s.008 + 0^s.003 = 0^s.011$. As the observer can always easily obtain his latitude within 1′ and the declination (even when the longitude is somewhat uncertain) within a few seconds, we may regard the method as practically free from the effects of any errors in these quantities. The accuracy of the result will therefore depend wholly upon the accuracy of the observations.

The accuracy of the observations depends in a measure upon the constancy of the instrument, but chiefly upon the skill of the observer. Each observer may determine the probable error of his observations by discussing them by the method of least squares. An example of such a discussion will be given in the following article.

The effect of an error in the altitude is given by (266). Since we have, A being the azimuth of the object,

$$\sin A = \frac{\cos\delta \sin t}{\cos h}$$

the formula may also be written

$$\Delta' T_0 = \frac{\Delta h}{30 \cos \varphi \sin A}$$

which will be least when the denominator is greatest, *i.e.* when $A = 90°$ or $270°$, or when the object is near the prime vertical. From this we deduce the practical precept to *take the observations when the object is nearly east or west.* This rule, however, must not be carried so far as to include observations at very low altitudes, where anomalies in the refraction may produce unknown differences in the altitudes. If the star's declination is very nearly equal to the latitude, it will be in the prime vertical only when quite near to the meridian, and then both observations may be obtained within a brief interval of time; and this circumstance is favorable to accuracy, inasmuch as the instrument will be less liable to changes in this short time.

144. *Probable error of observation.*—The error of observation is composed of two errors, one arising from imperfect setting of the index of the sextant, the other from imperfect noting of the time; but these are inseparable, and can only be discussed as a single error in the observed time. The individual observations are also affected by any irregularity of graduation of the sextant, but this error does not affect the mean of a *pair* of observations on opposite sides of the meridian; and therefore the error of observation proper will be shown by comparing the mean of the several pairs with the mean of these means. If, then, the mean of a pair of observed times be called a, the mean of all these means a_0, the probable error of a single pair, supposing all to be of the same weight, is*

$$r = q\sqrt{\frac{\Sigma(a - a_0)^2}{n - 1}}$$

in which $n =$ the number of pairs, and $q = 0.6745$ is the factor to reduce mean to probable errors. The probable error of the final mean a_0 is

$$r_0 = \frac{r}{\sqrt{n}}$$

* See Appendix, *Least Squares.*

EXAMPLE.—At the U. S. Naval Academy, June 18, 1849, the following series of equal altitudes of the sun was observed.

Chro. A.M.	Chro. P.M.	a	$a - a_0$	$(a - a_0)^2$
$0^h\ 43^m\ 53^s.$	$9^h\ 44^m\ 3^s.5$	$5^h\ 13^m\ 58^s.25$	$+\ 0^s.12$	0.0144
44 19.	43 38.	58.50	+ 0.37	.1369
44 45.	43 11.5	58.25	+ 0.12	.0144
45 11	42 46.3	58.65	+ 0.52	.2704
45 37.	42 19.7	58.35	+ 0.22	.0484
46 1.7	41 53.5	57.60	— 0.53	.2809
46 28.5	41 27.	57.75	— 0.38	1444
46 55.	41 0.5	57.75	— 0.38	.1444
47 19.7	40 36.5	58.10	— 0.03	.0009

$$a_0 = 5\ 13\ 58\ 13 \qquad \Sigma (a - a_0)^2 = 1.0551$$

$$n = 9 \qquad r = \pm\ q\sqrt{\frac{\Sigma (a - a_0)^2}{n - 1}} = 0^s245$$

$$n - 1 = 8 \qquad r_0 = \frac{r}{\sqrt{n}} = 0^s.082$$

A similar discussion of a number of sets of equal altitudes of the sun taken by the same observer gave $0^s.23$ as the probable error of a single pair for that observer, and consequently the probable error of the result of six observations on each side of the meridian would be only $0^s.23 \div \sqrt{6} = 0^s.094$. This, however, expresses only the *accidental error of observation*, and does not include the effect of changes in the state of the sextant between the morning and afternoon observations. Such changes are not unfrequently produced by the changes of temperature to which it is exposed in observations of the sun; it is important, therefore, to guard the instrument from the sun's rays as much as possible, and to expose it only during the few minutes required for each observation. The determination of the time by stars is mostly free from difficulties of this kind, but the observation is not otherwise so accurate as that of the sun, except in the hands of very skilful observers.

THIRD METHOD.—BY A SINGLE ALTITUDE, OR ZENITH DISTANCE.

145. Let the altitude of any celestial body be observed with the sextant or any altitude instrument, and the time noted by the clock. For greater precision, observe several altitudes in quick succession, noting the time of each, and take the mean of the altitudes as corresponding to the mean of the times. But

in taking the mean of several observations in this way, it must not be forgotten that we assume that the altitude varies in proportion to the time, which is theoretically true only in the exceptional case where the observer is on the equator and the star's declination is zero. It is, however, practically true for an interval of a few minutes when the star is not too near the meridian. The observations themselves will generally show the limit beyond which it will not be safe to apply this rule. When the observations have been extended beyond this limit, a correction for the unequal change in altitude (*i.e.* for second differences) can be applied, which will be treated of below.

With the altitude and azimuth instrument we generally obtain zenith distances directly. In all cases, however, we may suppose the observation to give the zenith distance. Having then corrected the observation for instrumental errors, for refraction, &c., Arts. 135, 136, let ζ be the resulting true or geocentric zenith distance. Let φ be the latitude of the place of observation, δ the star's declination, t the star's hour angle. The three sides of the spherical triangle formed by the zenith, the pole, and the star may be denoted by $a = 90° - \varphi$, $b = \zeta$, $c = 90° - \delta$, and the angle at the pole by $B = t$, and hence, Art. 22, we deduce

$$\sin \tfrac{1}{2} t = \sqrt{\left(\frac{\sin \frac{1}{2}[\zeta + (\varphi - \delta)] \sin \frac{1}{2}[\zeta - (\varphi - \delta)]}{\cos \varphi \cos \delta}\right)} \qquad (267)$$

which gives t by a very simple logarithmic computation. From t we deduce, by Art. 55, the local time, which compared with the observed clock time gives the clock correction required.

It is to be observed that the double sign belonging to the radical in (267) gives two values of $\sin \frac{1}{2} t$, the positive corresponding to a west and the negative to an east hour angle; since any given zenith distance may be observed on either side of the meridian. To distinguish the true solution, the observer must of course note on which side of the meridian he has observed.

If the object observed is the sun, the moon, or a planet, its declination is to be taken from the Ephemeris, for the time of the observation (referred to the meridian of the Ephemeris); but, as this time is itself to be found from the observation, we must at first assume an approximate value of it, with which an approximate declination is found. With this declination a first compu-

tation by the formula gives an approximate value of t, and hence a more accurate value of the time, and a new value of the declination, with which a second computation by the formula gives a still more accurate value of t. Thus it appears that the solution of our problem is really indirect, and theoretically involves an infinite series of successive approximations; in practice, however, the observer generally possesses a sufficiently precise value of his clock correction for the purpose of taking out the declination of the sun or planets. The moon is never employed for determining the local time except at sea, and when no other object is available.*

EXAMPLE.—At the U. S. Naval Academy, in Latitude $\varphi = 38° 58' 53''$ N., Longitude $5^h\ 5^m\ 57^s.5$ W., December 9, 1851, the following double altitudes of the sun west of the meridian were observed with a sextant and artificial horizon, the times being noted by a Greenwich mean time chronometer:

	Chronometer.	2 ⨀†	
	$7^h\ 35^m\ 14^s.5$	33° 30′	Barom. 30.28 inches.
	35 55 .	" 20	Att. Therm. 55° F.
	36 35 .5	" 10	Ext. Therm. 50° F.
	37 15 .5	" 0	Index correction of the
	37 55 .	32 50	sextant $= -1' 10''$
Means	7 36 35 .1	33 10	

The approximate correction of the chronometer was assumed to be $+ 9^m\ 40^s$. Find its true correction.

With the assumed chronometer correction we obtain the approximate Greenwich time $= 7^h\ 46^m\ 15^s$, with which we take from the Ephemeris

$$\delta = -22° 50' 27'' \quad \text{Sun's semidiameter } S = 16' 17''$$
$$\text{Eq. of time} = -7^m\ 25^s.80 \quad \text{" hor. parallax } \pi = 8''.7$$

We have then

* But the moon's altitude and the hour angle deduced from it may be used in finding the observer's longitude, as will be shown in the Chapter on Longitude.

† The symbol ⨀ is used for "observed altitude of the sun's lower limb," and 2 ⨀ for the double altitude from the artificial horizon. In a similar manner we use ♁, ♃, ☽.

Observed 2 ⊙	$= 33°\ 10'\ 0''$
Index corr.	$= -\ \ 1\ 10$
	$33\ \ 8\ 50$
App. altitude	$= 16\ 34\ 25$
z	$= 73\ 25\ 35$
(Table II.) r	$= +\ \ 3\ 15$
$\pi \sin z = p$	$= -\ \ \ \ 8$
S	$= -\ 16\ 17$
ζ	$= 73\ 12\ 25$

The computation by (267) is then as follows:

$\varphi =$	$38°\ 58'\ 53''$	log sec φ	0.109383
$\delta =$	$-\ 22\ 50\ 27$	log sec δ	0.035464
$\varphi - \delta =$	$61\ 49\ 20$	log sin ½ sum	9.965661
$\zeta =$	$73\ 12\ 25$	log sin ½ diff.	8.996455
½ sum =	$67\ 30\ 52.5$		19.106963
½ diff. =	$5\ 41\ 32.5$	log sin ½ t	9.553482

½ t	$= 20°\ 57'\ 25''.6$
Apparent time $= t$	$= 2^h\ 47^m\ 39^s.4$
Eq. of time	$= -\ \ 7\ \ 25.8$
Local mean time	$= 2\ 40\ 13.6$
Longitude	$= 5\ \ 5\ 57.5$
True Gr. Time $= T'$	$= 7\ 46\ 11.1$
T	$= 7\ 36\ 35.1$
ΔT	$= +\ 9\ 36.0$

agreeing so nearly with the assumed correction that a repetition of the computation is unnecessary.

146. If it is preferred to use the altitude instead of the zenith distance, put the true altitude $h = 90° - \zeta$, and the polar distance of the star $P = 90° - \delta$, then we have, in (267),

$$\sin \tfrac{1}{2}[\zeta - (\varphi - \delta)] = \sin \tfrac{1}{2}(90° - h - \varphi + 90° - P) = \cos \tfrac{1}{2}(h + \varphi + P)$$
$$\sin \tfrac{1}{2}[\zeta + \varphi - \delta] = \sin \tfrac{1}{2}(90° - h + \varphi - 90° + P) = \sin \tfrac{1}{2}(\varphi + P - h)$$

If then we put

$$s = \tfrac{1}{2}(h + \varphi + P)$$

the formula becomes

$$\sin \tfrac{1}{2} t = \sqrt{\left(\frac{\cos s \sin (s - h)}{\cos \varphi \sin P}\right)} \tag{268}$$

In this form we may always take $P =$ the distance from the elevated pole, and regard the latitude as always positive, and then no attention to the algebraic signs of the quantities in the second member is required. Thus, in the preceding example, we should proceed as follows:

$$
\begin{array}{rrrrll}
\text{App. alt.} = & 16^\circ & 34' & 25'' & & \\
r - p = & - & 3 & 7 & & \\
S = & & 16 & 17 & & \\
\hline
h = & 16 & 47 & 35 & & \\
\varphi = & 38 & 58 & 53 & \ldots\ldots \text{log sec} & 0.109383 \\
P = & 112 & 50 & 27 & \ldots\ldots \text{log cosec} & 0.035464 \\
\hline
2s = & 168 & 36 & 55 & & \\
s = & 84 & 18 & 27.5 & \ldots\ldots \text{log cos} & 8.996455 \\
s - h = & 67 & 30 & 52.5 & \ldots\ldots \text{log sin} & 9.965661 \\
\hline
 & & & & & 19.106963
\end{array}
$$

and the computation is finished as in the preceding article.

147. If we aim at the greatest degree of precision which the logarithmic tables can afford, we should find the angle $\frac{1}{2}t$ by its tangent, since the logarithms of the tangent always vary more rapidly than those of the other functions. For this purpose we deduce

$$
\left.\begin{aligned}
s &= \tfrac{1}{2}(\zeta + \varphi + \delta) \\
\tan \tfrac{1}{2} t &= \sqrt{\left(\frac{\sin(s-\varphi)\sin(s-\delta)}{\cos s \cos(s-\zeta)}\right)}
\end{aligned}\right\} \quad (269)
$$

or, if the altitude is used,

$$
\left.\begin{aligned}
s &= \tfrac{1}{2}(h + \varphi + P) \\
\tan \tfrac{1}{2} t &= \sqrt{\left(\frac{\cos s \sin(s-h)}{\sin(s-\varphi)\cos(s-P)}\right)}
\end{aligned}\right\} \quad (270)
$$

148. If a number of observations of the same star at the same place are to be individually computed, it will be most readily done by the fundamental equation

$$\cos t = \frac{\cos \zeta - \sin \varphi \sin \delta}{\cos \varphi \cos \delta}$$

for the logarithms of $\sin\varphi\sin\delta$ and $\cos\varphi\cos\delta$ will be constant, and for each observation we shall only have to take from the trigonometric table the log. of $\cos\zeta$; the logarithm of the numerator will then be found by the aid of ZECH's Addition or Subtraction Table, which is included in HÜLSSE's edition of VEGA's Tables. The addition or the subtraction table will be used according as $\sin\varphi\sin\delta$ is positive or negative.

149. *Effect of errors in the data upon the time computed from an altitude.*—We have from the differential equation (51), Art. 35, multiplying dt by 15 to reduce it to seconds of arc,

$$\sin q\cos\delta\,(15\,dt) = d\zeta - \cos A\,d\varphi + \cos q\,d\delta$$

where $d\zeta$, $d\varphi$, $d\delta$, may denote small errors of ζ, φ, δ, and dt the corresponding error of t; A is the star's azimuth, q the parallactic angle, or angle at the star.

If the zenith distance alone is erroneous, we have, by putting $d\varphi = 0$, and $d\delta = 0$,

$$15\,dt = \frac{d\zeta}{\sin q\cos\delta} = \frac{d\zeta}{\cos\varphi\sin A}$$

from which it follows that a given error in the zenith distance will have the least effect upon the computed time when the azimuth is 90° or 270°; that is, when the star is on the prime vertical; for we then have $\sin A = \pm 1$, and the denominator of this expression obtains its maximum numerical value. Also, since $\cos\varphi$ is a maximum for $\varphi = 0$, it follows that observations of zenith distances for determining the time give the most accurate results when the place is on the equator. On the other hand, the least favorable position of the star is when it is on the meridian, and the least favorable position of the observer is at the pole.

By putting $d\zeta = 0$, $d\delta = 0$, $\sin q\cos\delta = \cos\varphi\sin A$ we have

$$15\,dt = -\frac{d\varphi}{\cos\varphi\tan A}$$

by which we see that an error in the latitude also produces the least effect when the star is on the prime vertical, or when the observer is on the equator. Indeed, when the star is exactly in

the prime vertical, a small error in φ has *no* appreciable effect: since, then, $\tan A = \infty$, and hence when the latitude is uncertain, we may still obtain good results by observing only stars near the prime vertical.

By putting $d\zeta = 0$, $d\varphi = 0$, we have

$$15\,dt = \frac{d\delta}{\cos\delta\,\tan q}$$

which shows that the error in the declination of a given star produces the least effect when the star is on the prime vertical;* and of different stars the most eligible is that which is nearest to the equator.

As very great zenith distances (greater than 80°) are, if possible, to be avoided on account of the uncertainty in the refraction, the observer will often be obliged, especially in high latitudes, to take his observations at some distance from the prime vertical, in which case small errors of zenith distance, latitude, or declination may have an important effect upon the computed *clock correction.* Nevertheless, *constant* errors in these quantities will have no sensible effect upon the *rate* of the clock deduced from zenith distances of the same star on different days, if the star is observed at the same or nearly the same azimuth, on the same side of the meridian; for all the clock corrections will be increased or diminished by the same quantities, so that their differences, and consequently the rate, will be the same as if these errors did not exist. The errors of eccentricity and graduation of the instrument are among the constant errors which may thus be eliminated.

But if the same star is observed both east and west of the meridian, and at the same distance from it, $\sin A$ or $\tan A$, and $\tan q$, will be positive at one observation and negative at the other, and, having the same numerical value, constant errors $d\varphi$, $d\delta$, and $d\zeta$ will give the same numerical value of dt with opposite signs. Hence, while one of the deduced clock corrections will be too great, the other will be too small, and their mean will be the true correction at the time of the star's transit

* From the equation $\sin q = \frac{\cos\phi}{\cos\delta}\sin A$, it follows that $\sin q$ is a maximum (for constant values of ϕ and δ) when $\sin A = 1$, and $\tan q$ is a maximum in the same case.

over the meridian. Hence, it follows again, as in Art. 143, that small errors in the latitude and declination have no sensible effect upon the time computed from equal altitudes.

150. *To find the change of zenith distance of a star in a given interval of time, having regard to second differences.*

The formula

$$d\zeta = \cos\varphi \sin A\, dt$$

is strictly true only when $d\zeta$ and dt are infinitesimals. But the complete expression of the finite difference $\Delta\zeta$ in terms of the finite difference Δt involves the square and higher powers of Δt. Let ζ be expressed as a function of t of the form

$$\zeta = ft$$

then, to find any zenith distance $\zeta + \Delta\zeta$ corresponding to the hour angle $t + \Delta t$, we have, by TAYLOR'S Theorem,

$$\zeta + \Delta\zeta = f(t + \Delta t) = ft + \frac{dft}{dt}\cdot\Delta t + \frac{d^2ft}{dt^2}\cdot\frac{\Delta t^2}{2} + \ldots$$

or, taking only second differences,

$$\Delta\zeta = \frac{d\zeta}{dt}\cdot\Delta t + \frac{d^2\zeta}{dt^2}\cdot\frac{\Delta t^2}{2}$$

We have already found

$$\frac{d\zeta}{dt} = \cos\varphi \sin A$$

which gives, since A varies with t, but φ is constant,

$$\frac{d^2\zeta}{dt^2} = \cos\varphi \cos A \cdot \frac{dA}{dt}$$

But from the second of equations (51) we have, since $d\delta$ and $d\varphi$ are here zero,

$$\frac{dA}{dt} = \frac{\cos q \cos\delta}{\sin\zeta} = \frac{\cos q \sin A}{\sin t}$$

whence

$$\frac{d^2\zeta}{dt^2} = \frac{\cos\varphi \sin A \cos A \cos q}{\sin t}$$

and the expression for $\Delta\zeta$ becomes

$$\Delta\zeta = \cos\varphi \sin A \,.\, \Delta t + \frac{\cos\varphi \sin A \cos A \cos q}{\sin t} \,.\, \frac{\Delta t^2}{2}$$

Since $\Delta\zeta$ and Δt are here supposed to be expressed in parts of the radius, if we wish to express them in seconds of arc and of time respectively, we must substitute for them $\Delta\zeta \sin 1''$ and $15\Delta t \sin 1''$, and the formula becomes

$$\Delta\zeta = \cos\varphi \sin A\,(15\,\Delta t) + \frac{\cos\varphi \sin A \cos A \cos q}{\sin t} \,.\, \frac{(15\,\Delta t)^2 \sin 1''}{2} \qquad (271)$$

But in so small a term as the last we may put

$$\frac{(15\,\Delta t)^2 \sin 1''}{2} = \frac{2 \sin^2 \frac{1}{2}\Delta t}{\sin 1''} = m$$

the value of which is given in our Table V., and its logarithm in Table VI.; so that if we put also

$$a = \cos\varphi \sin A, \qquad k = \frac{\cos A \cos q}{\sin t}$$

we shall have

$$\Delta\zeta = 15\,a\Delta t + akm \qquad (272)$$

151. *A number of zenith distances being observed at given clock times, to correct the mean of the zenith distances or of the clock times for second differences.*—The first term of the above value of $\Delta\zeta$ varies in proportion to Δt, but the second term varies in proportion to Δt^2; and hence, when the interval is sufficiently great to render this second term sensible, equal intervals of time correspond to unequal differences of zenith distance, and *vice versa*: in other words, we shall have second differences either of the zenith distance or of the time. Two methods of correction present themselves.

1st. *Reduction of the mean of the zenith distances to the mean of the times.*—Let T_1, T_2, T_3, &c. be the observed clock times; ζ_1, ζ_2, ζ_3, &c. the corresponding observed zenith distances; T the mean of the times; ζ_0 the mean of the zenith distances; ζ the zenith distance corresponding to T. The change $\zeta_1 - \zeta$ corresponds to the interval $T_1 - T$, $\zeta_2 - \zeta$ to $T_2 - T$, &c.; so that if we put

$$T_1 - T = \tau_1\,, \quad T_2 - T = \tau_2, \text{ \&c.}$$

we have, by (272),

$$\begin{aligned}\zeta_1 - \zeta &= 15\,a\,\tau_1 + akm_1 \\ \zeta_2 - \zeta &= 15\,a\,\tau_2 + akm_2 \\ \zeta_3 - \zeta &= 15\,a\,\tau_3 + akm_3 \\ \&c. &\qquad \&c.\end{aligned}$$

in which $m_1 = \frac{2\sin^2\frac{1}{2}\tau_1}{\sin 1''}$, $m_2 = \frac{2\sin^2\frac{1}{2}\tau_2}{\sin 1''}$, &c., are found by Tab. V. with the arguments τ_1, τ_2, &c. The mean of these equations, observing that

$$\tau_1 + \tau_2 + \tau_3 + \&c. = 0$$

gives

$$\zeta = \zeta_0 - ak \cdot \frac{m_1 + m_2 + m_3 + \&c}{n}$$

in which $n =$ the number of observations. Or, denoting the mean of the values of m from the table by m_0, that is, putting

$$m_0 = \frac{m_1 + m_2 + m_3 + \&c.}{n}$$

we have

$$\zeta = \zeta_0 - akm_0 \tag{273}$$

2d. *Reduction of the mean of the times to the mean of the zenith distances.*—Let T_0 be the clock time corresponding to the mean of the zenith distances, then $\zeta_0 - \zeta$ is the change of zenith distance in the interval $T_0 - T$, and, since this interval is very small, we shall have sensibly

$$15\,a\,(T_0 - T) = \zeta_0 - \zeta = akm_0$$

whence

$$T_0 = T + \tfrac{1}{15}\,km_0 \tag{274}$$

We have, then, only to compute the true time T_0' from the mean of the zenith distances in the usual manner, and the clock correction will then be found, as in other cases, by the formula

$$\Delta T = T_0' - T_0$$

To compute k, we must either first find q and A, or, which is preferable, express it by the known quantities. We have

$$\begin{aligned}\cos q \cos A &= \cos t - \sin q \sin A \cos \zeta \\ &= \cos t - \frac{\sin^2 t}{\sin^2 \zeta} \cos\varphi \cos\delta \cos\zeta\end{aligned}$$

whence

$$T_0 = T + \tfrac{1}{15}\, m_0 \cot t - \tfrac{1}{15}\, m_0\, \frac{\sin t \cos \varphi \cos \delta}{\sin \zeta \tan \zeta} \tag{275}$$

in which we employ for ζ and t the mean zenith distance and the computed hour angle.

This mode of correction is evidently more simple and direct than the first.

EXAMPLE.—In St. Louis, Lat. 38° 38′ 15″ N., Long. $6^h\ 1^m\ 7^s$ W., the following double altitudes of the sun were observed with a Pistor and Martin prismatic sextant, the index correction of which was + 20″. The assumed correction of the chronometer to mean local time was + $2^m\ 12^s$. Barom. 30.25 inches, Att. Therm. 80°, Ext. Therm. 81°.

St. Louis, June 24, 1861.

2☉	Chronom.	τ	m
125° 15′ 10″.	$22^h\ 14^m\ 30^s.5$	$6^m\ 42^s$	88″.14
125 49 10	16 7 .5	5 5	50 .73
126 23 0	17 46 .0	3 26	23 .14
126 41 40	18 39 .5	2 33	12 .76
127 32 30	21 6 .5	0 6	0 .02
127 57 45	22 22 .	1 10	2 .67
128 22 0	23 33 .5	2 21	10 .84
128 51 50	25 1 .2	3 49	28 .60
129 8 35	25 51 .3	4 39	42 .45
129 33 0	27 3 .5	5 51	67 .19

Mean	127 33 28	$T =$	22 21 12 .15	$m_0 =$	32 .65
	+ 20	Correction for second diff. = −	1 .67	log m_0	1.5139
	127 33 48			log. $\frac{1}{15}$	8.8239
Obs'd ☉	63 46 54	$T_0 =$	22 21 10 .48	log cot t	n0.3367
(*) $r =$	− 27 .2	$T_0' =$	22 23 22 .94	$-\ 4^s.73$	n0.6745
$p =$	+ 3 .7	$\Delta T =$ +	2 12 .46		
$S =$	+ 15 46 .3			log $\frac{1}{15}\, m_0$	0.3378
$h_0 =$	64 2 16 .8			log sin t	n9.6215
$\zeta_0 =$	25 57 43 .2			log cos ϕ	9.8927
$\phi =$	38 38 15 .			log cos δ	9.9627
$\delta =$	23 23 49 .3			log cosec ζ_0	0.3588
$t =$	− 24° 43′ 48″.4			log cot ζ_0	0.3125
=	− $1^h\ 38^m\ 55^s.23$			$-\ 3^s.06$	n0.4860
App. time =	22 21 4 .77			− 1 .67	
Eq. of time =	+ 2 18 .17				
$T_0' =$	22 23 22 .94				

* The refraction should here be the mean of the refractions computed for the

The correction for second differences is particularly useful in reducing series of altitudes observed with the repeating circle;* for with this instrument we do not obtain the several altitudes, but only their mean. (See Vol. II.) When the several altitudes are known, we can avoid the correction by computing each observation, or by dividing the whole series into groups of such extent that within the limits of each the second differences will be insensible, and computing the time from the mean of each group.

FOURTH METHOD.—BY THE DISAPPEARANCE OF A STAR BEHIND A TERRESTRIAL OBJECT.

152. The *rate* of the clock may be found by this method with considerable accuracy without the aid of astronomical instruments. The terrestrial object should have a sharply defined vertical edge, behind which the disappearance is to be observed, and the position of the eye of the observer should be precisely the same at all the observations. If the star's right ascension and declination are constant, the difference between the sidereal clock times T_1 and T_2 of two disappearances is the rate δT in the interval, or

$$\delta T = T_1 - T_2$$

but if the right ascension α has increased in the interval by $\Delta\alpha$, then the rate is

$$\delta T = T_1 - T_2 + \Delta\alpha$$

To find the correction for a small change of declination $= \Delta\delta$,

---

several altitudes or zenith distances, but for small zenith distances the difference will be insensible. At great zenith distances we should compute the several refractions, but under 80° we may take the refraction r for the mean apparent zenith distance z_0, and correct it as follows: Take the difference between z_0 and each z, and the mean m_0 of the values of

$$m = \frac{2 \sin^2 \frac{1}{2} (z - z_0)}{\sin 1''}$$

from Table V. (converting the argument $z - z_0$ into time); then the mean of the refractions will be found by the formula

$$r_0 = r + 2m_0 \sin r \sec^2 z_0$$

The difference $z - z_0$ should not much exceed 1°.

* This method was frequently practised in the geodetic survey of France. See *Nouvelle Description Géométrique de la France* (PUISSANT), Vol. I. p. 96.

we have, by the second equation of (51), since the azimuth is here constant as well as the latitude, so that $dA = 0$ and $d\varphi = 0$,

$$\Delta t = -\frac{\Delta\delta \tan q}{15 \cos \delta}$$

and hence the rate in the interval will be

$$\delta T = T_1 - T_2 + \Delta\alpha - \frac{\Delta\delta \tan q}{15 \cos \delta} \qquad (276)$$

The angle q will be found with sufficient precision from an approximate value of t by (19) or (20).

If we know the absolute azimuth of the object, we can find the hour angle by Art. 12, and hence also the clock correction.

TIME OF RISING AND SETTING OF THE STARS.

153. *To find the time of true rising or setting,*—that is, the instant when the star is in the true horizon,—we have only to compute the hour angle by the formula (28)

$$\cos t = -\tan \varphi \tan \delta$$

and then deduce the local time by Art. 55.

154. *To find the time of apparent rising or setting,*—that is, the instant when the star appears on the horizon of the observer,—we must allow for the horizontal refraction. Denoting this refraction by r_0, the true zenith distance of the star at the time of apparent rising or setting is $90° + r_0$, and, employing this value for ζ, we compute the hour angle by (267).

Since the altitude $h = 90° - \zeta$, we have in this case $h = -r_0$, with which we can compute the hour angle by the formula (268).

In common life, by the time of sunrise or sunset is meant the instant when the sun's upper limb appears in the horizon. The true zenith distance of the centre is, then, $\zeta = 90° + r_0 - \pi + S$ (where π = the horizontal parallax and S = the semidiameter), with which we compute the hour angle as before. The same form is to be used for the moon.

TIME OF THE BEGINNING AND ENDING OF TWILIGHT.

155. Twilight begins in the morning or ends in the evening when the sun is 18° below the horizon, and consequently the

zenith distance is then $\zeta = 90° + 18°$, or $h = -18°$, with which we can find the hour angle by (267) or (268).

Note.—Methods of finding at once both the time and the latitude from observed altitudes will be treated of under Latitude, in the next chapter.

FINDING THE TIME AT SEA.

First Method.—By a Single Altitude.

156. This is the most common method among navigators, as altitudes from the sea horizon are observed with the greatest facility with the sextant. Denoting the observed altitude corrected for the index error of the sextant by H, the dip of the horizon by D, we have the apparent altitude $h' = H - D$; then, taking the refraction r for the argument h', the true altitude of a star is $h = h' - r$. A planet is observed by bringing the estimated centre of its reflected image upon the horizon, so that no correction for the semidiameter is employed; the parallax is computed by the simple formula (π being the horizontal parallax)

$$p = \pi \cos h'$$

and hence for a planet

$$h = h' - r + \pi \cos h'$$

The moon and sun are observed by bringing the reflected image of either the upper or the lower limb to touch the horizon. As very great precision is neither possible nor necessary in these observations, the compression of the earth is neglected, and the parallax is computed by the formula

$$p = \pi \cos (h' - r)$$

and then, S being the semidiameter,

$$h = h' - r + \pi \cos (h' - r) \pm S$$

In nautical works, the whole correction of the moon's altitude for parallax and refraction $= \pi \cos (h' - r) - r$ is given in a table with the arguments apparent altitude (h') and horizontal parallax (π). In the construction of this table the mean refraction is used, but the corrections for the barometer and thermometer are given in a very simple table, although they are not usually of sufficient importance to be regarded in correcting altitudes of the moon which are taken to determine the local time.

The hour angle is usually found by (268).

It is important at sea, where the latitude is always in some degree uncertain, to find the time by altitudes near the prime vertical, where the error of latitude has little or no effect (Art. 149).

157. The instant when the sun's limb touches the sea horizon may be observed, instead of measuring an altitude with the sextant. In this case the refraction should be taken for the zenith distance $90° + D$, but, on account of the uncertainty in the horizontal refraction, great precision is not to be expected, and the mean horizontal refraction r_0 may be used. We then have $\zeta = 90° + D + r_0 - \pi \pm S$, with which we proceed by (267). In so rude a method, π may be neglected, and we may take $16'$ as the mean value of S, $36'$ as the value of r_0, $4'$ as the average value of D from the deck of most vessels; then for the lower limb we have $\zeta = 90° 56'$, and for the upper limb $\zeta = 90° 24'$. If both limbs have been observed and the mean of the times is taken, the corresponding hour angle will be found by taking $\zeta = 90° 40'$.

Second Method.—By Equal Altitudes.

158. The method of equal altitudes as explained in Arts. 139 and 140 may be applied at sea by introducing a correction for the ship's change of place between the two observations. If, however, the ship sails due east or west between the observations, and thus without changing her latitude, no correction for her change of place is necessary, for the middle time will evidently correspond to the instant of transit of the star over the middle meridian between the two meridians on which the equal altitudes are observed. But, if the ship changes her latitude, let

$\Delta\varphi$ = the increase of latitude at the second observation;

then (Art. 149) the effect upon the second hour angle is

$$\Delta t = -\frac{\Delta\varphi}{15 \cos \varphi \tan A}$$

which is the correction subtractive from the second observed time to reduce it to that which would have been observed if the

ship had not changed her latitude or had run upon a parallel. Hence $\frac{1}{2}\Delta t$ is to be subtracted from the mean of the chronometer times to obtain the chronometer time of the star's transit over the middle meridian.

In this formula we must observe the sign of tan A. It will be more convenient in practice to disregard the signs, and to apply the numerical value of the correction to the middle time according to the following simple rule:—add the correction when the ship has *receded* from the sun; subtract it when the ship has *approached* the sun.

The azimuth may be found by the formula

$$\sin A = \frac{\sin t \cos \delta}{\cos h}$$

in which for t we take one-half the elapsed time.

The sun being the only object which is employed in this way, we should also apply the equation of equal altitudes, Art. 140; but, as the greatest change of the sun's declination in one hour is about 1′, and the change of the ship's latitude is generally much greater, the equation is commonly neglected as relatively unimportant in a method which at sea is necessarily but approximate. But, if required, the equation may be computed and applied precisely as if the ship had been at rest.

Example.—At sea, March 20, 1856, the latitude at noon being 39° N., the same altitude was observed A.M. and P.M. as follows, by a chronometer regulated to mean Greenwich time:

Obsd. $\odot$	30° 0′		A.M. Chro. time	$= 11^h\ 39^m\ 33$
Index corr.	− 2		P.M. " "	$= 6\ 20\ 17$
Dip	− 4		Elapsed time $= 2t =$	$6\ 40\ 44$
Refraction	− 2		Middle time	$= 2\ 59\ 55$
Semidiam.	+ 16		Chron. correction	$= -2\ 12$
	$h = 30\ \ 8$		Green. time of noon	$= 2\ 57\ 43$

The ship changed her latitude between the two observations by $\Delta\varphi = -20' = -1200''$. For the Greenwich date March 20, $2^h.58^m$, the Ephemeris gives $\delta = +0°\ 4'$, and we have $t = 3^h\ 20^m\ 22^s = 50°\ 5'\ 30''$, $\varphi = 39°\ 0'$. Hence

$\log \sin t$	9.8848	$\log \frac{1}{30}$	8.5229
$\log \cos \delta$	0.0000	$\log \Delta\varphi$	3.0792
$\log \sec h$	0.0631	$\log \sec \varphi$	0.1095
$\log \sin A$	9.9479	$\log \cot A$	9.7165
		$\log 26^s.8$	1.4281

The ship has approached the sun, and hence $26^s.8$ must be subtracted from the middle time.

If we wish to apply the equation of equal altitudes, we have further from the Ephemeris $\Delta'\delta = +\ 59''$, and hence, by Art. 140,

	$\log A$	$n9.4628$		$\log B$	9.2698
	$\log \Delta'\delta$	1.7709		$\log \Delta'\delta$	1.7709
	$\log \tan \varphi$	9.9084		$\log \tan \delta$	7.0658
$a = -\ 13^s.9$	$\log a$	$n1.1421$	$b = +\ 0^s.0$	$\log b$	8.1065

Hence we have

Chro. middle time	$=$	2^h	59^m	$55^s.$
Corr. for change of lat.	$=$		$-$	26.8
Equation of eq. alts.	$=$		$-$	13.9
Chro. time app. noon	$=$	2	59	14.3

At sea, instead of using the observation to find the chronometer correction, we use it to determine the ship's longitude (as will be fully shown hereafter); and therefore, to carry the operation out to the end, we shall have

Chro. time app. noon	$=$	2^h	59^m	14^s
Corr. of chronom.	$= -$		2	12
Green. mean time noon	$=$	2	57	2
Equation of time	$= -$		7	48
Greenwich app. time at the local noon	$=$	2	49	14

which is the longitude of the middle meridian, or the longitude of the ship at noon.

159. In low latitudes (as within the tropics) observations for the time may be taken when the sun is very near the meridian, for the condition that the sun should be near the prime vertical may then be satisfied within a few minutes of noon; and in case the ship's latitude is *exactly* equal to the declination, it will be satisfied *only* when the sun is on the meridian in the zenith. In such cases the two equal altitudes may be observed within a few minutes of each other, and all corrections, whether for change of latitude or change of declination, may be disregarded.

CHAPTER VI.

FINDING THE LATITUDE BY ASTRONOMICAL OBSERVATIONS.

160. By the definition, Art. 7, the latitude of a place on the surface of the earth is the *declination of the zenith.* It was also shown in Art. 8 to be equal to the *altitude of the north pole* above the horizon of the place. In adopting the latter definition, it is to be remembered that a depression below the horizon is a negative altitude, and that south latitude is negative. The south latitude of a place, considered numerically, or without regard to its algebraic sign, is equal to the elevation of the south pole.

It is to be remembered, also, that the latitude thus defined is not an angle at the centre of the earth measured by an arc of the meridian, as it would be if the earth were a sphere; but it is the angle which the vertical line at the place makes with the plane of the equator, Art. 81.

We have seen, Art. 86, that there are abnormal deviations of the plumb line, which make it necessary to distinguish between the *geodetic* and the *astronomical* latitude. We shall here treat exclusively of the methods of determining the astronomical latitude; for this depends only upon the actual position of the plumb line, and is merely the declination of that point of the heavens towards which the plumb line is directed.

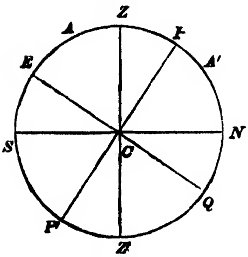
FIRST METHOD.—BY MERIDIAN ALTITUDES OR ZENITH DISTANCES.

161. Let the altitude or zenith distance of a star of known declination be observed at the instant when it is on the meridian. Deduce the true geocentric zenith distance ζ, and let δ be the geocentric declination, φ the astronomical latitude.

Let the celestial sphere be projected on the plane of the meridian, and let ZNZ', Fig. 24, be the celestial meridian; C the centre of the sphere coincident with that of the earth; PCP' the axis of the sphere; P the north pole; and ECQ the projection

of the plane of the equinoctial. Let CZ be parallel to the vertical line of the observer; then the point Z of the celestial sphere, being the vanishing point of all lines parallel to CZ, is the astronomical zenith of the observer, and ZE = the astronomical latitude $= \varphi$. If, then, A is the position of the star on the meridian, north of the equator but south of the zenith, we have $ZA = \zeta$, $AE = \delta$, and hence

Fig. 24.



$$\varphi = \delta + \zeta \tag{277}$$

This equation may be treated as entirely general by attending to the signs of δ and ζ. Since in deducing it we supposed the star to be north of the equator, it holds for the case where it is south by giving the declination in that case the negative sign, according to the established practice; and, since we supposed the star to be south of the zenith, the equation will hold for the case where it is north of the zenith by giving ζ in that case the negative sign. If the star is so far north of the zenith as to be below the pole, or at its lower culmination, the equation will still hold, provided we still understand by δ the star's distance north of the equator, measured from E *through the zenith and elevated pole*, or the arc EA'. This arc is the supplement of the declination; and we may here remark that, in general, any formula deduced for the case of a star above the pole will apply to the case where it is below the pole by employing the supplement of the declination instead of the declination itself; that is, by reckoning the declination *over the pole.*

The case of a star below the pole is, however, usually considered under the following simple form. Put

$P = PA'$ = the star's polar distance,
$h = NA'$ = " true altitude,

then

$$\varphi = P + h \tag{278}$$

in which for south latitude P must be the star's south polar distance, and the sum of P and h is only the numerical value of φ.

The declination is to be found for the instant of the meridian trans t by Art. 60 or 62.

In the observatory, instruments are employed which give

directly the zenith distance, or its supplement, the nadir distance. With a meridian circle *perfectly adjusted in the meridian*, the instant of transit would be known without reference to the clock, and the observation would be made at the instant the star passed the middle thread of the reticule; but when the instrument is not exactly in the meridian, or when the observation is not made on the middle thread, the observed zenith distance must be reduced to the meridian, for which see Vol. II., Meridian Circle.

With the sextant or other portable instruments the meridian altitude of a fixed star may be distinguished as the greatest altitude, and no reference to the time is necessary. But, as the sun, moon, and planets constantly change their declination, their greatest altitudes may be reached either before or after the meridian passage;* and in order to observe a strictly meridian altitude the clock time of transit must be previously computed and the altitude observed at that time.

EXAMPLE 1.—On March 1, 1856, in Long. $10^h\ 5^m\ 32^s$ E., suppose the apparent meridian altitude of the sun's lower limb, north of the zenith, is 63° 49′ 50″, Barom. 30. in., Ext. Therm. 50°; what is the latitude?

$$\begin{array}{rrrrr}
\text{App. zen. dist.}\ \underline{\odot} = & 26^\circ & 10' & 10''. & \\
r = & + & & 28 & .7 \\
p = & - & & 3 & .8 \\
S = & + & 16 & 10 & .3 \\
\hline
\zeta = - & 26 & 26 & 45 & .2 \\
\delta = - & 7 & 33 & 5 & .8 \\
\hline
\varphi = - & 33 & 59 & 51 & .0
\end{array}$$

EXAMPLE 2.—July 20, 1856, suppose that at a certain place the true zenith distances of *α Aquilæ* south of the zenith, and *α Cephei* north of the zenith, have been obtained as follows:

$$\begin{array}{ll}
\quad\ \alpha\ \textit{Aquilæ} & \quad\ \alpha\ \textit{Cephei} \\
\zeta = +\ 26^\circ\ 34'\ 27''.5 & \zeta = -\ 26^\circ\ 54'\ 28''.3 \\
\delta = +\ \ 8\ \ 29\ \ 22\ .7 & \delta = +\ 61\ \ 58\ \ 21\ .1 \\
\hline
\varphi = +\ 35\ \ \ 3\ \ 50\ .2 & \varphi = +\ 35\ \ \ 3\ \ 52\ .8
\end{array}$$

The mean latitude obtained by the two stars is, therefore, $\varphi = +\ 35^\circ\ 3'\ 51''.5$. In this example, the stars being at nearly

* See Art. 172 for the method of finding the time of the sun's greatest altitude, which may also be used for the moon or a planet.

the same zenith distance, but on opposite sides of the zenith, any constant though unknown error of the instrument, peculiar to that zenith distance, is eliminated in taking the mean. Thus, if the zenith distance in both cases had been $10''$ greater, we should have found from α *Aquilæ* $\varphi = 35° \ 4' \ 0''.2$, but from α *Cephei* $\varphi = 35° \ 3' \ 42''.8$, but the mean would still be $\varphi = 35° \ 3' \ 51''.5$.

It is evident, also, that errors in the refraction, whether due to the tables or to constant errors of the barometer and thermometer, or to any peculiar state of the air common to the two observations, are nearly or quite eliminated by thus combining a pair of stars the mean of whose declinations is nearly equal to the declination of the zenith. The advantages of such a combination do not end here. If we select the two stars so that the difference of their zenith distances is so small that it may be measured with a micrometer attached to a telescope which is so mounted that it may be successively directed upon the two stars without disturbing the angle which it makes with the vertical line, we can dispense altogether with a graduated circle, or, at least, the result obtained will be altogether independent of its indications. For, let ζ and ζ' be the zenith distances, δ and δ' the declinations of the two stars, the second of which is north of the zenith; then, if ζ' denotes only the numerical value of the zenith distance, we have

$$\varphi = \delta + \zeta$$
$$\varphi = \delta' - \zeta'$$

the mean of which is

$$\varphi = \tfrac{1}{2}(\delta + \delta') + \tfrac{1}{2}(\zeta - \zeta') \tag{279}$$

so that the result depends only upon the given declinations and the observed *difference* of zenith distance which is measured with the micrometer. Such is the simple principle of the method first introduced by Captain TALCOTT, and now extensively used in this country. To give it full effect, the instrument formerly known as the *Zenith Telescope* in England has received several important modifications from our Coast Survey. It will be fully treated of, in its present improved form, in Vol. II., where also will be found a discussion of TALCOTT's method in all its details.

162. *Meridian altitudes of a circumpolar star observed both above and below the pole.*—Every star whose distance from the elevated

pole is less than the latitude may be observed at both its upper and lower culminations. If we put

h = the true altitude at the upper culmination.
h_1 = " " " lower "
p = the star's polar distance at the upper culmination,
p_1 = " " " " lower "

we have, evidently,

$$\varphi = h - p,$$
$$\varphi = h_1 + p_1$$

the mean of which is

$$\varphi = \tfrac{1}{2}(h + h_1) + \tfrac{1}{2}(p_1 - p) \tag{280}$$

whence it appears that by this method the absolute values of p and p_1 are not required, but only their difference $p_1 - p$. The change of a star's declination by precession and nutation is so small in 12^h as usually to be neglected, but for extreme precision ought to be allowed for. This method, then, is free from any error in the declination of the star, and is, therefore, employed in all fixed observatories.

Example.—With the meridian circle of the Naval Academy the upper and lower transits of *Polaris* were observed in 1853 Sept. 15 and 16, and the altitudes deduced were as below:

Upper Transit.

Sept. 15, App. alt.			40° 28′ 25″.42
Barom.	30.005	Ref.	1 6 .34
Att. Therm.	65°.2		
Ext. "	63 .8		
		h =	40 27 19 .08
		p =	1 28 26 .04
		ϕ =	38 58 53 .04

Lower Transit.

Sept. 16,			37° 31′ 39″.76
Barom.	30.146	Ref.	1 12 .45
Att. Therm.	75°		
Ext. "	74 .6		
		h_1 =	37 30 27 .31
		p_1 =	1 28 25 .86
		ϕ =	38 58 53 .17
			" " 53 .04
		Mean ϕ =	38 58 53 .11

In order to compare the results, each observation is carried out separately. By (280) we should have

$$\tfrac{1}{2}(h + h_1) = 38° \; 58' \; 53''.20$$
$$\tfrac{1}{2}(p_1 - p) = \quad - \; 0 \; .09$$
$$\varphi = 38 \;\; 58 \;\; 53 \;.11$$

This method is still subject to the whole error in the refraction,

which, however, in the present state of the tables, will usually be very small.

If the latitude is greater than 45°, and the star's declination less than 45°, the upper transit occurs on the opposite side of the zenith from the pole. In that case h must still represent the distance of the star from the point of the horizon below the pole, and will exceed 90°. Thus, among the Greenwich observations we find

$$\begin{array}{lrl} \text{1837 June 14, Capella} & h_1 = & 7^\circ\ 18'\ 7''.94 \\ & h = & 95\ \ 39\ \ \ 7\ .91 \\ \hline & \varphi = & 51\ \ 28\ \ 37\ .93 \end{array}$$

163. *Meridian zenith distances of the sun observed near the summer and winter solstices.*—When the place of observation is near the equator, the lower culminations of stars can no longer be observed, and, consequently, the method of the preceding article cannot be used. The latitude found from stars observed at their upper culminations only is dependent upon the tabular declination, and is, therefore, subject to the error of this declination. If, therefore, an observatory is established on or near the equator, and its latitude is to be fixed independently of observations made at other places, the meridian zenith distances of stars cannot be employed. The only independent method is then by meridian observations of the sun near the solstices.

Let us at first suppose that the observations can be obtained exactly at the solstice, and the obliquity (ε) of the ecliptic is constant. The declination of the sun at the summer solstice is $= +\,\varepsilon$, and at the winter solstice it is $= -\,\varepsilon$; hence, from the meridian zenith distances ζ and ζ' observed at these times, we should have

$$\varphi = \zeta + \varepsilon$$
$$\varphi = \zeta' - \varepsilon$$

the mean of which is

$$\varphi = \tfrac{1}{2}(\zeta + \zeta')$$

a result dependent only upon the data furnished by the observations.

Now, the sun will not, in general, pass the meridian of the observer at the instant of the solstice, or when the declination is at its maximum value ε; nor is the obliquity of the ecliptic constant. But the *changes* of the declination near the solstices are very small, and hence are very accurately obtained from the

solar tables (or from the Ephemeris which is based on these tables), notwithstanding small errors in the absolute value of the obliquity. The small change in the obliquity between two solstices is also very accurately known. If then $\Delta\varepsilon$ is the unknown correction of the tabular obliquity, and the tabular values at the two solstices are ε and ε', the true values are $\varepsilon + \Delta\varepsilon$ and $\varepsilon' + \Delta\varepsilon$; and if the tabular declinations at two observations near the solstices are $\varepsilon - x$ and $-(\varepsilon' - x')$, the true declinations will be $\delta = \varepsilon + \Delta\varepsilon - x$ and $\delta' = -(\varepsilon' + \Delta\varepsilon - x')$, and by the formula $\varphi = \zeta + \delta$ we shall have for the two observations

$$\begin{aligned} \varphi &= \zeta + \varepsilon + \Delta\varepsilon - x \\ \varphi &= \zeta' - \varepsilon' - \Delta\varepsilon + x' \end{aligned}$$

the mean of which is

$$\varphi = \tfrac{1}{2}(\zeta + \zeta') + \tfrac{1}{2}(\varepsilon - \varepsilon') - \tfrac{1}{2}(x - x')$$

a result which depends upon the small changes $\varepsilon - \varepsilon'$ and $x - x'$, both of which are accurately known.

It is plain that, instead of computing these changes directly, it suffices to deduce the latitude from a number of observations near each solstice by employing the apparent declinations of the solar tables or the Ephemeris; then, if φ' is the mean value of the latitude found from all the observations at the northern solstice, and φ'' the mean from all at the southern solstice, the true latitude will be

$$\varphi = \tfrac{1}{2}(\varphi' + \varphi'')$$

Every observation should be the mean of the observed zenith distances of both the upper and the lower limb of the sun, in order to be independent of the tabular semidiameter and to eliminate errors of observation as far as possible.

SECOND METHOD.—BY A SINGLE ALTITUDE AT A GIVEN TIME.

164. At the instant when the altitude is observed, the time is noted by the clock. The clock correction being known, we find the true local time, and hence the star's hour angle, by the formula

$$t = \Theta - \alpha$$

in which Θ is the sidereal time and α the star's right ascension.

If the sun is observed, t is simply the apparent solar time. We have, then, by the first equation of (14),

$$\sin \varphi \sin \delta + \cos \varphi \cos \delta \cos t = \sin h$$

in which φ is the only unknown quantity. To determine it, assume d and D to satisfy the conditions

$$d \sin D = \sin \delta$$

$$d \cos D = \cos \delta \cos t$$

then the above equation becomes

$$d \cos (\varphi - D) = \sin h$$

which determines $\varphi - D$, and hence also φ. For practical convenience, however, put

$$\varphi - D = \pm \gamma$$

then, by eliminating d, the solution may be put under the following form:

$$\left.\begin{aligned} \tan D &= \tan \delta \sec t \\ \cos \gamma &= \sin h \sin D \operatorname{cosec} \delta \\ \varphi &= D \pm \gamma \end{aligned}\right\} \quad (281)$$

The first of these equations fully determines D, which will be taken numerically less than 90°, positive or negative according to the sign of its tangent. As t should always be less than 90°, or 6^h, D will have the same sign as δ.

The second equation is indeterminate as to the sign of γ, since the cosine of $+\gamma$ and $-\gamma$ are the same. Hence we obtain by the third equation two values of the latitude. Only one of these values, however, is admissible when the other is numerically greater than 90°, which is the maximum limit of latitudes. When both values are within the limits + 90° and − 90°, the true solution is to be distinguished as that which agrees best with the approximate latitude, which is always sufficiently well known for this purpose, except in some peculiar cases at sea.

EXAMPLE 1.—1856 March 27, in the assumed latitude 23° S. and longitude 43° 14′ W., the double altitude of the sun's lower

limb observed with the sextant and artificial horizon was 114° 40′ 30″ at $4^h\ 21^m\ 15^s$ by a Greenwich Chronometer, which was *fast* $2^m\ 30^s$. Index Correction of Sextant = − 1′ 12″, Barom. 29.72 inches, Att. Therm. 61° F., Ext. Therm. 61° F. Required the true latitude.

Sextant reading	= 114° 40′ 30″	Chronometer	$4^h\ 21^m\ 15^s$
Index corr.	= − 1 12	Correction	− 2 30
	114 39 18	Gr. date, March 27,	4 18 45
App. alt. ☉	= 57 19 39	Longitude	= 2 52 56
Semidiameter	= + 16 3	Local mean t.	= 1 25 49
Ref. and par.	= − 31	Eq. of time	= − 5 19
h	= 57 35 11	App. time, t	= 1 20 30
δ	= +2 51 30		= 20° 7′ 30″

log sec t	0.027360		
log tan δ	8.698351		
log tan D	8.725711	log cosec δ	1.302190
		log sin D	8.725098
D =	+ 3° 2′ 38″	log sin h	9.926445
γ =	25 58 49	log cos γ	9.953733
$D - \gamma = \varphi$ =	− 22 56 11		

EXAMPLE 2.—1856 Aug. 22; suppose the true altitude of *Fomalhaut* is found to be 29° 10′ 0″ when the local sidereal time is $21^h\ 49^m\ 44^s$; what is the latitude?

We have $\alpha = 22^h\ 49^m\ 44^s$, whence $t = -1^h\ 0^m\ 0^s$; $\delta = -30° 22' 47''.5$; $D = -31° 15' 13''$, $\gamma = \pm 60° 0' 6''$, $\varphi = + 28° 44' 53''$. The negative value of γ here gives $\varphi = -91° 15' 19''$; which is inadmissible.

165. The observation of equal altitudes east and west of the meridian may be used not only for determining the time (Art. 139), but also the latitude. For the half elapsed sidereal time between two such altitudes of a fixed star is at once the hour angle required in the method of the preceding article. When the sun is used in this way, the half difference between the apparent times of the observations is the hour angle, and the declination must be taken for noon, or more strictly for the mean of the times of observation. By thus employing the mean of the A.M. and P.M. hour angles and the mean of the corresponding declinations, we obtain sensibly the same result

as by computing each observation separately with its proper hour angle and declination and then taking the mean of the two resulting latitudes; and an error in the clock correction does not affect the final result. The clock rate, however, must be known, as it affects the elapsed interval. See also Art. 182.

166. *Effect of errors in the data upon the latitude computed from an observed altitude.*—From the first of the equations (51) we find

$$d\varphi = \frac{d\zeta}{\cos A} - \frac{\sin q \cos \delta}{\cos A}\, dt + \frac{\cos q}{\cos A}\, d\delta$$

or, since $h = 90° - \zeta$, $dh = - d\zeta$, and $\sin q \cos \delta = \cos \varphi \sin A$,

$$d\varphi = - \sec A \,.\, dh - \cos \varphi \tan A \,.\, dt + \cos q \sec A \,.\, d\delta$$

whence it appears that errors of altitude and time will have the least effect when $A = 0$ or $180°$, that is, when the observation is in the meridian, and the greatest effect when the observation is on the prime vertical. If the same star is observed on both sides of the meridian and at equal distances from it, the coefficient of dt will have opposite signs at the two observations, and hence a small error in the time will be wholly eliminated by taking the mean of the values of the latitude found from two such observations. It is advisable, therefore, in taking a series of observations, to distribute them symmetrically with respect to the meridian. When they are all taken very near to the meridian, a special method of reduction is used, which will be treated of below as our *Third Method* of finding the latitude.

The sign of $\sec A$ is different for stars north and south of the zenith: hence errors of altitude will be at least partially eliminated by taking the mean of the results found from stars near the meridian, both north and south of the zenith. A *constant* error of the instrument may thus be *wholly* eliminated.

As for the effect of the error $d\delta$, its coefficient is zero only when $q = 90°$ and $\sec A$ is not infinite. This occurs when a circumpolar star is observed at its elongation, where we have, Art. 18,

$$\sec A = \frac{\cos \varphi}{\sqrt{[\sin (\delta + \varphi) \sin (\delta - \varphi)]}}$$

which shows that $\sec A$ diminishes as δ increases. In order, therefore, to reduce the effect of an error in the declination

at the same time with that of errors of altitude and time, we should select a star as near the pole as possible, and observe it at or near its greatest elongation, on either side of the meridian. The proximity of the star to the pole enables us to facilitate the reduction of a series of observations, and we shall therefore treat specially of this case as our *Fourth Method* below.

167. When several altitudes not very far from the meridian are observed in succession, if we wish to use their mean as a single altitude, the correction for second differences (Art. 151) must be applied. It is, however, preferable to incur the labor of a separate reduction of each altitude, as we shall then be able to compare the several results, and to discuss the probable errors of the observations and of the final mean. When the observations are very near to the meridian, this separate reduction is readily effected, with but little additional labor, by the following method:

THIRD METHOD.—BY REDUCTION TO THE MERIDIAN WHEN THE TIME IS GIVEN.

168. *To reduce an altitude, observed at a given time, to the meridian.*—This is done in various ways.

(A.) If in the formula, employed in Art. 164,

$$\sin\varphi\sin\delta + \cos\varphi\cos\delta\cos t = \sin h$$

we substitute

$$\cos t = 1 - 2\sin^2\tfrac{1}{2}t$$

it becomes

$$\sin\varphi\sin\delta + \cos\varphi\cos\delta - 2\cos\varphi\cos\delta\sin^2\tfrac{1}{2}t = \sin h$$

But

$$\sin\varphi\sin\delta + \cos\varphi\cos\delta = \cos(\varphi - \delta) \text{ or } \cos(\delta - \varphi)$$

Hence, if we put

$$\zeta_1 = \varphi - \delta, \qquad \text{or } \zeta_1 = \delta - \varphi$$

the above equation may be written

$$\cos\zeta_1 = \sin h + \cos\varphi\cos\delta\,(2\sin^2\tfrac{1}{2}t) \tag{282}$$

If the star does not change its declination, ζ_1 is the zenith distance of the star at its meridian passage; and, being found by

this equation, we then have the latitude as from a meridian observation by the formula

$$\varphi = \delta + \zeta_1, \quad \text{or} \quad \varphi = \delta - \zeta_1$$

according as the zenith is north or south of the star.

When the star changes its declination, this method still holds *if we take δ for the time of observation*, as is evident from our formulæ, in which δ is the declination at the instant when the true altitude is h.

To compute the second member, a previous knowledge of the latitude is necessary. As the term $\cos\varphi \cos\delta\,(2\sin^2\frac{1}{2}t)$ decreases with t, if the observations are not too far from the meridian, the error produced by using an approximate value of φ will be relatively small, so that the latitude found will be a closer approximation than the assumed one; and if the computation be repeated with the new value, a still closer approximation may be made, and so on until the exact value is found.

This method is only convenient where the computer is provided with a table of natural sines and cosines, as well as a table of log. versed sines, or the logarithmic values of $2\sin^2\frac{1}{2}t$.

Example.—Same as Example 1, Art. 164. $h = 57° 35' 11''$, $\delta = +2° 51' 30''$, $t = 1^h 20^m 30^s$. Approximate value of $\varphi = -23°$.

		log $(2\sin^2\frac{1}{2}t)$	8.785726
		log $\cos\varphi$	9.964026
nat. sin h	0.844201	log $\cos\delta$	9.999459
nat. no.	0.056132	. . . . log	8.749211
nat. cos ζ_1	0.900333		

$\zeta_1 = -25° 47' 54''$ (zenith south of sun.)
$\delta = +\ \ 2\ \ 51\ \ 30$
$\varphi = -22\ \ 56\ \ 24$

differing but 13″ from the true value, although the assumed latitude was in error nearly 4′. Repeating the computation with $-22° 56' 24''$ as the approximate latitude, we find $\varphi = -22° 56' 11''$, exactly as in Art. 164.

169. (B.) We may also compute directly the reduction of the observed altitude to the meridian altitude. Putting

$$h_1 = \text{meridian altitude} = 90° - \zeta_1$$

the formula (282) gives

$$\sin h_1 - \sin h = 2 \cos \varphi \cos \delta \sin^2 \tfrac{1}{2} t$$

But we have

$$\sin h_1 - \sin h = 2 \cos \tfrac{1}{2}(h_1 + h) \sin \tfrac{1}{2}(h_1 - h)$$

and hence

$$\sin \tfrac{1}{2}(h_1 - h) = \frac{\cos \varphi \cos \delta \sin^2 \tfrac{1}{2} t}{\cos \tfrac{1}{2}(h_1 + h)} \tag{283}$$

which gives the difference $h_1 - h$, or the correction of h to reduce it to h_1; but it requires in the second member an approximate value both of φ and of h_1, the latter being obtained from the assumed value of φ by the equation $h_1 = 90° - (\varphi - \delta)$; or, if the zenith is south of the star, by the equation $h_1 = 90° - (\delta - \varphi)$.

EXAMPLE.—Same as the above.

	$\delta =$	2° 51′ 30″	$\log \sin^2 \frac{1}{2} t$	8.484696
Approx.	$\varphi =$	− 23 00 00	$\log \cos \varphi$	9.964026
"	$\zeta_1 =$	25 51 30	$\log \cos \delta$	9.999459
"	$h_1 =$	64 8 30	$\log \sec \frac{1}{2}(h_1 + h)$	0.312573
"	$\frac{1}{2}(h_1 + h) =$	60 51 50	$\log \sin \frac{1}{2}(h_1 - h)$	8.760754
	$h_1 - h =$	6 36 33		
	$h =$	57 35 11	$\delta =$	2° 51′ 30″
	$h_1 =$	64 11 44	$\zeta_1 =$	− 25 48 16
			$\varphi =$	− 22 56 46

This method does not approximate so rapidly as the preceding, but the objection is of little weight when the observations are very near the meridian. On the other hand, it has the great advantage of not requiring the use of the table of natural sines.

170. (C.) *Circummeridian altitudes.*—When a number of altitudes are observed very near the meridian,* they are called *circummeridian* altitudes. Each altitude reduced to the meridian gives nearly as accurate a result as if the observation were taken on the meridian.

An approximate method of reducing such observations with the greatest ease is found by regarding the small arc $\frac{1}{2}(h_1 - h)$ as sensibly equal to its sine; that is, by putting

$$\sin \tfrac{1}{2}(h_1 - h) = \tfrac{1}{2}(h_1 - h) \sin 1''$$

* *How near* to the meridian will be determined in Art. 175.

and taking h_1 for $\frac{1}{2}(h_1 + h)$, from which it differs very little, so that (283) may be put under the form

$$h_1 - h = \frac{\cos\varphi\cos\delta}{\cos h_1}\cdot\frac{2\sin^2\frac{1}{2}t}{\sin 1''} \tag{284}$$

The value in seconds of

$$m = \frac{2\sin^2\frac{1}{2}t}{\sin 1''}$$

is given in Table V. with the argument t. If h', h'', h''', &c. are the observed altitudes (corrected for refraction, etc.); t', t'', t''', &c., the hour angles deduced from the observed clock times; m', m'', m''', &c., the values of m from the table; and we put the constant factor

$$\left.\begin{aligned} A = \frac{\cos\varphi\cos\delta}{\cos h_1} &= \frac{\cos\varphi\cos\delta}{\sin\zeta_1}, \\ \text{we have}\qquad h_1 &= h' + Am' \\ h_1 &= h'' + Am'' \\ h_1 &= h''' + Am''' \\ &\&\text{c.} \end{aligned}\right\} \tag{285}$$

and the mean of all these equations gives

$$h_1 = \frac{h' + h'' + h''' + \text{etc.}}{n} + A\,\frac{m' + m'' + m''' + \&\text{c.}}{n}$$

in which n is the number of observations; or

$$h_1 = h_0 + Am_0 \tag{286}$$

in which h_0 denotes the mean of the observed altitudes corrected for refraction, &c., and m_0 the mean of the values of m.

When h_1 has been thus found, the latitude is deduced as from any meridian altitude, only observing that for the sun the declination to be used is that which corresponds to the mean of the times of observation, as has already been remarked in Art. 168.

Example.—At the U. S. Naval Academy, 1849 June 22, circummeridian altitudes of β *Ursae Minoris* were observed with a Troughton sextant from an artificial horizon, as in the following table. The times were noted by a sidereal chronometer which

was fast 1^m $45^s.7$. The index correction of the sextant was $-14' 58''$, Barometer, 30.81 inches, Att. Therm. 65° F., Ext. Therm. 64° F.

The right ascension of the star was	14^h	51^m	$14^s.0$
Chronometer fast		$+1$	45.7
Chronometer time of star's transit	14	52	59.7

The hour angles in the column t are found by taking the differ ence between each observed chronometer time and this chronometer time of transit.

	2 Alt. ✱	Chronom.	t	m
	108° 39′ 40″	14^h 45^m 47^s.	7^m $12^s.7$	102.1
	39 50	47 1.	5 58.7	70.2
	40 40	48 54.5	4 5.2	32.8
	41 0	51 29.5	1 30.2	4.4
	41 0	54 36.5	1 36.8	5.1
	40 30	56 22.	3 22.3	22.3
	40 20	57 43.	4 43.3	43.8
	40 0	58 47.5	5 47.8	66.0
	40 0	15 0 17.5	7 17.8	104.5
	39 20	2 10.	9 10.3	165.1
Mean	108 40 14			$m_0 =$ 61.63
Ind. corr.	$-$ 14 58			
	108 25 16	Assumed $\phi = 38° 59' 0''$		
	54 12 38	$\delta = 74$ 46 36 .9		
Refr.	$-$ 42 .0	Approx. $\zeta_1 = 35$ 47 36 .9	log cos ϕ	9.8906
Am_0	$+$ 21 .5		log cos δ	9.4193
$h_1 =$	54 12 17 .5		log cosec ζ_1	0.2329
$\zeta_1 =$	$-$ 35 47 42 .5		log A	9.5428
$\delta =$	74 46 36 .9		log m_0	1.7898
$\phi =$	38 58 54 .4		log Am_0	1.3326

REMARK 1.—The reduction $h_1 - h$ increases as the denominator of A decreases, that is, as the meridian zenith distance decreases. The preceding method, therefore, as it supposes the reduction to be small, should not be employed when the star passes very near the zenith, unless at the same time the observations are restricted to very small hour angles. It can be shown, however, from the more complete formulæ to be given presently, that so long as the zenith distance is not less than 10°, the reduction computed by this method may amount to 4′ 30″ without being in error more than 1″; and this degree of accuracy suffices for even the best observations made with the sextant.

REMARK 2.—If in (284) we put $\sin \frac{1}{2} t = \frac{1}{2} \sin 1''. t$ (t being in seconds of time), we have

$$h_1 - h = \frac{\cos \varphi \cos \delta}{\cos h_1} \cdot \frac{225}{2} \sin 1''. t^2 = at^2 \qquad (287)$$

in which a denotes the product of all the *constant* factors. It follows from this formula that *near the meridian the altitude varies as the square of the hour angle*, and not simply in proportion to the time. Hence it is that near the meridian we cannot reduce a number of altitudes by taking their mean to correspond to the mean of the times, as is done (in most cases without sensible error) when the observations are remote from the meridian. The method of reduction above exemplified amounts to separately reducing each altitude and then taking the mean of all the results.

171. (D.) *Circummeridian altitudes more accurately reduced.*—The small correction which the preceding method requires will be obtained by developing into series the rigorous equation (282). This equation, when we put $\zeta = 90° - h$ = true zenith distance deduced from the observation, may be put under the form

$$\cos \zeta = \cos \zeta_1 - 2 \cos \varphi \cos \delta \sin^2 \tfrac{1}{2} t$$

which developed in series* gives, neglecting sixth and higher powers of $\sin \frac{1}{2} t$,

---

* If we put $y = 2 \cos \phi \cos \delta \sin^2 \frac{1}{2} t$, the equation to be developed is

$$\cos \zeta = \cos \zeta_1 - y \qquad (a)$$

in which ζ_1 is constant and ζ may be regarded as a function of y; so that by MACLAURIN'S Theorem

$$\zeta = fy = (f) + \left(\frac{df}{dy}\right) y + \tfrac{1}{2}\left(\frac{d^2f}{dy^2}\right) y^2 + \&c. \qquad (b)$$

in which (f), $\left(\frac{df}{dy}\right)$ &c. denote the values of fy and its differential coefficients when $y = 0$. The equation (a) gives, by differentiation,

$$\sin \zeta \frac{d\zeta}{dy} = 1 \qquad \frac{d\zeta}{dy} = \frac{1}{\sin \zeta}$$

$$\frac{d^2\zeta}{dy^2} = -\frac{\cos \zeta}{\sin^2 \zeta} \cdot \frac{d\zeta}{dy} = -\frac{\cot \zeta}{\sin^2 \zeta} \quad \&c.$$

$$\zeta_1 = \zeta - \frac{\cos\varphi\cos\delta}{\sin\zeta_1}\cdot\frac{2\sin^2\frac{1}{2}t}{\sin 1''} + \left(\frac{\cos\varphi\cos\delta}{\sin\zeta_1}\right)^2\cdot\frac{2\cot\zeta_1\sin^4\frac{1}{2}t}{\sin 1''} \qquad (288)$$

By this formula, first given by DELAMBRE, the reduction to the meridian consists of two terms, the first of which is the same as that employed in the preceding method, and the second is the small correction which that method requires. These two terms will be designated as the "1st Reduction" and "2d Reduction." Putting

$$m = \frac{2\sin^2\frac{1}{2}t}{\sin 1''} \qquad n = \frac{2\sin^4\frac{1}{2}t}{\sin 1''}$$

$$A = \frac{\cos\varphi\cos\delta}{\sin\zeta_1} \qquad B = A^2\cot\zeta_1$$

we have

$$\zeta_1 = \zeta - Am + Bn \qquad (289)$$

If a number of observations are taken, we have a number of equations of this form, the mean of which will be

$$\zeta_1 = \zeta_0 - Am_0 + Bn_0$$

in which ζ_0 is the arithmetical mean of the observed zenith distances, m_0 and n_0 the arithmetical means of the values of m and n corresponding to the values of t. The values of n are also given in Table V.

Having found ζ_1, we have the latitude, as before, by the formula

$$\varphi = \delta + \zeta_1$$

in which we must give ζ_1 the negative sign when the zenith is south of the star, and it must be remembered that for the sun (or any object whose proper motion is sensible) δ must be the mean of the declinations belonging to the several observations,

---

But when $y = 0$ we have, by (a), $\zeta = \zeta_1$, so that (b) becomes

$$\zeta = \zeta_1 + \frac{y}{\sin\zeta_1} - \frac{y^2\cot\zeta_1}{2\sin^2\zeta_1} + \tfrac{1}{6}(1 + 3\cot^2\zeta_1)\frac{y^3}{\sin^3\zeta_1} - \&c \qquad (c)$$

Restoring the value of y, this gives the development used in the text, observing that as ζ and ζ_1 are supposed to be in seconds of arc, the terms of the series are divided by sin 1″ to reduce them to the same unit.

or, which is the same, the declination corresponding to the mean of the times of observation.*

Finally, if the star is near the meridian below the pole, the hour angles should be reckoned from the instant of the lower transit. Recurring to the formula

$$\cos\zeta = \sin\varphi\sin\delta + \cos\varphi\cos\delta\cos t$$

in which t is the hour angle reckoned from the upper transit, we observe that if this angle is reckoned from the lower transit we must put $180° - t$ instead of t, or $-\cos t$ for $+\cos t$, and then we have

$$\cos\zeta = \sin\varphi\sin\delta - \cos\varphi\cos\delta\cos t$$

and, substituting as before,

$$\cos t = 1 - 2\sin^2\tfrac{1}{2}t$$

this gives

$$\cos\zeta = -\cos(\varphi + \delta) + 2\cos\varphi\cos\delta\sin^2\tfrac{1}{2}t$$

or, since for lower culminations we have $\zeta_1 = 180° - (\varphi + \delta)$ and $\cos\zeta_1 = -\cos(\varphi + \delta)$,

$$\cos\zeta = \cos\zeta_1 + 2\cos\varphi\cos\delta\sin^2\tfrac{1}{2}t$$

which developed gives

$$\zeta_1 = \zeta + \frac{\cos\varphi\cos\delta}{\sin\zeta_1}\cdot\frac{2\sin^2\frac{1}{2}t}{\sin 1''} + \left(\frac{\cos\varphi\cos\delta}{\sin\zeta_1}\right)^2\cdot\frac{2\cot\zeta_1\sin^4\frac{1}{2}t}{\sin 1''}$$

or

$$\zeta_1 = \zeta + Am + Bn \text{ (sub polo)} \qquad (290)$$

which is computed by the same table, but both first and second reductions here have the same sign.

If a star is observed with a sidereal chronometer the daily rate of which is so small as to be insensible during the time of

---

* To show that the mean declination is to be used, we may observe that for each observation we have put $\zeta_1 = \phi - \delta$, and that if δ', δ'', &c., are the several declinations, the several equations of the form (289) will give

$$\phi = \delta' + \zeta' - Am' + A^2\cot\zeta_1 n'$$
$$\phi = \delta'' + \zeta'' - Am'' + A^2\cot\zeta_1 n''$$
&c.,

the mean of which, if δ = mean of δ', δ'', &c., will be

$$\phi = \delta + \zeta_0 - Am_0 + A^2\cot\zeta_1 n_0 = \delta + \zeta_1$$

the observations, the hour angles t are found by merely taking the difference between each noted time and the chronometer time of the star's transit, as in the example of Article 170. But if we wish to take account of the rate of the chronometer, it can be done without separately correcting each hour angle, as follows: Let δT be the rate of the chronometer in 24^h (δT being positive for losing rate, Art. 137); then, if t is the hour angle given directly by the chronometer, and t' the true hour angle, we have

$$t' : t = 24^h : 24^h - \delta T = 86400^s : 86400^s - \delta T$$

whence

$$t' = t \cdot \left[\frac{1}{1 - \frac{\delta T}{86400}}\right]$$

Instead of $\sin \frac{1}{2} t$ we must use $\sin \frac{1}{2} t'$; for which we shall have, with all requisite precision,

$$\sin \tfrac{1}{2} t' = \sin \tfrac{1}{2} t \cdot \frac{t'}{t}, \text{ or } \sin^2 \tfrac{1}{2} t' = \sin^2 \tfrac{1}{2} t \cdot \left(\frac{t'}{t}\right)^2$$

Hence, if we put

$$k = \left[\frac{1}{1 - \frac{\delta T}{86400}}\right]^2 = \left(\frac{t'}{t}\right)^2$$

we shall have

$$Am = k \cdot \frac{\cos \varphi \cos \delta}{\sin \zeta_1} \cdot \frac{2 \sin^2 \frac{1}{2} t}{\sin 1''}$$

so that if we compute A by the formula

$$A = k \cdot \frac{\cos \varphi \cos \delta}{\sin \zeta_1}$$

we can take $m = \dfrac{2 \sin^2 \frac{1}{2} t}{\sin 1''}$ for the actual chronometer intervals, and no further attention to the rate is required.

The factor k can be given in a small table with the argument "rate," in connection with the table for m, as in our Table V.

If a star is observed with a mean time chronometer, the intervals are not only to be corrected for rate, but also to be reduced

from mean to sidereal intervals by multiplying them by $\mu = 1.00273791$ (Art. 49); so that for $\sin^2 \frac{1}{2} t$ we must substitute $k \sin^2 (\frac{1}{2} . \mu t)$, or, with sufficient precision, $k\mu^2 \sin^2 \frac{1}{2} t$.

If the sun is observed with a mean time chronometer, the intervals are both to be corrected for rate and reduced from mean solar to apparent solar intervals. The mean interval differs from the apparent only by the change in the equation of time during the interval, and this change may be combined with the rate of the chronometer. Denoting by δE the *increase* of the equation of time in 24^h (remembering that E is to be regarded as positive when it is additive to apparent time), and by δT the rate of the chronometer on mean time, we may regard $\delta T - \delta E$ as the rate of the chronometer on apparent time. Instead of the factor k we shall then have a factor k', which is to be found by the formula

$$k' = \left[\frac{1}{1 - \frac{\delta T - \delta E}{86400}}\right]^2$$

which may be taken from the table for k by taking $\delta T - \delta E$ as the argument.

Finally, if the sun is observed with a sidereal chronometer, we must multiply $\sin^2 \frac{1}{2} t$ not only by k' but by $\frac{1}{\mu^2}$.

Denoting μ^2 by i and $\frac{1}{\mu^2}$ by i', these rules may be collected, for the convenience of reference, as follows:

$$\left.\begin{array}{lll}
\text{Star by sidereal chron.,} & A = k \cdot \dfrac{\cos\varphi \cos\delta}{\sin\zeta_1} & \\[2ex]
\text{Star by mean time chron.,} & A = ki \cdot \dfrac{\cos\varphi \cos\delta}{\sin\zeta_1} & [\log i = 0.002375] \\[2ex]
\text{Sun by mean time chron.,} & A = k' \cdot \dfrac{\cos\varphi \cos\delta}{\sin\zeta_1} \cdot & \\[2ex]
\text{Sun by sidereal chron.,} & A = k' i' \dfrac{\cos\varphi \cos\delta}{\sin\zeta_1} & [\log i' = 9.997625]
\end{array}\right\} \quad (291)$$

for which $\log k$ will be taken from Table V. with the argument rate of the chronometer $= \delta T$; and $\log k'$ from the same table

with the argument $\delta T - \delta E =$ daily rate of the chronometer diminished by the daily increase of the equation of time.

EXAMPLE.—1856 March 15, at a place assumed to be in latitude 37° 49′ N. and longitude 122° 24′ W., suppose the following zenith distances of the sun's lower limb to have been observed with an Ertel universal instrument,* Barom. 29.85 inches, Att. Therm. 65° F., Ext. Therm. 63° F. The chronometer, regulated to the local mean time, was, at noon, slow $11^m\ 20^s.8$, with a daily losing rate of $6^s.6$.

	Obs'd zen. dist.	Chronometer.	t	m	n
	40° 8′ 40″.7	$23^h\ 37^m\ 35^s.$	$-19^m\ 58^s.8$	783″.3	1″.49
	40 2 16 .5	42 3 .	−15 30.8	472 .4	0 .54
	39 57 28 .3	46 29.5	−11 4.3	240 .6	0 .14
	39 54 17 .2	50 46.5	− 6 47.3	90 .5	0 .02
	39 52 33 .	55 16 .	− 2 17.8	10 .4	0 .00
	39 52 34 .5	0 0 37.5	+ 3 3.7	18 .4	0 .00
	39 54 28 .6	5 13 .	7 39.2	115 .0	0 .03
	39 58 9 .8	9 49.5	12 15.7	295 .1	0 .21
	40 3 0 .3	14 8 .	16 34.2	538 .9	0 .70
	40 9 36 .	18 31 .	20 57.2	861 .4	1 .80
Means	39 59 18 .5		$t_0 = +$ 0 29.1	$m_0 =$ 342 .60	$n_0 =$ 0 .49

The equation of time at the local noon being $+ 8^m\ 54^s.6$, we have

Mean time of app. noon	=	$0^h\ 8^m\ 54^s.6$
Chronometer slow	=	11 20.8
Chr. time of app. noon	=	23 57 33.8

The difference between this and the observed chronometer times gives the hour angles t as above.

The mean of the hour angles being $+ 29^s.1$, the declination is to be taken for the local apparent time $0^h\ 0^m\ 29^s.1$, or for the Greenwich mean time March 15, $8^h\ 18^m\ 59^s.7$; whence

$$\delta = -\ 1°\ 48'\ 8''.8$$
$$\text{(Approximate)}\ \varphi = +\ 37\ \ 49\ \ 0\ .$$
$$\text{"}\qquad \zeta_1 = 39\ \ 37\ \ 8\ .8$$

The increase of the equation of time in 24^h is $\delta E = - 17^s.4$,

---

* See Vol. II., *Altitude and Azimuth Instrument*, for the method of observing the zenith distances.

and, the chronometer rate being $\delta T = +6^s.6$, we have $\delta T - \delta E = +24^s.0$, with which as the argument "rate" in Table V. we find $\log k' = 0.00024$.

The computation of the latitude is now carried out as follows:

$\log \cos \varphi$	9.89761	Mean observed zen. dist. ☉		=	39° 59′ 18″.5
$\log \cos \delta$	9.99979			$r - p$ =	+ 41 .8
$\log \operatorname{cosec} \zeta_1$	0.19540	$\log A^2$	0.1861	S =	− 16 6 .5
$\log k'$	0.00024	$\log \cot \zeta_1$	0.0821	Am_0 =	− 7 4 .4
$\log A$	0.09304	$\log B$	0.2682	Bn_0 =	+ 0 .9
$\log m_0$	2.53479	$\log n_0$	9.6902	ζ_1 =	39 36 50 .3
$\log Am_0$	2.62783	$\log Bn_0$	9.9584	δ =	−1 48 8 .8
				φ =	37 48 41 .5

The assumed value of φ being in error, the value of A is not quite correct; but a repetition of the computation with the value of φ just found does not in this case change the result so much as $0''.1$.

172. (E.) *Gauss's method of reducing circummeridian altitudes of the sun.*—The preceding method of reduction is both brief and accurate, and the latitude found is the mean of all the values that would be found by reducing each observation separately. This separate reduction, however, is often preferred, notwithstanding the increased labor, as it enables us to compare the observations with each other, and to discuss the probable error of the final result; and it is also a check against any gross error. But, if we separately reduce the observations by the preceding method, we must not only interpolate the refraction for each altitude, but also the declination for each hour angle. Gauss proposed a method by which the latter of these interpolations is avoided. He showed that if we reckon the hour angles, not from apparent noon, but from *the instant when the sun reaches its maximum altitude*, we can compute the reduction by the method above given, and use the *meridian declination* for all the observations. This method is, indeed, not quite so exact as the preceding; but I shall show how it may be rendered quite perfect in practice by the introduction of a small correction.

In the rigorous formula

$$\cos \zeta = \sin \varphi \sin \delta + \cos \varphi \cos \delta \cos t$$

δ is the declination corresponding to the hour angle t. If then

$\Delta\delta =$ the hourly increase of the declination, *positive* when the sun is moving *northward*,
$\delta_1 =$ the declination at noon,

and if t is expressed in seconds of time, we have

$$\delta = \delta_1 + \frac{t \,.\, \Delta\delta}{3600} = \delta_1 + x$$

where, since $\Delta\delta$ never exceeds $60''$, x will not exceed $30''$ so long as $t < 30^m$. Hence we may substitute, with great accuracy,

$$\begin{aligned} \sin\delta &= \sin\delta_1 + \cos\delta_1 \sin x \\ \cos\delta &= \cos\delta_1 - \sin\delta_1 \sin x \end{aligned}$$

and the above formula becomes

$$\begin{aligned} \cos\zeta = \sin\varphi \sin\delta_1 + \cos\varphi \cos\delta_1 \cos t + \sin(\varphi - \delta_1) \sin x \\ + 2 \cos\varphi \sin\delta_1 \sin^2 \tfrac{1}{2} t \sin x \end{aligned}$$

The last term is extremely small, rarely affecting the value of ζ by as much as $0''.1$; and since x is proportional to the hour angle, and therefore has opposite signs for observations on different sides of the meridian, the effect of this term will nearly or quite disappear from the mean of a series of observations properly distributed before and after the meridian passage. Now, we have

$$\sin x = \frac{t\,\Delta\delta \sin 1''}{3600} = 15\, t \sin 1'' \cdot \frac{\Delta\delta}{54000}$$

Let

$$\sin\vartheta = \frac{\Delta\delta}{54000} \cdot \frac{\sin(\varphi - \delta_1)}{\cos\varphi \cos\delta_1}$$

then, taking

$$15\, t \sin 1'' = \sin t + \tfrac{1}{6} \sin^3 t$$

we have

$$\sin x = (\sin t + \tfrac{1}{6} \sin^3 t) \sin\vartheta \cdot \frac{\cos\varphi \cos\delta_1}{\sin(\varphi - \delta_1)}$$

and the formula for $\cos\zeta$ becomes, by omitting the last term,

$$\begin{aligned} \cos\zeta = \sin\varphi \sin\delta_1 + \cos\varphi \cos\delta_1 (\cos t + \sin t \sin\vartheta) \\ + \tfrac{1}{6} \cos\varphi \cos\delta_1 \sin^3 t \sin\vartheta \end{aligned}$$

The last term involving $\sin^3 t$ multiplied by the small quantity $\sin\vartheta$ is even less than the term above rejected. Like that, also, it has opposite signs for observations on different sides of the meridian, and will not affect the mean result of a properly arranged series of observations. Rejecting it, therefore, our formula becomes

$$\cos\zeta = \sin\varphi\sin\delta_1 + \cos\varphi\cos\delta_1\cos(t-\vartheta) + 2\cos\varphi\cos\delta_1\sin^2\tfrac{1}{2}\vartheta$$

The last term here must also be rejected if we wish to obtain the method as proposed by GAUSS; but, as it is always a positive term and affects all the observations alike, I shall retain it, especially as it can be taken into account in an extremely simple manner.

The maximum value of $\cos\zeta$, which corresponds to the maximum altitude, is given immediately by the above formula by putting $t = \vartheta$. Hence ϑ *is the hour angle of the maximum altitude.* Putting

$$t' = t - \vartheta$$

we have

$$\cos\zeta = \cos(\varphi - \delta_1) - 2\cos\varphi\cos\delta_1\sin^2\tfrac{1}{2}t' + 2\cos\varphi\cos\delta_1\sin^2\tfrac{1}{2}\vartheta$$

Let

$$\delta' = \delta_1 + \frac{\cos\varphi\cos\delta_1}{\sin(\varphi-\delta_1)}\cdot\frac{2\sin^2\frac{1}{2}\vartheta}{\sin 1''}$$

then our formula becomes

$$\cos\zeta = \cos(\varphi - \delta') - 2\cos\varphi\cos\delta_1\sin^2\tfrac{1}{2}t'$$

This equation is of the same form as that from which (288) was obtained, and therefore when developed gives

$$\zeta_1 = \zeta - \frac{\cos\varphi\cos\delta_1}{\sin\zeta_1}\cdot\frac{2\sin^2\frac{1}{2}t'}{\sin 1''} + \left(\frac{\cos\varphi\cos\delta_1}{\sin\zeta_1}\right)^2\cdot\frac{2\cot\zeta_1\sin^4\frac{1}{2}t'}{\sin 1''}$$

in which $\zeta_1 = \varphi - \delta'$. Putting then, as before,

$$A = \frac{\cos\varphi\cos\delta_1}{\sin\zeta_1} \qquad B = A^2\cot\zeta_1 \tag{292}$$

and taking m and n from Table V., or their logarithms from Table VI., with the argument t', which is the hour angle reckoned

from the instant the sun reaches its maximum altitude, we have

$$\zeta_1 = \zeta - Am + Bn \tag{293}$$

Since ζ_1 differs from the latitude by the constant quantity δ', its value found from each observation should be the same. Taking its mean value, we have

$$\varphi = \zeta_1 + \delta'$$

The angle ϑ, being very small, may be found with the utmost precision by the formula

$$\vartheta = \frac{1}{810000 \sin 1''} \cdot \frac{\Delta \delta}{A} = [9.40594] \frac{\Delta \delta}{A} \tag{294}$$

which gives ϑ in seconds of the chronometer when A has been computed by the formula (292).

The most simple method of finding the corrected hour angles t' will be to add ϑ to the chronometer time of apparent noon, and then take the difference between this corrected time and each observed time.

If we put $\delta' = \delta_1 + y$, we have

$$y = A \cdot \frac{2 \sin^2 \frac{1}{2} \vartheta}{\sin 1''} \tag{295}$$

which requires only one new logarithm to be taken, namely, the value of log m from Table VI. with the argument ϑ. We then have, finally,

$$\varphi = \zeta_1 + \delta_1 + y \tag{296}$$

Example.—The same as that of the preceding article. We have there employed the assumed latitude 37° 49′; but, since even a rough computation of two or three observations will give a nearer value, let us suppose we have found as a first approximation $\varphi = 37° \, 48' \, 45''$. With this and the meridian declination $\delta_1 = -1° \, 48' \, 9''.2$, and log $k' = 0.00024$ as before, we now find, by (292),

$$\log A = 0.09310 \qquad \log B = 0.2683$$

We have also there found the chronometer time of apparent

noon $= 23^h\ 57^m\ 33^s.8$. We now take from the Ephemeris $\Delta\delta = + 59''.22$, and hence, by (294),

	log $\Delta\delta$	1.7725
	ar. co. log A	9.9069
	const. log	9.4059
$\vartheta = + 12^s.2$	log ϑ	1.0853

Hence the chronometer time of the maximum altitude is $23^h\ 57^m\ 33^s.8 + 12^s.2 = 23^h\ 57^m\ 46^s$, which gives the hour angles t' as below:

t'	log m	log Am	log n	log Bn
$-\ 20^m\ 11^s.$	2.90274	2.99584	0.1900	0.4583
15 43 .	2.68558	2.77868	9.7557	0.0240
11 16 .5	2.39718	2.49028	9.1776	9.4459
6 59 .5	1.98216	2.07526	8.3487	8.6170
$-$ 2 30 .	1.08891	1.18201		
$+$ 2 51 .5	1.20525	1.29835		
7 27 .	2.03730	2.13040	8.4553	8.7236
12 3 .5	2.45551	2.54861	9.2955	9.5638
16 22 .	2.72077	2.81387	9.8260	0.0943
20 45 .	2.92677	3.01987	0.2381	0.5064

The refraction may be computed from the tables first for a mean zenith distance, and then with its variation in one minute (which will be found with sufficient accuracy from the table of mean refraction) its value for each zenith distance is readily found. The parallax, which is here sensibly the same ($= 5''.54$) for all the observations, is subtracted from the refraction, and the results are given in the column $r - p$ of the following computation. The numbers in the 3d and 4th columns are found from their logarithms above; and the last column contains the several values of the minimum zenith distance of the sun's lower limb, formed by adding together the numbers of the preceding columns. To the mean of these we then apply the sun's semidiameter, the meridian declination, and the correction y, which are all constant for the whole series of observations.

Obs'd ζ	$r - p$	Am	Bn	ζ_1
40° 8′ 40″.7	+ 42″.1	− 16′ 30″.5	+ 2″.9	39° 52′ 55″.2
40 2 16 .5	41 .9	10 0 .7	1 .1	58 .8
39 57 28 .3	41 .8	5 9 .2	0 .3	61 .2
39 54 17 .2	41 .7	1 58 .9	0 .0	60 .0
39 52 33 .	41 .6	0 15 .2	0 .0	59 .4
39 52 34 .5	41 .6	0 19 .9	0 .0	56 .2
39 54 28 .6	41 .7	2 15 .0	0 .1	55 .4
39 58 9 .8	41 .8	5 53 .7	0 .4	58 .3
40 3 0 .3	41 .9	10 51 .4	1 .2	52 .0
40 9 36 .	42 .1	17 26 .8	3 .2	54 .5

(Lower limb) Mean ζ_1 = 39 52 57 .10
Semidiameter = − 16 6 .49
δ_1 = − 1 48 9 .20
y = + 0 .10
φ = 37 48 41 .51

$\log \dfrac{2 \sin^2 \frac{1}{2} \vartheta}{\sin 1''}$ 8.9090
$\log A$ 0.0931
$\log y$ 9.0021

This result agrees precisely with that found before. If we suppose all the observations to be of the same weight, we can now determine the probable error of observation. Denoting the difference between each value of ζ_1 and the mean of all by v, and the sum of the squares of v by $[vv]$, according to the notation used in the method of least squares, we have

v	vv
− 1″.9	3.61
+ 1 .7	2.89
+ 4 .1	16.81
+ 2 .9	8.41
+ 2 .3	5.29
− 0 .9	.81
− 1 .7	2.89
+ 1 .2	1.44
− 5 .1	26.01
− 2 .6	6.76
$n = 10$, $[vv] =$	74.92

Mean error of a single observation $= \sqrt{\dfrac{[vv]}{n-1}} = 2''.89$

Mean error of the final value of $\varphi = \dfrac{2.89}{\sqrt{10}} = 0''.91$

Probable error of a single obs. $= 2''.89 \times 0.6745 = 1''.95$
" " of φ $= 0''.91 \times 0.6745 = 0''.61$

It must not be forgotten that the probable error 1″.95 here represents the probable error of *observation* only: a constant error of the instrument, affecting all the observations, will form no part of this error; and the same is true of an error in the refraction.

173. For the most refined determinations of the latitude, standard stars are to be preferred to the sun. Their declinations are somewhat more precisely known; the instrument is in night observations less liable to the errors produced by changes of temperature during the observations; constant instrumental errors and errors of refraction may be eliminated to a great extent by combining north and south stars; or errors of declination may be avoided by employing only circumpolar stars at or near their upper and lower culminations. In general, errors of circummeridian altitudes are eliminated under the same conditions as those of meridian observations, since the former are reduced to the meridian with the greatest precision. See the next following article.

For a great number of nice determinations of the latitude by circummeridian altitudes of stars north and south of the zenith and of circumpolar stars, see PUISSANT, *Nouvelle Description Géométrique de la France.*

174. *Effect of errors of zenith distance, declination, and time, upon the latitude found by circummeridian altitudes.*—Differentiating (289), regarding A as constant, and neglecting the variations of the last term, which is too small to be sensibly affected by small errors of t, we have, since $d\varphi = d\zeta_1 + d\delta$,

$$d\varphi = d\zeta + d\delta - \frac{A \sin t}{\sin 1''} (15\, dt)$$

The errors $d\zeta$ and $d\delta$ affect the resulting latitude by their whole amount. The coefficient of dt has opposite signs for east and west hour angles; and therefore, if the observations are so taken as to consist of a number of pairs of equal zenith distances east and west of the meridian, a small constant error in the hour angles, arising from an imperfect clock correction, will be eliminated in the mean. This condition is in practice nearly satisfied when the same number of observations are taken on each side of the meridian, the intervals of time between the successive observations being made as nearly equal as practicable.

An error in the assumed latitude which affects A is eliminated by repeating the computation with the latitude found by the first computation. An error in the declination which would affect A is not to be supposed.

175. *To determine the limits within which the preceding methods of reducing circummeridian altitudes are applicable.—First.* In the method of Art. 170 we employ only the "first reduction" ($= Am$), which is the first term of the more complete reduction expressed by (288). The error of neglecting the "second reduction" ($= Bn$) increases with the hour angle, and if this method is to be used it becomes necessary to determine the value of the hour angle at which this reduction would be sensible. We have

$$Bn = A^2 \cot \zeta_1 \frac{2 \sin^4 \frac{1}{2} t}{\sin 1''}$$

whence if we put b for Bn and

$$F = \sqrt{\tfrac{1}{2} \sin 1'' \tan \zeta_1}$$

we derive

$$\sin^2 \tfrac{1}{2} t = \frac{F}{A} \sqrt{b} \tag{298}$$

Since $\zeta_1 = \varphi - \delta$, F and A are but functions of φ and δ; and therefore by this formula we can compute the values of t for any assigned value of b, and for a series of values of φ and δ. Table VII.A gives the values of t in minutes computed by (298) when $b = 1''$. That is, calling t_1 the tabular hour angle and t the hour angle for any assigned limit of error b, we have

$$\sin^2 \tfrac{1}{2} t_1 = \frac{F}{A} \qquad\qquad \sin^2 \tfrac{1}{2} t = \sin^2 \tfrac{1}{2} t_1 \sqrt{b}$$

As the limits are not required with great precision, we may substitute for the last equation the following:

$$t = t_1 \sqrt[4]{b}$$

If we take $b = 0''.1$, we have $\sqrt[4]{b} = 0.56$, or nearly $\frac{1}{2}$: hence *the limiting hour angle at which the second reduction amounts to* $0''.1$ is *about* $\frac{1}{2}$ *the angle given in* Table VII.A.

Example.—How far from the meridian may the observations in the example p. 237 be extended before the error of the method of reduction there employed amounts to $1''$? With $\varphi = + 39°$, $\delta = + 75°$, Table VII.A gives $t_1 = 30^m$. Hence

the method is in that example correct within 1″ if the observations are taken within 30^m of the meridian, and correct within 0″.1 if they are taken within 15^m of the meridian.

Second.—In the more exact methods of reduction given in Arts. 171 and 172, we have neglected the last term of the development given in the note on page 239, which may be called a "third reduction." Denoting it by c, we have

$$c = \frac{4}{3}\left(\frac{1 + 3\cot^2\zeta_1}{\sin 1''}\right)A^3 \sin^6 \tfrac{1}{2}t$$

whence, if we put

$$K = \sqrt[3]{\frac{\frac{3}{4}\sin 1''}{1 + 3\cot^2\zeta_1}}$$

we deduce

$$\sin^2 \tfrac{1}{2}t = \frac{K}{A}\sqrt[3]{c} \tag{299}$$

Table VII.B gives the values of t, computed by this formula, for $c = 1''$. Denoting the tabular value of t by t_1, we have

$$\sin^2 \tfrac{1}{2}t_1 = \frac{K}{A} \qquad \sin^2 \tfrac{1}{2}t = \sin^2 \tfrac{1}{2}t_1 \sqrt[3]{c}$$

or, with sufficient accuracy in most cases,

$$t = t_1 \sqrt[6]{c}$$

For $c = 0''.1$ we have $\sqrt[6]{c} = 0.68$, or nearly $\frac{2}{3}$; and hence the limiting hour angle at which the third reduction (omitted in our most exact methods) would amount to 0″.1 is about $\frac{2}{3}$ the angle given in Table VII.B.

Example.—How far from the meridian may the observations in the example p. 243 be extended before the error of the method of reduction there employed amounts to 0″.1? With $\varphi = 38°$, $\delta = -2°$, Table VII.B gives $t_1 = 39^m$, and $\frac{2}{3}$ of this is $t = 26^m$: so that the method is in that example correct within 1″ when the observations are taken within 39^m of the meridian; and it is correct within 0″.1 when the observations are taken within 26^m of the meridian.

The limiting hour angle for a given limit of error diminishes

rapidly with the zenith distance; and hence in general very small zenith distances are to be avoided. But the observations may be extended somewhat beyond the limits of our tables, since the errors which affect only the extreme observations are reduced in taking the mean of a series.

FOURTH METHOD.—BY THE POLE STAR.

176. The latitude may be deduced with accuracy from an altitude of the pole star observed at any time whatever, when this time is known. The computation may be performed by (281); but when a number of successive observations are to be reduced, the following methods are to be preferred. If we put

$$p = \text{the star's polar distance,}$$

we have, by (14),

$$\sin h = \sin \varphi \cos p + \cos \varphi \sin p \cos t$$

in which the hour angle t and the altitude h are derived from observation and φ is the required latitude. Now, p being small (at present less than 1° 30′), we may develop φ in a series of ascending powers of p, and then employ as many terms as we need to attain any given degree of precision. The altitude cannot differ from the latitude by more than p: if, then, we put

$$\varphi = h - x$$

x will be a small correction of the same order of magnitude as p We have*

$$\begin{aligned}
\sin \varphi &= \sin (h - x) = \sin h - x \cos h - \tfrac{1}{2} x^2 \sin h + \tfrac{1}{6} x^3 \cos h + \&c. \\
\cos \varphi &= \cos (h - x) = \cos h + x \sin h - \tfrac{1}{2} x^2 \cos h - \tfrac{1}{6} x^3 \sin h + \&c. \\
\sin p &= p - \tfrac{1}{6} p^3 + \&c. \\
\cos p &= 1 - \tfrac{1}{2} p^2 + \&c.
\end{aligned}$$

which substituted in the above formula for sin h give

$$\sin h = \sin h - x \cos h + p \cos t \cos h - \tfrac{1}{2} (x^2 - 2\, xp \cos t + p^2) \sin h + \&c.$$

and from this we obtain the following general expression of the correction:

* Pl. Trig. (403) and (406).

$$\begin{aligned}x = p\cos t &- \tfrac{1}{2}(x^2 - 2\,xp\cos t + p^2)\tan h \\ &+ \tfrac{1}{6}(x^3 - 3\,x^2 p\cos t + 3\,xp^2 - p^3\cos t) \\ &+ \tfrac{1}{24}(x^4 - 4\,x^3 p\cos t + 6\,x^2 p^2 - 4\,xp^3\cos t + p^4)\tan h \\ &- \&c. \end{aligned} \tag{a}$$

For a first approximation, we take

$$x = p\cos t \tag{b}$$

and, substituting this in the second term of (a), we find for a second approximation, neglecting the third powers of p and x,

$$x = p\cos t - \tfrac{1}{2}p^2\sin^2 t\tan h \tag{c}$$

Substituting this value in the second and third terms of (a), we find, as a third approximation,

$$x = p\cos t - \tfrac{1}{2}p^2\sin^2 t\tan h + \tfrac{1}{3}p^3\cos t\sin^2 t \tag{d}$$

This value, substituted in the second, third, and fourth terms of (a), gives, as a fourth approximation,

$$\begin{aligned}x = p\cos t - \tfrac{1}{2}p^2\sin^2 t\tan h + \tfrac{1}{3}p^3\cos t\sin^2 t - \tfrac{1}{8}p^4\sin^4 t\tan^3 h \\ + \tfrac{1}{24}p^4(4 - 9\sin^2 t)\sin^2 t\tan h\end{aligned} \tag{e}$$

In these formulæ, to obtain x in seconds when p is given in seconds, we must multiply the terms in p^2, p^3, and p^4 by $\sin 1''$, $\sin^2 1''$, $\sin^3 1''$, respectively.

In order to determine the relative accuracy of these formulæ, let us examine the several terms of the last, which embraces all the others. The value of t, which makes the last term of (e) a maximum, will be found by putting the differential coefficient of $(4 - 9\sin^2 t)\sin^2 t$ equal to zero; whence

$$4\sin t\cos t\,(2 - 9\sin^2 t) = 0$$

which is satisfied by $t = 0$, $t = 90°$, or $\sin^2 t = \frac{2}{9}$, the last of which alone makes the second differential coefficient negative. The maximum value of the term is, then, $\frac{1}{54}p^4\sin^3 1''\tan h$, which for $p = 1°\,30' = 5400''$ is $0''.0018\tan h$. This can amount to $0''.01$ only when h is nearly $80°$,—that is, when the latitude is nearly $80°$. This term, therefore, is wholly inappreciable in every practical case.

The term $\frac{1}{8}p^4 \sin^3 1'' \sin^4 t \tan^3 h$ is a maximum for $\sin t = 1$, in which case, for $p = 5400''$, it is $0''.0121 \tan^3 h$. This amounts to $0''.1$ when $h = 64°$, and to $1''$. when $h = 77°$.

For the maximum of the term $\frac{1}{3} p^3 \sin^2 1'' \cos t \sin^2 t$ we have, by putting the differential coefficient of $\cos t \sin^2 t$ equal to zero,

$$\sin t\,(2 - 3 \sin^2 t) = 0$$

which gives $\sin^2 t = \frac{2}{3}$, and consequently $\cos t = \sqrt{\frac{1}{3}}$; and hence the maximum value of this term is $\frac{2}{9} p^3 \sin^2 1'' \sqrt{\frac{1}{3}} = 0''.475$. Since the maximum values of this and the following terms do not occur for the same value of t, their aggregate will evidently never amount to $1''$ in any practical case.

Hence, *to find the latitude by the pole star within* $1''$, we have the formula

$$\varphi = h - p \cos t + \tfrac{1}{2} p^2 \sin 1'' \sin^2 t \tan h \tag{300}$$

The hour angle t is to be deduced from the sidereal time Θ of the observation and the star's right ascension α, by the formula

$$t = \Theta - \alpha$$

To facilitate the computation of the formula (300), tables are given in every volume of the British Nautical Almanac and the Berlin Jahrbuch; but the formula is so simple that a direct computation consumes very little more time than the use of these tables, and it is certainly more accurate.

Example.—(From the Nautical Almanac for 1861).—On March 6, 1861, in Longitude 37° W., at $7^h\ 43^m\ 35^s$ mean time, suppose the altitude of *Polaris*, when corrected for the error of the instrument, refraction, and dip of the horizon, to be 46° 17′ 28″: required the latitude.

Mean time		$7^h\ 43^m\ 35^s$.
Sid. time mean noon, March 6,		22 56 47.9
Reduction for $7^h\ 43^m\ 35^s$		1 16.2
Reduction for Long. $2^h\ 28^m$		24.3
Sidereal time	$\Theta =$	6 42 3.4
March 6, $p = 1°\ 25'\ 32''.7$	$\alpha =$	1 7 39.0
	$t =$	5 34 24.4
	$=$	83° 36′ 6″

Hence, by formula (300),

$\log p$	3.71035	$\log p^2$	7.4207
$\log \cos t$	9.04704	$\log \sin^2 t$	9.9946
log 1st term	2.75739	$\log \tan h$	0.0196
		$\log \frac{1}{2} \sin 1''$	4.3845
		log 2d term	1.8194

$$
\begin{aligned}
h &= 46^\circ\ 17'\ 28'' \\
\text{1st term} &= -\quad 9\ \ 32\ .0 \\
\text{2d} \quad " &= +\quad 1\ \ \ 6\ .0 \\
\varphi &= 46\quad 9\ \ \ 2\ .0
\end{aligned}
$$

By the Tables in the Almanac, $\varphi = 46^\circ\ 9'\ 1''$

177. If we take all the terms of (e) except the last, which we have shown to be insignificant, we have the formula

$$
\begin{aligned}
\varphi = h - p \cos t &+ \tfrac{1}{2} p^2 \sin 1'' \sin^2 t \tan h \\
&- \tfrac{1}{3} p^3 \sin^2 1'' \cos t \sin^2 t + \tfrac{1}{8} p^4 \sin^3 1'' \sin^4 t \tan^3 h \qquad (301)
\end{aligned}
$$

which is exact within $0''.01$ for all latitudes less than 75°, and serves for the reduction of the most refined observations with first-class instruments.

If we have taken a number of altitudes in succession, the separate reduction of each by this formula will be very laborious. To facilitate the operation, PETERSEN has computed very convenient tables, which are given in SCHUMACHER'S *Hülfstafeln*, edited by WARNSTORFF. These tables give the values of the following quantities:

$$
\begin{aligned}
\alpha &= p_0 \cos t + \tfrac{1}{3} p_0^3 \sin^2 1'' \cos t \sin^2 t \\
\beta &= \tfrac{1}{2} p_0^2 \sin 1'' \sin^2 t \\
\lambda &= \tfrac{1}{3} p (p^2 - p_0^2) \sin^2 1'' \cos t \sin^2 t \\
\mu &= \tfrac{1}{8} p^4 \sin^3 1'' \sin^4 t \tan^3 h
\end{aligned}
$$

in which $p_0 = 1^\circ\ 30' = 5400''$. Then, putting

$$A = \frac{p}{p_0}$$

$$\log A = \log p - 3.7323938$$

the formula (301) becomes

$$\varphi = h - (A\alpha + \lambda) + A^2\beta \tan h + \mu$$

Putting then

$$\left.\begin{aligned} x &= A\alpha + \lambda \\ y &= A^2\beta \tan h + \mu \\ \text{we have}\qquad \varphi &= h - x + y \end{aligned}\right\} \quad (302)$$

or, when the zenith distance ζ is observed,

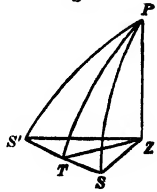
$$\left.\begin{aligned} x &= A\alpha + \lambda \\ y &= A^2\beta \cot \zeta + \mu \\ 90^\circ - \varphi &= \zeta + x - y \end{aligned}\right\} \quad (303)$$

The first table gives α with the argument t; the second, β with the argument t; the third, λ with the arguments p and t; and the fourth, μ with the arguments y and φ, φ being used for h in so small a term.

To reduce a series of altitudes or zenith distances by these tables, we take for h or ζ the mean of the true altitudes or zenith distances; for α and β the means of the tabular numbers corresponding to the several hour angles, with which we find $A\alpha$ and $A^2\beta \cot \zeta$. The mean values of the very small quantities λ and μ are sensibly the same as the values corresponding to the mean of the hour angles; so that λ is taken out but once for the arguments polar distance and mean hour angle, and μ is taken with the arguments φ and the approximate value of $y = A^2\beta \cot \zeta$. Illustrative examples are given in connection with the tables.

FIFTH METHOD.—BY TWO ALTITUDES OF THE SAME STAR, OR DIFFERENT STARS, AND THE ELAPSED TIME BETWEEN THE OBSERVATIONS.

Fig. 25.



178. Let S and S', Fig. 25, be any two points of the celestial sphere, Z the zenith of the observer, P the pole. Suppose that the altitudes of stars seen at S and S', either at the same time or different times, are observed, and that the observer has the means of determining the angle SPS'; also that the right ascensions and declinations of the stars are known. From these data we can find *both the latitude and the local time.* A graphic solution (upon an artificial globe) is indeed quite simple, and it will throw light upon the analytic solution. With the known polar distances of the stars and the angle SPS',

let the triangle SPS' be constructed. From S and S' as poles describe small circles whose radii (on the surface of the sphere) are the given zenith distances of S and S': these small circles intersect in the zenith Z of the observer, and, consequently, determine the distance PZ, or the co-latitude, and at the same time the hour angles ZPS and ZPS', from either of which with the star's right ascension we deduce the local time. This graphic solution shows clearly that the problem has, in general, two solutions; for the small circles described from S and S' as poles intersect in two points, and thus determine the zenith of another observer who at the same instants of time might have observed the same altitudes of the same stars. The analytic solution will, therefore, also give two values of the latitude; but in practice the observer always has an approximate knowledge of the latitude, which generally suffices to distinguish the true value.

Let us consider first the most general case.

(A.) *Two different stars observed at different times.*—Let

h, h' = the true altitudes, corrected for refraction, &c.,
T, T' = the clock times of observation,
$\Delta T, \Delta T'$ = the corresponding corrections of the clock,
α, α' = the right ascensions, and
δ, δ' = the declinations of the stars at the times of the observations respectively,
t, t' = the hour angles of the stars at the times of the observations respectively,
$\lambda = t' - t$ = the difference of the hour angles,
φ = the latitude of the observer:

then we have, if the clock is sidereal,

$$\begin{aligned} t &= T + \Delta T - \alpha \\ t' &= T' + \Delta T' - \alpha' \\ \lambda &= (T' - T) + (\Delta T' - \Delta T) - (\alpha' - \alpha) \end{aligned} \tag{304}$$

a formula which does not require a knowledge of the absolute values of ΔT and $\Delta T'$, but only of their *difference;* that is, of the *rate* of the clock in the interval between the two observations.

If the clock is regulated to mean time, the interval $T' - T + \Delta T' - \Delta T$ is to be converted into a sidereal interval by adding the acceleration, Art. 49.

We have next to obtain formulæ for determining φ and t or t'

from the data h, h', δ, δ', and λ. I shall give two general solutions, the first of which is the one commonly known.

First Solution.—Let the observed points S and S' be joined by an arc of a great circle SS'. In the triangle PSS' there are given the sides $PS = 90° - \delta$, $PS' = 90° - \delta'$, and the angle $SPS' = \lambda$, from which we find the third side $SS' = B$, and the angle $PS'S = P$, by the formulæ [𝔄 of Art. 10]

$$\begin{aligned} \cos B &= \sin\delta' \sin\delta + \cos\delta' \cos\delta \cos\lambda \\ \sin B \cos P &= \cos\delta' \sin\delta - \sin\delta' \cos\delta \cos\lambda \\ \sin B \sin P &= \cos\delta \sin\lambda \end{aligned}$$

or, adapted for logarithmic computation,

$$\left.\begin{aligned} m \sin M &= \sin\delta \\ m \cos M &= \cos\delta \cos\lambda \\ \cos B &= m \cos(M - \delta') \\ \sin B \cos P &= m \sin(M - \delta') \\ \sin B \sin P &= \cos\delta \sin\lambda \end{aligned}\right\} \qquad (305)$$

In the triangle ZSS' there are now known the three sides $ZS = 90° - h$, $ZS' = 90° - h'$, $SS' = B$, and hence the angle $ZS'S = Q$, by the formula employed in Art. 22:

$$\sin\tfrac{1}{2}Q = \sqrt{\left(\frac{\cos\frac{1}{2}(h' + h + B)\sin\frac{1}{2}(h' - h + B)}{\cos h' \sin B}\right)} \qquad (306)$$

Now, putting

$$q = P - Q$$

there are known in the triangle PZS' the sides $PS' = 90° - \delta'$, $ZS' = 90° - h'$, and the angle $PS'Z = q$, from which the side $PZ = 90° - \varphi$, and the angle $S'PZ = t'$, are found by the formulæ

$$\begin{aligned} \sin\varphi &= \sin\delta' \sin h' + \cos\delta' \cos h' \cos q \\ \cos\varphi \cos t' &= \cos\delta' \sin h' - \sin\delta' \cos h' \cos q \\ \cos\varphi \sin t' &= \cos h' \sin q \end{aligned}$$

or, adapted for logarithmic computation,

$$\left.\begin{aligned} n \sin N &= \sin h' \\ n \cos N &= \cos h' \cos q \\ \sin\varphi &= n \cos(N - \delta') \\ \cos\varphi \cos t' &= n \sin(N - \delta') \\ \cos\varphi \sin t' &= \cos h' \sin q \end{aligned}\right\} \qquad (307)$$

The formulæ (305) and (307) leave no doubt as to the quadrant in which the unknown quantities are to be taken. But we may take the radical in (306) with either the positive or the negative sign, and $\frac{1}{2}Q$, therefore, either in the first or fourth quadrant. If we take $\frac{1}{2}Q$ always in the first quadrant, the values of q will be

$$q = P \mp Q$$

and either value may be used in (307); whence two values of φ and t'. That value of φ, however, which agrees best with the known approximate latitude is to be taken as the true value. There is also another method of distinguishing which value of q will give the true solution; for, if A' and A are the azimuths of S' and S, we have in the triangle ZSS' the angle $SZS' = A' - A$, and

$$\sin Q = \sin (A' - A) \frac{\cos h}{\sin B}$$

in which $\cos h$ and $\sin B$ are always positive. Hence Q is positive or negative according as $A' - A$ is less or greater than 180°. The observer will generally be able to decide this: the only cases of doubt will be those where A' and A are very nearly equal or where $A' - A$ is very nearly 180°; but, as we shall see hereafter, these cases are very unfavorable for the determination of the latitude, and are, therefore, always to be avoided in practice

If the great circle SS' passes north of the zenith, we shall have $A' - A$ negative, or greater than 180°: hence we have also this criterion: *we must take $q = P - Q$ or $q = P + Q$ according as the great circle SS' passes south or north of the zenith.*

Second Solution.—Bisect the arc SS', Fig. 25, in T; join PT and ZT, and put

$$\begin{aligned}
C &= ST = S'T, \\
D &= \text{the declination of } T = 90° - PT, \\
H &= \text{the altitude of } T = 90° - ZT, \\
\tau &= \text{the hour angle of } T = ZPT, \\
P &= \text{the angle } PTS, \\
Q &= \text{the angle } ZTS, \\
q &= \text{the angle } PTZ.
\end{aligned}$$

We have in the triangle PSS' [Sph. Trig. (25)]

$$\sin^2 C = \sin^2 \tfrac{1}{2}(\delta - \delta') \cos^2 \tfrac{1}{2}\lambda + \cos^2 \tfrac{1}{2}(\delta + \delta') \sin^2 \tfrac{1}{2}\lambda$$

or, adapted for logarithmic computation, by introducing an auxiliary angle E,

$$\left.\begin{aligned}\sin C \sin E &= \sin \tfrac{1}{2}(\delta - \delta') \cos \tfrac{1}{2}\lambda \\ \sin C \cos E &= \cos \tfrac{1}{2}(\delta + \delta') \sin \tfrac{1}{2}\lambda\end{aligned}\right\} \quad (308)$$

In the triangle SPT we have the angle $PTS = P$, and therefore in the triangle $S'PT$ we have the angle $PTS' = 180° - P$, the cosine of which will be $= -\cos P$: hence, from these triangles we have the equations

$$\begin{aligned}\sin D \cos C + \cos D \sin C \cos P &= \sin \delta \\ \sin D \cos C - \cos D \sin C \cos P &= \sin \delta'\end{aligned}$$

whence

$$\begin{aligned}2 \sin D \cos C &= \sin \delta + \sin \delta' \\ 2 \cos D \sin C \cos P &= \sin \delta - \sin \delta'\end{aligned}$$

$$\left.\begin{aligned}\sin D &= \frac{\sin \tfrac{1}{2}(\delta + \delta') \cos \tfrac{1}{2}(\delta - \delta')}{\cos C} \\ \cos P &= \frac{\cos \tfrac{1}{2}(\delta + \delta') \sin \tfrac{1}{2}(\delta - \delta')}{\cos D \sin C}\end{aligned}\right\} \quad (309)$$

which determine D and P after C has been found from (308).

In precisely the same manner we derive from the triangles ZTS and ZTS' the equations

$$\left.\begin{aligned}\sin H &= \frac{\sin \tfrac{1}{2}(h + h') \cos \tfrac{1}{2}(h - h')}{\cos C} \\ \cos Q &= \frac{\cos \tfrac{1}{2}(h + h') \sin \tfrac{1}{2}(h - h')}{\cos H \sin C}\end{aligned}\right\} \quad (310)$$

Then in the triangle PTZ we have the angle PTZ, by the formula

$$q = P - Q$$

and hence the equations

$$\begin{aligned}\sin \varphi &= \sin D \sin H + \cos D \cos H \cos q \\ \cos \varphi \cos \tau &= \cos D \sin H - \sin D \cos H \cos q \\ \cos \varphi \sin \tau &= \cos H \sin q\end{aligned}$$

To adapt these for logarithmic computation, let β and γ be determined by the conditions*

$$\left.\begin{aligned} \cos\beta\sin\gamma &= \cos H\cos q \\ \cos\beta\cos\gamma &= \sin H \\ \sin\beta &= \cos H\sin q \end{aligned}\right\} \tag{311}$$

then φ and τ are found by the equations

$$\left.\begin{aligned} \sin\varphi &= \cos\beta\sin(D+\gamma) \\ \cos\varphi\cos\tau &= \cos\beta\cos(D+\gamma) \\ \cos\varphi\sin\tau &= \sin\beta \end{aligned}\right\} \tag{312}$$

To find the hour angles t and t', let

$$x = \tau - \tfrac{1}{2}(t' + t)$$

then, since $\frac{1}{2}\lambda = \frac{1}{2}(t' - t)$, we have

$$\begin{aligned} \tfrac{1}{2}\lambda + x &= \tau - t = \text{the angle } TPS, \\ \tfrac{1}{2}\lambda - x &= t' - \tau = \text{the angle } TPS', \end{aligned}$$

and from the triangles PTS and PTS' we have

$$\frac{\sin(\frac{1}{2}\lambda + x)}{\sin C} = \frac{\sin P}{\cos\delta} \qquad \frac{\sin(\frac{1}{2}\lambda - x)}{\sin C} = \frac{\sin P}{\cos\delta'}$$

whence

$$\frac{\sin(\frac{1}{2}\lambda + x) - \sin(\frac{1}{2}\lambda - x)}{\sin(\frac{1}{2}\lambda + x) + \sin(\frac{1}{2}\lambda - x)} = \frac{\cos\delta' - \cos\delta}{\cos\delta' + \cos\delta}$$

and, consequently,

$$\tan x = \tan\tfrac{1}{2}(\delta + \delta')\tan\tfrac{1}{2}(\delta - \delta')\tan\tfrac{1}{2}\lambda \tag{313}$$

Hence, finally,

$$\left.\begin{aligned} t &= \tau - x - \tfrac{1}{2}\lambda \\ t' &= \tau - x + \tfrac{1}{2}\lambda \end{aligned}\right\} \tag{314}$$

As in the first solution, the value of q will become $= P + Q$ when the arc joining the observed places of the stars passes north of the zenith.

EXAMPLE.—1856 March 5, in the assumed Latitude 39° 17′ N. and Longitude $5^h\ 6^m\ 36^s$ W., suppose the following altitudes

* The equations (311) can always be satisfied, since the sum of their squares gives the identical equation $1 = 1$.

already corrected for refraction) to have been obtained; the time being noted by a mean solar chronometer whose daily rate was $10^s.32$ *losing*. The star *Arcturus* was not far from the prime vertical east of the meridian; *Spica* was near the meridian.

Arcturus, $h =$	$18°\ 6'\ 30''$	Chronometer T	$=$	$9^h\ 40^m\ 24^s.8$
Spica, $h' =$	$40\ \ 4\ \ 43$	" T'	$=$	$14\ \ 38\ \ 16.7$
		$T' - T$	$=$	$4\ \ 57\ \ 51.9$
$\delta =$	$+19°\ 55'\ 44''.6$	Corr. for rate	$=$	$+\ \ 2.1$
$\delta' =$	$-10\ \ 24\ \ 39\ .5$	Red. to sid. time	$=$	$+\ \ 48.9$
		Sid. interval	$=$	$4\ \ 58\ \ 42.9$
$\alpha =$	$14^h\ \ 9^m\ \ 6^s.79$	$\alpha - \alpha'$	$=$	$0\ \ 51\ \ 29.1$
$\alpha' =$	$13\ \ 17\ \ 37.72$	λ	$=$	$5\ \ 50\ \ 12.0$
			$=$	$87°\ 33'\ \ 0''.$

According to our *first solution*, we obtain,

by (305), $B = 91°\ 15'\ 52''.5$ $\quad P = 69°\ 57'\ 54''.7$
and, by (306), $\quad Q = 64\ \ 51\ \ 24\ .8$
whence $\quad q = 5\ \ \ 6\ \ 29\ .9$

Then, by (307), we find

$$\varphi = 39°\ 17'\ 20'' \qquad t' = 5°\ 3'\ 0'' = 0^h\ 20^m\ 12^s.$$

	α'	$= 13\ \ 17\ \ 37.72$
Sidereal time of the observation of *Spica*		$= 13\ \ 37\ \ 49.72$
Sidereal time at mean Greenwich noon		$= 22\ \ 53\ \ 39.83$
		$14\ \ 44\ \ \ 9.89$
Acceleration for $14^h\ 44^m\ 9^s.89$		$= -\ \ \ 2\ \ 24.85$
" longitude		$= -\ \ \ \ \ \ \ 50.23$
Mean time of the observation of *Spica*		$= 14\ \ 40\ \ 54.81$
Chronometer correction at that time, $\Delta T'$		$= +\ \ \ 2^m\ 38^s.11$

According to the *second solution*, we prepare the quantities

$$\tfrac{1}{2}\lambda = 43°\,46'\,30'' \quad \tfrac{1}{2}(\delta+\delta') = 4°\,45'\,32''.6 \quad \tfrac{1}{2}(h+h') = 29°\ \ 5'\,36''.5$$
$$\tfrac{1}{2}(\delta-\delta') = 15\ \ 10\ \ 12\ .1 \quad \tfrac{1}{2}(h-h') = -10\ \ 59\ \ 6\ .5$$

with which we find, by (308), (309), and (310),

$\log\tan E = 9.437854$	$D = 6°\ 34'\ 32''.0$
$\log\sin C = 9.854225$	$P = 68\ \ 27\ \ 22\ .2$
$\log\cos C = 9.844639$	$Q = 108\ \ 35\ \ 12\ .1$
$\log\sin H = 9.834176$	$q = -40\ \ \ 7\ \ 49\ .9$
$\log\cos H = 9.863785$	

(The auxiliaries C and H are not themselves required: we take their cosines from the table, employing the sines as arguments.) We now find, by (311), (312), (313), and (314).

$\log \sin \beta = n9.673029$	$\tau = 322° \ 30' \ 51''.3$
$\log \cos \beta = \ \ 9.945532$	$x = \ \ 1 \ \ 14 \ \ 21 .3$
$\gamma = 39° \ 18' \ \ 4''.0$	$T - x = 321 \ \ 16 \ \ 30$
$D + \gamma = 45 \ \ 52 \ \ 36 .0$	$= 21^h \ 25^m \ \ 6^s$
$\varphi = 39 \ \ 17 \ \ 20 .$	$\tfrac{1}{2}\lambda = \ \ 2 \ \ 55 \ \ \ 6$
	$t = 18 \ \ 30 \ \ \ 0$
	$t' = \ \ 0 \ \ 20 \ \ 12$

agreeing precisely with the results of the first solution.

179. In the observation of lunar distances, as we shall see hereafter, the altitudes of the moon and a star are observed at the same instant with the distance of the objects. The observed distance is reduced to the true geocentric distance, which is the arc B of the above *first solution*, or $2\,C$ of the *second*. The observation of a lunar distance with the altitudes of the objects furnishes, therefore, the data for finding the latitude, the local time, and the longitude.

180. (B.) *A fixed star observed at two different times.*—In this case the declination is the same at both observations, and the general formulæ of the preceding articles take much more simple forms. The hour angle λ is in this case merely the elapsed sidereal time between the observations, the formula (304), since $\alpha = \alpha'$, becoming here

$$\lambda = (T' - T) + (\Delta T' - \Delta T) \tag{315}$$

Putting δ' for δ in (308) and (309), we find $E = 0$, $\cos P = 0$, $P = 90°$; and C and D are found by the equations

$$\sin C = \cos \delta \sin \tfrac{1}{2}\lambda, \qquad \sin D = \frac{\sin \delta}{\cos C} \tag{316}$$

Since we have $q = P - Q = 90° - Q$, the last two equations of (311) give

$$\sin \beta = \cos H \cos Q, \qquad \cos \gamma = \sin H \sec \beta$$

which, by (310), become*

$$\left.\begin{aligned}\sin\beta &= \frac{\cos\frac{1}{2}(h+h')\sin\frac{1}{2}(h-h')}{\sin C}\\ \cos\gamma &= \frac{\sin\frac{1}{2}(h+h')\cos\frac{1}{2}(h-h')}{\cos\beta\cos C}\end{aligned}\right\}\quad(317)$$

Then φ and τ are found by (312), or rather by the following:

$$\left.\begin{aligned}\sin\varphi &= \cos\beta\sin(D+\gamma)\\ \tan\tau &= \frac{\tan\beta}{\cos(D+\gamma)}\quad\text{or}\ \sin\tau = \frac{\sin\beta}{\cos\varphi}\end{aligned}\right\}\quad(318)$$

The hour angles at the two observations are

$$\left.\begin{aligned}t &= \tau - \tfrac{1}{2}\lambda\\ t' &= \tau + \tfrac{1}{2}\lambda\end{aligned}\right\}\quad(319)$$

Here γ, being determined by its cosine, may be either a positive or a negative angle, and we obtain two values of the latitude by taking either $D+\gamma$ or $D-\gamma$ in (318). The first value will be taken when the great circle joining the two positions of the star passes north of the zenith; the second, when it passes south of the zenith.

The solution may be slightly varied by finding D by the formula

$$\tan D = \frac{\tan\delta}{\cos\frac{1}{2}\lambda}\quad(320)$$

obtained directly from the triangle *PTS* (Fig. 25), which is right-angled at T when the declinations are equal. We can then dispense with C by writing the formulæ (317) as follows:

$$\left.\begin{aligned}\sin\beta &= \frac{\cos\frac{1}{2}(h+h')\sin\frac{1}{2}(h-h')}{\cos\delta\sin\frac{1}{2}\lambda}\\ \cos\gamma &= \frac{\sin\frac{1}{2}(h+h')\cos\frac{1}{2}(h-h')\sin D}{\cos\beta\sin\delta}\end{aligned}\right\}\quad(321)$$

* The formulæ (315), (316), and (317) are essentially the same as those first given for this case by M. Caillet in his *Manuel du Navigateur*, Nantes, 1818.

181. (C.) *The sun observed at two different times.*—In this case the hour angle λ is the elapsed apparent solar time. If then the times T and T' are observed by a mean solar chronometer, and the equation of time at the two observations is e and e' (positive when additive to apparent time), we have

$$\lambda = (T' - T) + (\Delta T' - \Delta T) - (e' - e) \tag{322}$$

Taking then the declinations δ and δ' for the two times of observation, we can proceed by the general methods of Art. 178.

But, as the declinations differ very little, we can assume their mean—*i.e.* the declination at the middle instant between the observations—as a constant declination, and compute at least an approximate value of the latitude by the simple formulæ for a fixed star in the preceding article. We can, however, readily correct the resulting latitude for the error of this assumption. To obtain the correction, we recur to the rigorous formulæ of our second solution in Art. 178. The change of the sun's declination being never greater than 1′ per hour, we may put $\cos \frac{1}{2}(\delta - \delta') = 1$. Making this substitution in (308) and (309), and putting δ for $\frac{1}{2}(\delta + \delta')$ so that δ will signify the mean of the declinations, and $\Delta\delta$ for $\frac{1}{2}(\delta' - \delta)$ so that $\Delta\delta$ will signify one-half the *increase* of the sun's declination from the first to the second observation (positive when the sun is moving northward), we shall have

$$\Delta\delta = - \tfrac{1}{2}(\delta - \delta')$$

$$\tan E = - \frac{\sin \Delta\delta}{\cos \delta \tan \frac{1}{2}\lambda}$$

But $\Delta\delta$ diminishes with λ, so that E always remains a small quantity of the same order as $\Delta\delta$; and hence we may also put $\cos E = 1$. Thus the second equation of (308) gives

$$\sin C = \cos \delta \sin \tfrac{1}{2}\lambda$$

and the first of (309)

$$\sin D = \frac{\sin \delta}{\cos C}$$

which are the same as (316), as given for the case where the declination is absolutely invariable. Hence no sensible error is produced in the values of C and D by the use of the mean de-

clination. But by the second equation of (309) P will no longer be exactly 90°: if then we put

$$P = 90° + \Delta P$$

we have, by that equation,

$$\sin \Delta P = \frac{\cos \delta \sin \Delta\delta}{\cos D \sin C} = -\frac{\sin \Delta\delta}{\cos D \sin \frac{1}{2}\lambda}$$

or simply

$$\Delta P = -\frac{\Delta\delta}{\cos D \sin \frac{1}{2}\lambda}$$

Then, since $q = P - Q$, we have

$$q = 90° - Q + \Delta P$$

The rigorous formula for the latitude is

$$\sin \varphi = \sin D \sin H + \cos D \cos H \cos q$$

in which when we use the mean declination we take $q = 90° - Q$: therefore, to find the correction of φ corresponding to a correction of $q = \Delta P$, we have by differentiating this equation, D and H being invariable,

$$\cos \varphi . \Delta\varphi = -\cos D \cos H \sin q . \Delta P$$

$$= -\frac{\Delta\delta \cos H \cos Q}{\sin \frac{1}{2}\lambda}$$

We have found in the preceding article $\sin \beta = \cos H \cos Q$; and hence

$$\Delta\varphi = -\frac{\Delta\delta \sin \beta}{\cos \varphi \sin \frac{1}{2}\lambda} \qquad (323)$$

In the case of the sun, therefore, we compute the approximate latitude φ by the formulæ (316), (317), and (318), employing for δ the mean declination. We then find $\Delta\varphi$ by (323) (which involves very little additional labor, since the logarithms of $\sin \beta$ and $\sin \frac{1}{2}\lambda$ have already occurred in the previous computation), and then we have the true latitude

$$\varphi' = \varphi + \Delta\varphi$$

If we wish also to correct the hour angle τ found by this method, we have, from the second equation of (47) applied to

the triangle PTZ (taking b and c to denote the sides PT and ZT, which are here constant),

$$\Delta\tau = \frac{\cos H \cos A}{\cos \varphi} \cdot \Delta P$$

in which A is the azimuth of the point T. But we have in the triangle PTZ

$$\cos H \cos A = \sin \varphi \cos D \cos \tau - \cos \varphi \sin D$$

Substituting this and the value of ΔP, we have

$$\Delta\tau = \frac{\Delta\delta}{\sin \frac{1}{2}\lambda} (\tan \varphi \cos \tau - \tan D)$$

and, substituting the value of $\tan D$ (320),

$$\Delta\tau = \frac{\Delta\delta}{\sin \frac{1}{2}\lambda} \left(\tan \varphi \cos \tau - \frac{\tan \delta}{\cos \frac{1}{2}\lambda} \right)$$

When this correction is added to τ, we have the value that would be found by the rigorous formulæ, and we have then to apply also the correction x according to (314). In the present case we have, by (313),

$$x = - \Delta\delta \tan \delta \tan \tfrac{1}{2}\lambda$$

and the complete formulæ for the hour angles t and t' become

$$\begin{aligned} t &= \tau + \Delta\tau - x - \tfrac{1}{2}\lambda \\ t' &= \tau + \Delta\tau - x + \tfrac{1}{2}\lambda \end{aligned}$$

Putting

$$y = \Delta\tau - x$$

we find, by substituting the above values of $\Delta\tau$ and x,

$$y = \Delta\delta \cdot \left(\frac{\tan \varphi \cos \tau}{\sin \frac{1}{2}\lambda} - \frac{\tan \delta}{\tan \frac{1}{2}\lambda} \right) \tag{324}$$

and then we have

$$\left.\begin{aligned} t &= \tau + y - \tfrac{1}{2}\lambda \\ t' &= \tau + y + \tfrac{1}{2}\lambda \end{aligned}\right\} \tag{325}$$

The corrections $\Delta\varphi$ and y are computed with sufficient accuracy with four-place logarithms, and, therefore, add but little to the labor of the computation. Nevertheless, when both latitude and time are required with the greatest precision, it will be preferable to compute by the rigorous formulæ.

EXAMPLE.—1856 March 10, in Lat. 24° N., Long. 30° W., suppose we obtain two altitudes of the sun as follows, all corrections being applied: find the latitude.

At app. time	$0^h\ 30^m$	$h = 61°\ 11'\ 38''.3$	$(\delta) = -3°\ 51'\ 52''.8$
"	$4\ \ 30$	$h' = 18\ \ 46\ \ 35\ .8$	$(\delta') = -3\ \ 47\ \ 57\ .4$
$\frac{1}{2}\lambda =$	$2^h\ \ 0^m$	$\frac{1}{2}(h + h') = 39\ \ 59\ \ 7\ .1$	$\delta = -3\ \ 49\ \ 55\ .1$
	$= 30°\ \ 0'$	$\frac{1}{2}(h - h') = 21\ \ 12\ \ 31\ .3$	$\Delta\delta = +\ \ 1'\ 57''.7$

The following is the form of computation by the formulæ (316), (317), and (318), employed by BOWDITCH in his *Practical Navigator*, the reciprocals of the equations (316) and of the second of (317) being used to avoid taking arithmetical complements. This form is convenient when the tables give the secants and cosecants, as is usual in nautical works.

cosec $\frac{1}{2}\lambda$ 0.301030					
sec δ 0.000972					cosec n1.175024
cosec C 0.302002	cos 9.937854				cos 9.937854
cos $\frac{1}{2}(h + h')$ 9.884347	cosec 0.192065	$D =$	$-\ 4°\ 25'\ 21''.3$		cosec n1.112878
sin $\frac{1}{2}(h - h')$ 9.558428	sec 0.030459				
sin β 9.744777	cos 9.919829				cos 9.919829
	sec 0.080207	$\gamma =$	$33\ \ 45\ \ 38\ .0$		
		$D + \gamma =$	$29\ \ 20\ \ 16\ .7$		sin 9.690161
		$\phi =$	$24°\ \ 2'\ 23''.2$		sin 9.609990

If the approximate latitude had not been given, we might also have taken $D - \gamma = -38°\ 10'\ 59''.3$, and then we should have had

$$
\begin{array}{lll}
 & \cos\beta & 9.919829 \\
 & \sin(D - \gamma) & n9.791113 \\
\varphi = -30°\ 55'\ 44''.3 & \sin\varphi & n9.710942
\end{array}
$$

To correct the first value of the latitude for the change of declination, we have, by (323),

$$
\begin{array}{lll}
 & \log\Delta\delta & 2.0708 \\
 & \sin\beta & 9.7448 \\
 & \text{cosec}\ \frac{1}{2}\lambda & 0.3010 \\
 & \sec\varphi & 0.0394 \\
\Delta\varphi = -143''.2 & \log\Delta\varphi & n2.1560
\end{array}
$$

and hence the true latitude is

$$\varphi' = 24°\ 2'\ 23''.2 - 2'\ 23''.2 = 24°\ 0'\ 0''$$

which agrees exactly with the value computed by the rigorous formulæ.

The approximate time is found by the last equation of (318) with but one new logarithm: we have

$$
\begin{array}{lr}
 & \sin\beta \;\; 9.744777 \\
 & \cos\varphi \;\; \underline{9.960596} \\
\tau = 37^\circ\ 28'\ 23'' & \sin\tau \;\; 9.784181
\end{array}
$$

By (324) and (325), we find

$$
\begin{array}{lrlr}
\log \Delta\delta & 2.0708 & \log\Delta\delta & 2.0708 \\
\operatorname{cosec}\frac{1}{2}\lambda & 0.3010 & \cot\frac{1}{2}\lambda & 0.2386 \\
\tan\varphi & 9.6494 & \tan\delta & \underline{n8.8259} \\
\cos\tau & \underline{9.8996} & -13''.7 & n1.1353 \\
+83''.3 & 1.9208 & &
\end{array}
$$

$$
\begin{aligned}
y &= +83''.3 - (-13''.7) = +97'' \\
\tau + y &= 37^\circ\ 30'\ 0'' = 2^h\ 30^m\ 0^s \\
t &= 0^h\ 30^m\ 0^s \qquad t' = 4^h\ 30^m\ 0^s
\end{aligned}
$$

which are perfectly exact.

182. (D.) *Two equal altitudes of the same star, or of the sun.*—This case is a very useful one in practice with the sextant when equal altitudes have been taken for determining the time by the method of Art. 140. By putting $h' = h$ in the formulæ (317), we find $\sin\beta = 0$, $\cos\beta = 1$, and hence (318) gives $\sin\varphi = \sin(D + \gamma)$, or $\varphi = D + \gamma$. We have, therefore, for this case, by (320) and (321),

$$
\left.
\begin{aligned}
\tan D = \frac{\tan\delta}{\cos\frac{1}{2}\lambda} \qquad \cos\gamma = \frac{\sin h \sin D}{\sin\delta} \\
\varphi = D \pm \gamma
\end{aligned}
\right\} \quad (326)
$$

which is the method of Art. 164 applied as proposed in Art. 165. We give γ the double sign, and obtain two values of the latitude, for the reasons already stated.

The time will be most conveniently found by Art. 140. The method there given is, however, embraced in the solution of the present problem. For, since we here have $\sin\beta = 0$, we also have $\tau = 0$, and the hour angles given by (325) become

$$
\begin{aligned}
t &= y - \tfrac{1}{2}\lambda \\
t' &= y + \tfrac{1}{2}\lambda
\end{aligned}
$$

the mean of which gives

$$\tfrac{1}{2}(t + t') - y = 0$$

that is, $-y$ is the correction of the mean of the times of observation to obtain the time of apparent noon $= 0^h$. This correction was denoted in Art. 140 by ΔT_0; and since $\cos \tau = 0$, the formula (324) gives, when divided by 15 to reduce it to seconds of time,

$$\Delta T_0 = -\frac{\Delta\delta \tan\varphi}{15 \sin\frac{1}{2}\lambda} + \frac{\Delta\delta \tan\delta}{15 \tan\frac{1}{2}\lambda}$$

which is identical with (262). Thus it appears that (262) is but a particular case of the formula (324), which I suppose to be new.

The latitude found by (326) will require no correction, since $\sin\beta = 0$, and therefore $\Delta\varphi = 0$.

NOTE.—The preceding solutions of the problem of finding the latitude and time by two altitudes leave nothing to be desired on the score of completeness and accuracy; but some brief approximative methods, used only by navigators, will be treated of among the methods of finding the latitude at sea, and in Chapter VIII.

183. I proceed to discuss the effect of errors in the data upon the latitude and time determined by two altitudes, and hence also the conditions most favorable to accuracy.

Errors of altitude.—Since the hour angles t and t' are connected by the relation $t' = t + \lambda$, the errors of altitude produce the same errors in both; for, λ being correct, we have $dt' = dt$; and for either of these we may also put $d\tau$, since we have, in the second general solution of Art. 178, $\tau - x = \frac{1}{2}(t + t')$, and x is not affected by errors of altitude. Now, we have the general relations

$$\left.\begin{aligned} \sin h &= \sin\varphi \sin\delta + \cos\varphi \cos\delta \cos t \\ \sin h' &= \sin\varphi \sin\delta' + \cos\varphi \cos\delta' \cos t' \end{aligned}\right\} \quad (327)$$

which, by differentiation relatively to h, φ, and t, give [see equations (51)]

$$\begin{aligned} dh &= -\cos A\, d\varphi - \cos\varphi \sin A\, d\tau \\ dh' &= -\cos A'\, d\varphi - \cos\varphi \sin A'\, d\tau \end{aligned}$$

in which A and A' denote the azimuths of the two stars, or of the same star at the two observations.

Eliminating $d\tau$ and $d\varphi$ successively, we find

$$\left.\begin{aligned} d\varphi &= -\frac{\sin A'}{\sin (A'-A)}\,dh + \frac{\sin A}{\sin (A'-A)}\,dh' \\ \cos\varphi\, d\tau &= \frac{\cos A'}{\sin (A'-A)}\,dh - \frac{\cos A}{\sin (A'-A)}\,dh' \end{aligned}\right\} \quad (328)$$

These equations show that, in order to reduce the effect of errors of altitude as much as possible, we must make $\sin (A'-A)$ as great as possible, and hence $A'-A$, the difference of the azimuths, should be as nearly 90° as possible.

If we suppose $A'-A=90°$, we have

$$\begin{aligned} d\varphi &= -\sin A'\,dh + \sin A\,dh' \\ \cos\varphi\, d\tau &= \cos A'\,dh - \cos A\,dh' \end{aligned}$$

and, under the same supposition, if one of the altitudes is near the meridian the other will be near the prime vertical. If, then, the altitude h is near the meridian, $\sin A$ will be small while $\sin A'$ is nearly unity, and the error $d\varphi$ will depend chiefly on the term $\sin A'dh$. At the same time, $\cos A$ will be nearly unity and $\cos A'$ small, so that the error $d\tau$ will depend chiefly on the term $\cos A\,dh'$. Hence the accuracy of the resulting latitude will depend chiefly upon that of the altitude near the meridian; and the accuracy of the time chiefly upon that of the altitude near the prime vertical.

If the observations are taken upon the same star observed at equal distances from the meridian. we have $A'=-A$, and the general expressions (328) become

$$d\varphi = -\frac{dh+dh'}{2\cos A}$$

$$\cos\varphi\, d\tau = -\frac{dh-dh'}{2\sin A}$$

The most favorable condition for determining *both* latitude and time from equal altitudes is $\sin A = \cos A$, or $A=45°$.

Errors in the observed clock times.—An error in the observed time may be resolved into an error of altitude; for if we say that dT is the error of T at the observation of the altitude h. it

amounts to saying either that the time $T - dT$ corresponds to the altitude h, or that T corresponds to the altitude $h + dh$, dh being the increase of altitude in the interval dT. We may, therefore, consider the time T as correctly observed while h is in error by the quantity $-dh$. Conversely, we may assume that the altitudes are correct while the times are erroneous. The discussion of the errors under the latter form, while it can lead to no new results, is, nevertheless, sufficiently interesting. We have, from the formula (304),

$$d\lambda = dT' - dT$$

The general equations (327), upon the supposition that h and h are correct, give

$$\begin{aligned} 0 &= -\cos A\, d\varphi - \cos\varphi \sin A\, dt \\ 0 &= -\cos A' d\varphi - \cos\varphi \sin A' (dt + d\lambda) \end{aligned}$$

where we put $dt + d\lambda$ for dt', since $t' = t + \lambda$. Eliminating dt, we find

$$d\varphi = \frac{\cos\varphi \sin A' \sin A}{\sin (A' - A)}\, d\lambda \tag{329}$$

Eliminating $d\varphi$,

$$dt = -\frac{\sin A' \cos A}{\sin (A' - A)}\, d\lambda$$

and similarly

$$dt' = -\frac{\sin A \cos A'}{\sin (A' - A)}\, d\lambda$$

But we have from the formula $\tau - x = \frac{1}{2}(t + t')$

$$d\tau = \tfrac{1}{2}(dt + dt')$$

and hence

$$d\tau = -\frac{\sin (A' + A)}{\sin (A' - A)} \cdot \frac{d\lambda}{2} \tag{330}$$

If we denote the clock correction at the time T by ϑ, we shall have

$$\vartheta = t + \alpha - T$$

and

$$d\vartheta = dt - dT$$

or, if we deduce ϑ from the second observation, the *rate* being supposed correct,

$$d\vartheta = dt' - dT'$$

The mean is

$$d\vartheta = d\tau - \tfrac{1}{2}(dT + dT')$$

Substituting the value of $d\tau$ and also $d\lambda = dT' - dT$, we find, after reduction,

$$d\vartheta = \frac{\sin A \cos A'}{\sin (A' - A)} dT - \frac{\sin A' \cos A}{\sin (A' - A)} dT' \qquad (331)$$

The conclusions above obtained as to the conditions favorable to the accurate determination of either the latitude or the time are, evidently, confirmed by the equations (329) and (331). In addition, we may observe that if $dT' = dT$, we have $d\varphi = 0$ and $d\vartheta = dT$: a *constant* error in noting the clock time produces no error in the latitude, but affects the clock correction by its whole amount.

Errors of declination.—These errors may also be resolved into errors of altitude. By differentiating the equations (327) relatively to h and δ, we find

$$dh = \cos q \, d\delta, \qquad dh' = \cos q' \, d\delta'$$

in which q and q' are the parallactic angles at the two observations respectively. If these values are substituted in (328), the resulting values of $d\varphi$ and $d\tau$ will be the *corrections* required in the latitude and hour angle for the *errors* $d\delta$ and $d\delta'$;* and, denoting these corrections by $\Delta\varphi$ and $\Delta\tau$, we have

$$\left.\begin{aligned} \Delta\varphi &= -\frac{\sin A' \cos q}{\sin (A' - A)} d\delta + \frac{\sin A \cos q'}{\sin (A' - A)} d\delta' \\ \cos \varphi \, \Delta\tau &= \frac{\cos A' \cos q}{\sin (A' - A)} d\delta - \frac{\cos A \cos q'}{\sin (A' - A)} d\delta' \end{aligned}\right\} \qquad (332)$$

If the observation h is on the meridian, and h' on the prime vertical, we have $\Delta\varphi = - d\delta$; and in the same case we have, by

* The error of a quantity and the correction for this error are too frequently confounded. They are numerically equal, but have opposite signs. If a should be $a - x$, it is *too great* by x; its error is $+ x$; but the correction to reduce it to its true value is $- x$.

(328), $d\varphi = + dh$, and the total correction of the latitude $= dh - d\delta$, precisely the same as if the meridian observation were the only one. Hence there is no advantage in combining an observation on the meridian with another remote from it, in the determination of latitude: a simple meridian observation, or, still better, a series of circummeridian observations, is always preferable if the time is approximately known.

When the sun is observed and the mean declination is employed, putting $\Delta\delta = \frac{1}{2}(\delta' - \delta)$, we have $d\delta = \Delta\delta$, $d\delta' = -\Delta\delta$, and, by (332),

$$\Delta\varphi = -\frac{\sin A' \cos q + \sin A \cos q'}{\sin (A' - A)} \cdot \Delta\delta$$

which, by substituting

$$\sin A' = \frac{\sin q' \cos \delta}{\cos \varphi} \qquad \sin A = \frac{\sin q \cos \delta}{\cos \varphi}$$

becomes

$$\Delta\varphi = -\frac{\sin (q' + q) \cos \delta}{\sin (A' - A) \cos \varphi} \cdot \Delta\delta \tag{333}$$

This is but another expression of the correction (323).

If, when the sun is observed, instead of employing the mean declination we employ the declination belonging to the *greater* altitude, which we may suppose to be h, we shall have $d\delta = 0$, $d\delta' = -2\,\Delta\delta$; and, denoting the correction of the latitude in this case by $\Delta'\varphi$, we have, by (332),

$$\Delta'\varphi = -\frac{2 \sin A \cos q'}{\sin (A' - A)} \cdot \Delta\delta = -\frac{2 \sin q \cos q' \cos \delta}{\sin (A' - A) \cos \varphi} \cdot \Delta\delta$$

Under what conditions will $\Delta'\varphi$ be numerically less than $\Delta\varphi$?

First. If both observations are on the same side of the meridian, the condition $\Delta'\varphi < \Delta\varphi$ gives

$$2 \sin q \cos q' < \sin (q' + q)$$

or

$$2 \sin q \cos q' < \sin q' \cos q + \cos q' \sin q$$

whence

$$\tan q < \tan q'$$

which condition is always satisfied when h is the greater altitude

Secondly. If the observations are on different sides of the

meridian, q and q' will have opposite signs, and we shall have, numerically, $\sin(q'-q)$ instead of $\sin(q'+q)$. The condition $\Delta'\varphi < \Delta\varphi$, then, requires that

$$2\sin q \cos q' < \sin q' \cos q - \cos q' \sin q$$

or

$$\tan q < \tfrac{1}{3}\tan q'$$

Therefore $\Delta\varphi'$ will be greater than $\Delta\varphi$ *only* when the observations are on opposite sides of the meridian and $\tan q > \frac{1}{3}\tan q'$. In cases where an approximate result suffices, as at sea, and the correction $\Delta\varphi$ is omitted to save computation, it will be expedient to employ the declination at the greater altitude, except in the single case just mentioned.* But to distinguish this case with accuracy we should have to compute the angles q and q'; and therefore an approximate criterion must suffice. Since the parallactic angles increase with the hour angles, we may substitute for the condition $\tan q > \frac{1}{3}\tan q'$ the more simple one $t > \frac{1}{3}t'$, which gives

$$t > \frac{t'-t}{2}$$

or (t and t' being only the numerical values of the hour angles)

$$t > \tau$$

Hence we derive this very simple practical rule: *employ the sun's declination at the greater altitude, except when the time of this altitude is farther from noon than the middle time, in which case employ the mean declination.*

The greatest error committed under this rule is (nearly) the value of $\Delta\varphi$ in (323), when $\tau = t$. But since in this case $3t = t'$, and $t + t' = \lambda$, we have $\tau = \frac{1}{4}\lambda$, and therefore $\sin\beta = \cos\varphi \sin\tau = \cos\varphi \sin\frac{1}{4}\lambda$. This reduces (323) to $\Delta\varphi = -\frac{1}{2}\Delta\delta \sec\frac{1}{4}\lambda$. Since λ will seldom exceed 6^h, $\Delta\delta$ will not exceed $3'$, and the maximum error will not exceed $1'.6$. In most cases the error will be under $1'$, a degree of approximation usually quite sufficient at sea. Nevertheless, the computation of the correction $\Delta\varphi$ by our formula (323) is so simple that the careful navigator

---

* BOWDITCH and navigators generally employ in all cases the mean declination: but the above discussion proves that, if the cases are not to be distinguished, it will be better always to employ the declination at the greater altitude.

will prefer to employ the mean declination and to obtain the exact result by applying this correction in all cases.

SIXTH METHOD.—BY TWO ALTITUDES OF THE SAME OR DIFFERENT STARS, WITH THE DIFFERENCE OF THEIR AZIMUTHS.

184. Instead of noting the times corresponding to the observed altitudes, we may observe the azimuths, both altitude and azimuth being obtained at once by the Altitude and Azimuth Instrument or the Universal Instrument. The instrument gives the *difference* of azimuths $= \lambda$. The computation of the latitude and the absolute azimuth A of one of the stars may then be performed by the formulæ of the preceding method, only interchanging altitudes and declinations and putting $180° - A$ for t. When A has been found, we may also find t by the usual methods.

SEVENTH METHOD.—BY TWO DIFFERENT STARS OBSERVED AT THE SAME ALTITUDE WHEN THE TIME IS GIVEN.

185. By this method the latitude is found from the declinations of the two stars and their hour angles deduced from the times of observation, *without employing the altitude itself*, so that the result is free from constant errors (of graduation, &c.) of the instrument with which the altitude is observed. Let

Θ, Θ' = the sidereal times of the observations,
α, α' = the right ascensions of the stars,
δ, δ' = the declinations " "
t, t' = the hour angles " "
h = the altitude of either star,
φ = the latitude;

then, the hour angles being found by the equations

$$t = \Theta - \alpha \qquad t' = \Theta' - \alpha'$$

we have

$$\sin h = \sin\varphi \sin\delta + \cos\varphi \cos\delta \cos t$$
$$\sin h = \sin\varphi \sin\delta' + \cos\varphi \cos\delta' \cos t'$$

From the difference of these we deduce

$$\begin{aligned}\tan\varphi\,(\sin\delta' - \sin\delta) &= \cos\delta \cos t - \cos\delta' \cos t' \\ &= \tfrac{1}{2}(\cos\delta - \cos\delta')(\cos t + \cos t') \\ &\quad + \tfrac{1}{2}(\cos\delta + \cos\delta')(\cos t - \cos t')\end{aligned}$$

and, by resolving the quantities in parentheses into their factors,

$$\begin{aligned}\tan\varphi = &\tan\tfrac{1}{2}(\delta' + \delta)\cos\tfrac{1}{2}(t' + t)\cos\tfrac{1}{2}(t' - t)\\ &+ \cot\tfrac{1}{2}(\delta' - \delta)\sin\tfrac{1}{2}(t' + t)\sin\tfrac{1}{2}(t' - t)\end{aligned}$$

If now we put

$$\left.\begin{aligned}m\sin M &= \sin\tfrac{1}{2}(t' - t)\cot\tfrac{1}{2}(\delta' - \delta)\\ m\cos M &= \cos\tfrac{1}{2}(t' - t)\tan\tfrac{1}{2}(\delta' + \delta)\end{aligned}\right\}\quad(334)$$

we have

$$\tan\varphi = m\cos\left[\tfrac{1}{2}(t' + t) - M\right]\qquad(335)$$

The equations (334) determine m and M, and then the latitude is found by (335) in a very simple manner.

It is important to determine the conditions which must govern the selection of the stars and the time at which they are to be observed. For this purpose we differentiate the above expressions for $\sin h$ relatively to φ and t. The error in the hour angles is composed of the error of observation and the error of the clock correction. If we put

T, T' = the observed (sidereal) clock time,
ΔT = the clock correction,
δT = the rate of the clock in a unit of clock time,

we shall have

$$t = T + \Delta T - \alpha,\qquad t' = T' + \Delta T + \delta T(T' - T) - \alpha'$$

Differentiating these, assuming that the *rate* of the clock is sufficiently well known, we have

$$dt = dT + d\Delta T,\qquad dt' = dT' + d\Delta T$$

in which dT, dT' are the errors in the observed times, and $d\Delta T$ the error in the clock correction. The differential equations are then

$$\begin{aligned}dh &= -\cos A\,d\varphi - \cos\varphi\sin A\,dT - \cos\varphi\sin A\,d\Delta T\\ dh &= -\cos A'\,d\varphi - \cos\varphi\sin A'\,dT' - \cos\varphi\sin A'\,d\Delta T\end{aligned}$$

in which A and A' are the azimuths of the stars. The difference of these equations gives

$$\frac{d\varphi}{\cos\varphi} = -\frac{\sin A}{\cos A - \cos A'}\,dT + \frac{\sin A'}{\cos A - \cos A'}\,dT' + \frac{\sin A' - \sin A}{\cos A' - \cos A}\,d\Delta T$$

The denominator $\cos A - \cos A'$ is a maximum when one of the azimuths is zero and the other 180°, that is, when one of the stars is south and the other north of the observer. To satisfy this condition as nearly as possible, two stars are to be selected the mean of whose declinations is nearly equal to the latitude, and the common altitude at which they are to be observed will be equal to or but little less than the meridian altitude of the star which culminates farthest from the zenith. It is desirable, also, that the difference of right ascensions should not be great.

The coefficient of $d\Delta T$ is equal to $-\cot \frac{1}{2}(A' + A)$, which is zero when $\frac{1}{2}(A' + A)$ is 90° or 270°: hence, when the observations are equally distant from the prime vertical, one north and the other south, small errors in the clock correction have no sensible effect.

When the latitude has been found by this method, we may determine the whole error of the instrument with which the altitude is observed; for either of the fundamental equations will give the true altitude, which increased by the refraction will be that which a perfect instrument would give.

186. With the zenith telescope (see Vol. II.) the two stars may be observed at *nearly* the same zenith distance, the small difference of zenith distance being determined by the level and the micrometer. The preceding method may still be used by correcting the time of one of the observations. If at the observed times T, T' the zenith distances are ζ and ζ', the times when the same altitudes would be observed are either

$$T \quad \text{and} \quad T' + \frac{\zeta - \zeta'}{\cos \varphi \sin A'}$$

or,

$$T + \frac{\zeta' - \zeta}{\cos \varphi \sin A} \quad \text{and} \quad T'$$

where $\zeta' - \zeta$ is given directly by the instrument. With the hour angles deduced from these times we may then proceed by (334) and (335).

But it will be still better to compute an approximate latitude by the formulæ (334) and (335), employing the actually observed times, and then to correct this latitude for the difference of zenith distance.

By differentiating the formula

$$\tan \varphi (\sin \delta' - \sin \delta) = \cos \delta \cos t - \cos \delta' \cos t'$$

relatively to φ and t', we have

$$\sec^2 \varphi (\sin \delta' - \sin \delta) \, d\varphi = \cos \delta' \sin t' \, dt' = \sin \zeta \sin A' \, dt'$$

Hence, substituting

$$dt' = dT' = \frac{\zeta - \zeta'}{\cos \varphi \sin A'}$$

we find

$$d\varphi = \frac{\frac{1}{2} (\zeta - \zeta') \sin \zeta \cos \varphi}{\sin \frac{1}{2} (\delta' - \delta) \cos \frac{1}{2} (\delta' + \delta)} \tag{336}$$

and the true latitude will be $\varphi + d\varphi$.

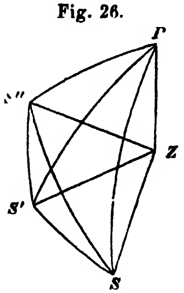
EIGHTH METHOD.—BY THREE OR MORE DIFFERENT STARS OBSERVED AT THE SAME ALTITUDE WHEN THE TIME IS NOT GIVEN.

187. *To find both the latitude and the clock correction from the clock times when three different stars arrive at the same altitude.*

As in the preceding method, we do not employ the common altitude itself; and, as we have one more observation, we can determine the time as well as the latitude.

Let S, S', S'', Fig. 26, be the three points of the celestial sphere, equidistant from the zenith Z, at which the stars are observed. Let

Fig. 26.



T, T', T'' = the clock times of the observations,
ΔT = the clock correction to sidereal time at the time T,
δT = the rate of the clock in a unit of clock time,
$\alpha, \alpha', \alpha''$ = the right ascensions of the stars,
$\delta, \delta', \delta''$ = the declinations " "
t, t', t'' = the hour angles " "
h = the altitude,
φ = the latitude.

Also, let

$$\lambda = t' - t = SPS',$$
$$\lambda' = t'' - t = SPS'';$$

then, since the sidereal times of the observations are

$$\begin{aligned}\Theta &= T + \Delta T \\ \Theta' &= T' + \Delta T + \delta T\,(T' - T) \\ \Theta'' &= T'' + \Delta T + \delta T\,(T'' - T)\end{aligned}$$

and the hour angles are

$$t = \Theta - \alpha, \qquad t' = \Theta' - \alpha', \qquad t'' = \Theta'' - \alpha'',$$

we have

$$\begin{aligned}\lambda &= T' - T + \delta T\,(T' - T) - (\alpha' - \alpha) \\ \lambda' &= T'' - T + \delta T\,(T'' - T) - (\alpha'' - \alpha)\end{aligned}$$

The angles λ and λ' are thus found from the observed clock times, the clock rate, and the right ascensions of the stars. The hour angles of the stars being t, $t + \lambda$, and $t + \lambda'$, we have

$$\begin{aligned}\sin h &= \sin\varphi \sin\delta + \cos\varphi\cos\delta\ \cos t \\ \sin h &= \sin\varphi\sin\delta' + \cos\varphi\cos\delta'\ \cos(t + \lambda) \\ \sin h &= \sin\varphi\sin\delta'' + \cos\varphi\cos\delta''\cos(t + \lambda')\end{aligned}$$

Subtracting the first of these from the second, we have an equation of the same form as that treated in Art. 185, only here we have $t + \lambda$ instead of t'; and hence, by (334), we have

$$\left.\begin{aligned}m\sin M &= \sin\tfrac{1}{2}\lambda\cot\tfrac{1}{2}(\delta' - \delta) \\ m\cos M &= \cos\tfrac{1}{2}\lambda\tan\tfrac{1}{2}(\delta' + \delta)\end{aligned}\right\} \tag{337}$$

and putting

$$N = \tfrac{1}{2}\lambda - M \tag{338}$$

we have, by (335),

$$\tan\varphi = m\cos(t + N) \tag{339}$$

In the same manner, from the first and third observations we have

$$\left.\begin{aligned}m'\sin M' &= \sin\tfrac{1}{2}\lambda'\cot\tfrac{1}{2}(\delta'' - \delta) \\ m'\cos M' &= \cos\tfrac{1}{2}\lambda'\tan\tfrac{1}{2}(\delta'' + \delta)\end{aligned}\right\} \tag{340}$$

$$N' = \tfrac{1}{2}\lambda' - M' \tag{341}$$

$$\tan\varphi = m'\cos(t + N') \tag{342}$$

The problem is then reduced to the solution of the two equations (339) and (342), involving the two unknown quantities φ and t. If we put

$$k\cos(t + N) = \frac{1}{m}$$

there follows also

$$k \cos (t + N') = \frac{1}{m'}$$

and these two equations are of the form treated of in Pl. Trig. Art. 179, according to which, if the auxiliary ϑ is determined by the condition

$$\tan \vartheta = \frac{m}{m'} \tag{343}$$

we shall have

$$\tan [t + \tfrac{1}{2} (N + N')] = \tan (45^\circ - \vartheta) \cot \tfrac{1}{2} (N' - N) \tag{344}$$

which determines t, from which the clock correction is found by the formula

$$\Delta T = \alpha + t - T$$

The latitude is then found from either (339) or (342).*

To determine the conditions which shall govern the selection of the stars, we have, as in Art. 185,

$$\begin{aligned}
dh &= -\cos A \; d\varphi - \cos \varphi \sin A \; dT \; - \cos \varphi \sin A \; d\Delta T \\
dh &= -\cos A' \; d\varphi - \cos \varphi \sin A' \; dT' - \cos \varphi \sin A' \; d\Delta T \\
dh &= -\cos A'' d\varphi - \cos \varphi \sin A'' dT'' - \cos \varphi \sin A'' d\Delta T
\end{aligned}$$

By the elimination of $d\Delta T$, we deduce the following;

$$\begin{aligned}
(\sin A \;\; - \sin A') \, dh &= \sin (A' - A \;) \, d\phi - \cos \phi \sin A' \sin A \;\; (dT' - dT \;) \\
(\sin A' - \sin A'') \, dh &= \sin (A'' - A') \, d\phi - \cos \phi \sin A'' \sin A' \, (dT'' - dT') \\
(\sin A'' - \sin A \;) \, dh &= \sin (A \;\; - A'') \, d\phi - \cos \phi \sin A \;\; \sin A'' (dT \;\; - dT'')
\end{aligned}$$

Adding these three equations together, and putting

$$2K = \sin (A' - A) + \sin (A'' - A') + \sin (A - A'')$$

we find

$$\frac{d\phi}{\cos \phi} = \frac{\sin A (\sin A'' - \sin A')}{2K} dT + \frac{\sin A' (\sin A - \sin A'')}{2K} dT' + \frac{\sin A'' (\sin A' - \sin A)}{2K} dT''$$

By eliminating $d\varphi$ from the same three equations, we shall find

* This simple and elegant solution is due to GAUSS, *Monatliche Correspondenz*, Vol XVIII. p. 287.

$$d\Delta T = \frac{\sin A\,(\cos A' - \cos A'')}{2K}\,dT + \frac{\sin A'\,(\cos A'' - \cos A)}{2K}\,dT'$$
$$+ \frac{\sin A''\,(\cos A - \cos A')}{2K}\,dT''$$

The denominator $2K$ is a maximum when the three differences of azimuth are each 120°,* which is therefore the most favorable condition for determining both the latitude and the time. In general, small differences of azimuth are to be avoided.

Gauss adds the following important practical remarks. It is clear that stars whose altitude varies slowly are quite as available as those which rise or fall rapidly; for the essential condition is not so much that the precise instant when the star reaches a supposed place should be noted, as that at the time which is noted the star should not be sensibly distant from that place. We may, therefore, without scruple select one of the stars near its culmination, or the Pole star at any time, and we can then easily satisfy the above condition. Moreover, at least one of the stars will always change its altitude rapidly when the condition of widely different azimuths is satisfied.

The stars proper to be observed may be readily selected with the aid of an artificial globe, and in general so that the intervals of time between the observations shall be so small that irregularities of the clock or an error in the assumed rate shall not have any sensible influence.

Having selected the stars, the clock times when they severally arrive at the assumed altitude are to be computed in advance within a minute or two, so that the observer may be ready for each observation at the proper time. This is quickly done with four-place logarithms by the formula (267), in which φ and ζ will have the same values for the three stars.

---

* For by putting $a = A' - A$, $a' = A'' - A'$, we have

$$2K = \sin a + \sin a' - \sin (a + a')$$

and, differentiating with reference to a and a', the conditions of maximum or minimum are

$$\cos a - \cos (a + a') = 0$$
$$\cos a' - \cos (a + a') = 0$$

which give either $a = a' = 0$ or $a = a' = 120°$; and the latter evidently belongs to the case of maximum.

If it is desired to compute the differential formulæ, the following form will be convenient. We have

$$K = -2 \sin \tfrac{1}{2}(A' - A) \sin \tfrac{1}{2}(A'' - A') \sin \tfrac{1}{2}(A - A'')$$

$$\frac{d\varphi}{15 \cos \varphi} = \frac{\sin A \cos \tfrac{1}{2}(A'' + A') \sin \tfrac{1}{2}(A'' - A')}{K} dT$$

$$+ \frac{\sin A' \cos \tfrac{1}{2}(A + A'') \sin \tfrac{1}{2}(A - A'')}{K} dT'$$

$$+ \frac{\sin A'' \cos \tfrac{1}{2}(A' + A) \sin \tfrac{1}{2}(A' - A)}{K} dT''$$

$$d \Delta T = \frac{\sin A \sin \tfrac{1}{2}(A'' + A') \sin \tfrac{1}{2}(A'' - A')}{K} dT$$

$$+ \frac{\sin A' \sin \tfrac{1}{2}(A + A'') \sin \tfrac{1}{2}(A - A'')}{K} dT'$$

$$+ \frac{\sin A'' \sin \tfrac{1}{2}(A' + A) \sin \tfrac{1}{2}(A' - A)}{K} dT''$$

where $d\varphi$ is divided by 15, since it will be expressed in seconds of arc, while dT, dT', and dT'' are in seconds of time. If we first compute the coefficients of the value of $d\Delta T$, those of $d\varphi$ will be found by multiplying the former respectively by $\cot \tfrac{1}{2}(A' + A'')$, $\cot \tfrac{1}{2}(A + A'')$, and $\cot \tfrac{1}{2}(A' + A)$, and also by $15 \cos \varphi$. It is well to remark, also, for the purpose of verification, that the sum of the three coefficients in the formula for $d\varphi$ must be $= 0$, and the sum of those in the formula for $d\Delta T$ must be $= -1$.

The substitution of $d\lambda$ for $dT' - dT$, and $d\lambda'$ for $dT'' - dT$, will reduce the above expressions to a more simple form, which I leave to the reader.

EXAMPLE.—To illustrate the above method, GAUSS took the following observations, with a sextant and mercurial horizon, at Göttingen, August 27, 1808. The double altitude on the sextant was 105° 18′ 55″. The time was noted by a sidereal clock whose rate was so small as not to require notice.

$$
\begin{array}{lll}
\alpha\ \textit{Andromedæ} & T & = 21^h\ 33^m\ 26^s \\
\alpha\ \textit{Ursæ Minoris} & T' & = 21\ \ 47\ \ 30 \\
\alpha\ \textit{Lyræ} & T'' & = 22\ \ \ 5\ \ 21
\end{array}
$$

The apparent places of the stars were as follows:

$$
\begin{array}{llll}
\alpha\ \textit{Andromedæ} & \alpha = 23^h\ 58^m\ 33^s.33 & \delta = 28^\circ\ \ 2'\ 14''.8 \\
\alpha\ \textit{Ursæ Minoris} & \alpha' = \ \ 0\ \ 55\ \ \ 4.70 & \delta' = 88\ \ 17\ \ \ 5\ .7 \\
\alpha\ \textit{Lyræ} & \alpha'' = 18\ \ 30\ \ 28.96 & \delta'' = 38\ \ 37\ \ \ 6\ .6
\end{array}
$$

Hence we find

$$
\begin{array}{ll|ll}
\tfrac{1}{2}\lambda = & -\ 5^\circ\ 18'\ 25''.28 & \tfrac{1}{2}\lambda' = & 44^\circ\ 59'\ 55''.28 \\
\tfrac{1}{2}(\delta'-\delta) = & 30\ \ \ 7\ \ 25\ .45 & \tfrac{1}{2}(\delta''-\delta) = & 5\ \ 17\ \ 25\ .90 \\
\tfrac{1}{2}(\delta'+\delta) = & 58\ \ \ 9\ \ 40\ .25 & \tfrac{1}{2}(\delta''+\delta) = & 33\ \ 19\ \ 40\ .70 \\
\log\cot\tfrac{1}{2}(\delta'-\delta) & 0.2363973 & \log\cot\tfrac{1}{2}(\delta''-\delta) & 1.0333869 \\
\log\sin\tfrac{1}{2}\lambda & n8.9661070 & \log\sin\tfrac{1}{2}\lambda' & 9.8494751 \\
\log m\sin M & n9.2025043 & \log m'\sin M' & 0.8828620 \\
\log\tan\tfrac{1}{2}(\delta'+\delta) & 0.2069331 & \log\tan\tfrac{1}{2}(\delta''+\delta) & 9.8179461 \\
\log\cos\tfrac{1}{2}\lambda & 9.9981343 & \log\cos\tfrac{1}{2}\lambda' & 9.8494949 \\
\log m\cos M & 0.2050674 & \log m'\cos M' & 9.6674410 \\
\log\tan M & n8.9974369 & \log\tan M' & 1.2154210 \\
\log\cos M & 9.9978645 & \log\sin M' & 9.9991963 \\
\log m & 0.2072029 & \log m' & 0.8836657 \\
M = & -\ 5^\circ\ 40'\ 37''.96 & M' = & 86^\circ\ 30'\ 55''.07 \\
\tfrac{1}{2}\lambda - M = N = & +\ 0\ \ 22\ \ 12\ .68 & \tfrac{1}{2}\lambda' - M' = N' = & -41\ \ 30\ \ 59\ .79
\end{array}
$$

$$
\begin{array}{rl|ll}
\vartheta = & 11^\circ\ 53'\ 41''.28 & \log\frac{m}{m'} = \log\tan\vartheta & 9.3235372 \\
45^\circ - \vartheta = & 33\ \ \ 6\ 18\ .72 & \log\tan(45^\circ-\vartheta) & 9.8142617 \\
\tfrac{1}{2}(N'-N) = & -20\ \ 56\ \ 36\ .24 & \log\cot\tfrac{1}{2}(N'-N) & n0.4171063 \\
t + \tfrac{1}{2}(N'+N) = & -59\ \ 35\ \ 14\ \ 71 & \log\tan[t+\tfrac{1}{2}(N'+N)] & n0.2313680 \\
\tfrac{1}{2}(N'+N) = & -20\ \ 34\ \ 23\ .56 & & \\
t = & -39\ \ \ 0\ 51\ .15 = & -\ 2^h\ 36^m\ \ 3^s.41 & \\
 & \alpha = & 23\ \ 58\ \ 33.33 & \\
 & t+\alpha = \Theta = & 21\ \ 22\ \ 29.92 & \\
 & T = & 21\ \ 33\ \ 26. & \\
 & \text{Clock correction } \Delta T = & -\ 10\ \ 56.08 &
\end{array}
$$

Then, to find the latitude, we have

$t + N = -38° 38' 38''.47$		$t + N' = -80° 31' 50''.94$	
$\log \cos (t + N)$	9.8926738	$\log \cos (t + N')$	9.2162110
$\log m$	0.2072029	$\log m'$	0.8836657
$\log \tan \varphi$	0.0998767	$\log \tan \varphi$	0.0998767

$$\varphi = 51° 31' 51''.46$$

If with these results we compute the true altitude of the stars, we find from each $h = 52° 37' 21''.2$. The refraction was $42''.7$, and hence the apparent altitude $= 52° 38' 3''.9$. The double altitude observed was, therefore, $105° 16' 7''.8$. The index correction of the sextant was $-3' 30''$, and hence the double altitude given by the instrument was $105° 15' 25''$, which was, consequently, too small by $43''$.

To compute the differential equations, we find

$$A = 293° 45'.2 \quad A' = 182° 9'.1 \quad A'' = 90° 17'.9$$

and hence

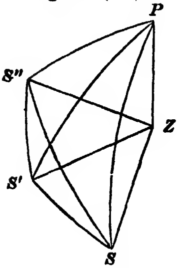
$$d\varphi = +3.808\, dT - 0.288\, dT' - 3.519\, dT''$$
$$d \Delta T = -0.391\, dT - 0.007\, dT' - 0.602\, dT''$$

by which we see that an error of one second in each of the times would produce at the most but $7''.6$ error in the latitude and one second in the clock correction.

188. *Solution of the preceding problem by* CAGNOLI'S *formulæ.*—After GAUSS had published the solution above given, he was himself the first to observe* that CAGNOLI'S formulæ for the solution of a very different problem† might be applied directly to this.

When the altitude is also computed, CAGNOLI'S formulæ have slightly the advantage over those of GAUSS. To deduce them, let q, q', q'' be the parallactic angles at the three stars, or (Fig. 26) let

Fig. 26. (*bis*).



$$q = PSZ, \qquad q' = PS'Z, \qquad q'' = PS''Z,$$

and also put

$$Q = \tfrac{1}{2}(PS''S' - PS'S'')$$
$$Q' = \tfrac{1}{2}(PS''S - PSS'')$$
$$Q'' = \tfrac{1}{2}(PS'S - PSS')$$

* *Monatliche Correspondenz*, Vol. XIX. p. 87.

† Namely, that of determining, from three heliocentric places of a solar spot, the position of the sun's equator, and the declination of the spot.—See CAGNOLI'S *Trigonométrie*, p. 488.

then, since ZSS', $ZS'S''$, and ZSS'' are isosceles triangles, we have

$$\begin{aligned} q + PSS' &= PS'S - q' \\ q' + PS'S'' &= PS''S' - q'' \\ q + PSS'' &= PS''S - q'' \end{aligned}$$

whence

$$\begin{aligned} q + q' &= 2Q'' \\ q' + q'' &= 2Q \\ q'' + q &= 2Q' \end{aligned}$$

$$q + q' + q'' = Q + Q' + Q''$$

$$\left.\begin{aligned} q &= -Q + Q' + Q'' \\ q' &= \quad Q - Q' + Q'' \\ q'' &= \quad Q + Q' - Q'' \end{aligned}\right\} \quad (345)$$

Now, Q, Q', Q'' are found from the triangles $PS''S'$, $PS''S$, and $PS'S$, by NAPIER's Analogies (Sph. Trig. Art. 73), thus:

$$\left.\begin{aligned} \tan Q &= \frac{\sin \frac{1}{2}(\delta'' - \delta')}{\cos \frac{1}{2}(\delta'' + \delta')} \cot \tfrac{1}{2}(\lambda' - \lambda) \\ \tan Q' &= \frac{\sin \frac{1}{2}(\delta'' - \delta)}{\cos \frac{1}{2}(\delta'' + \delta)} \cot \tfrac{1}{2}\lambda' \\ \tan Q'' &= \frac{\sin \frac{1}{2}(\delta' - \delta)}{\cos \frac{1}{2}(\delta' + \delta)} \cot \tfrac{1}{2}\lambda \end{aligned}\right\} \quad (346)$$

where λ, λ' are the angles at the pole found as in the preceding article. With these values of Q, Q', Q'', those of q, q', and q'' become known by (345).

We have also

$$\begin{aligned} \cos\varphi \sin(t + \lambda) &= \cos h \sin q' \\ \cos\varphi \sin t &= \cos h \sin q \end{aligned}$$

whence

$$\frac{\sin(t + \lambda)}{\sin t} = \frac{\sin q'}{\sin q}$$

and from this

$$\frac{\sin(t + \lambda) + \sin t}{\sin(t + \lambda) - \sin t} = \frac{\sin q' + \sin q}{\sin q' - \sin q}$$

or

$$\frac{\tan(t + \frac{1}{2}\lambda)}{\tan \frac{1}{2}\lambda} = \frac{\tan \frac{1}{2}(q' + q)}{\tan \frac{1}{2}(q' - q)}$$

Substituting the values of q and q' in terms of Q, this gives

$$\tan\,(t + \tfrac{1}{2}\lambda) = \tan\tfrac{1}{2}\lambda\,\tan Q''\cot\,(Q - Q')$$

or, substituting the value of $\tan Q''$,

$$\tan\,(t + \tfrac{1}{2}\lambda) = \frac{\sin\tfrac{1}{2}(\delta' - \delta)}{\cos\tfrac{1}{2}(\delta' + \delta)}\cot\,(Q - Q') \qquad (347)$$

which determines $t + \frac{1}{2}\lambda$, whence t and the clock correction. We can now find the latitude and altitude from any one of the triangles PSZ, $PS'Z$, $PS''Z$, by NAPIER'S Analogies (Sph. Trig. Art. 80): thus, from PSZ we have

$$\left.\begin{aligned} \tan\tfrac{1}{2}(\varphi + h) &= \frac{\cos\tfrac{1}{2}(t + q)}{\cos\tfrac{1}{2}(t - q)}\tan\,(45° + \tfrac{1}{2}\delta) \\ \tan\tfrac{1}{2}(\varphi - h) &= \frac{\sin\tfrac{1}{2}(t - q)}{\sin\tfrac{1}{2}(t + q)}\cot\,(45° + \tfrac{1}{2}\delta) \end{aligned}\right\} \qquad (348)$$

and then $\varphi = \frac{1}{2}(\varphi + h) + \frac{1}{2}(\varphi - h)$, $h = \frac{1}{2}(\varphi + h) - \frac{1}{2}(\varphi - h)$.

As all the angles are determined by their tangents, an ambiguity exists as to the semicircle in which they are to be taken; but, as GAUSS remarks, we may choose arbitrarily (taking, for example, Q, Q', Q'' always less than 90°, positive or negative according to the signs of their tangents), and then, according to the results, will have in some cases to make the following changes:

1. If the values of φ and h found by (348) are such that $\cos\varphi$ and $\sin h$ have opposite signs, we must substitute $180° + q$ for q and repeat the computation of these two equations. In this repetition the same logarithms will occur as before, but differently placed.
2. If the values of φ and h exceed 90°, we must take their supplements to the next multiple of 180°.
3. The latitude is to be taken as north or south according as $\sin\varphi$ and $\sin h$ have the same or different signs.

No ambiguity, however, exists in practice as to $t + \frac{1}{2}\lambda$, found by (347), since $Q - Q'$ can differ from its true value only by 180°, and this difference does not change the sign of $\cot(Q - Q')$: hence $\tan\,(t + \frac{1}{2}\lambda)$ will come out with its true sign; and between

the two values of $t + \frac{1}{2}\lambda$, differing by 180°, or 12^h, the observer will be at no loss to choose, as he cannot be uncertain of his time by 12^h.

EXAMPLE.—Taking the example of the preceding article, we shall find

$$Q = -37° 57' 9''.3 \quad Q' = +6° 17' 51''.66 \quad Q'' = -84° 25' 23''.81$$

$$\begin{aligned} q &= -Q + Q' + Q'' = -40° 10' 22''.85 \\ t &= -39 \quad 0 \ 51 \ .27 \end{aligned}$$

$$\begin{aligned} \tfrac{1}{2}(t+q) &= -39° 35' 37''.06 & \tfrac{1}{2}(t-q) &= +\ 0° 34' 45''.79 \\ \tfrac{1}{2}(\varphi+h) &= \quad 52 \quad 4 \ 36 .35 & \tfrac{1}{2}(\varphi-h) &= -\ 0 \ 32 \ 44 .84 \\ \varphi &= \quad 51 \ 31 \ 51 .5 & h &= \quad 52 \ 37 \ 21 .2 \end{aligned}$$

189. *If we have observed more than three stars at the same altitude,* we have more than sufficient data for the determination of the latitude; but by combining all the observations we may obtain a more accurate result than from only three. This combination is effected by the method of least squares, according to which we assume approximate values of the unknown quantities and then determine the most probable corrections of these values, or those which best satisfy all the observations.

Let T, T', T'', T''', &c. be the observed times by the clock when the several stars reach the same altitude. Let ΔT be the assumed clock correction at some assumed epoch $= T_0$; δT the *known* rate. Let φ and h be the assumed approximate values of the latitude and altitude. With φ and h, which will be the same for all the stars, and with the declinations δ, δ', δ'', &c., compute the hour angles t, t', t'', &c. and the azimuths A, A', A'', &c. If the assumed values were all correct and the observations perfect, we should have $\alpha + t = T + \Delta T + \delta T(T - T_0)$; for each of these quantities then represents the sidereal time of observation; but if φ, h, and ΔT require the corrections $d\varphi$, dh, and $d\Delta T$, and if dt is the corresponding correction of t, we shall have

$$\alpha + t + dt = T + \Delta T + d\Delta T + \delta T(T - T_0)$$

The relation between $d\varphi$, dh, and dt is

$$dh = -\cos A\, d\varphi - 15 \cos \varphi \sin A\, dt$$

and a similar equation of condition exists for each star. In all

these equations, dh and $d\varphi$ are the same, but dt is different for each. If we put

$$\left.\begin{aligned} f &= T + \Delta T + \delta T(T - T_0) - (\alpha + t) \\ f' &= T' + \Delta T + \delta T(T' - T_0) - (\alpha' + t') \\ f'' &= T'' + \Delta T + \delta T(T'' - T_0) - (\alpha'' + t'') \\ &\&c. \end{aligned}\right\} \qquad (349)$$

which are all known quantities, we have

$$dt = f + d\Delta T, \qquad dt' = f' + d\Delta T, \ \&c.$$

and the equations of condition become

$$\left.\begin{aligned} dh + \cos A\,.d\varphi + 15\cos\varphi\sin A\,.d\Delta T + 15\cos\varphi\sin A\,.f &= 0 \\ dh + \cos A'.d\varphi + 15\cos\varphi\sin A'.d\Delta T + 15\cos\varphi\sin A'.f' &= 0 \\ dh + \cos A''.d\varphi + 15\cos\varphi\sin A''.d\Delta T + 15\cos\varphi\sin A''.f'' &= 0 \\ \&c. \end{aligned}\right\} \qquad (350)$$

from which, by the method of least squares, the most probable values of dh, $d\varphi$, and $d\Delta T$ are determined. The true values of the altitude, latitude, and clock correction will then be $h + dh$, $\varphi + d\varphi$, $\Delta T + d\Delta T$.

The hour angles will be computed most accurately by (269), which is the same as the following:

$$\tan^2 \tfrac{1}{2} t = \frac{\sin \frac{1}{2}(\zeta - \varphi + \delta) \sin \frac{1}{2}(\zeta + \varphi - \delta)}{\cos \frac{1}{2}(\zeta + \varphi + \delta) \cos \frac{1}{2}(\zeta - \varphi - \delta)}$$

in which $\zeta = 90° - h$; and the azimuths by

$$\tan^2 \tfrac{1}{2} A = \frac{\sin \frac{1}{2}(\zeta - \varphi + \delta) \cos \frac{1}{2}(\zeta - \varphi - \delta)}{\cos \frac{1}{2}(\zeta + \varphi + \delta) \sin \frac{1}{2}(\zeta + \varphi - \delta)}$$

Since φ and ζ are constant, it will be convenient to put

$$b = \tfrac{1}{2}(\zeta + \varphi) \qquad c = \tfrac{1}{2}(\zeta - \varphi)$$

$$m = \frac{\sin(c + \frac{1}{2}\delta)}{\cos(b + \frac{1}{2}\delta)} \qquad n = \frac{\sin(b - \frac{1}{2}\delta)}{\cos(c - \frac{1}{2}\delta)}$$

then

$$\tan^2 \tfrac{1}{2} t = mn \qquad \tan^2 \tfrac{1}{2} A = \frac{m}{n} \qquad (351)$$

The barometer and thermometer should be observed with each

altitude, and if they indicate a sensible change in the refraction a correction for this change must be introduced into the equations of condition. Thus, if r_0 is the refraction for the altitude h for the mean height of the barometer and thermometer during the whole series, while for one of the stars it is r, then the assumed altitude requires for that star not only the correction dh, but also the correction $r - r_0$. Hence, if we find the refractions r, r', r'', &c. for all the observations, and take their mean r_0, we have only to add to the equations of condition respectively the quantities $r - r_0$, $r' - r_0$, $r'' - r_0$, &c.

If any one of the stars is observed at an altitude h_1 slightly different from the common altitude h, we correct the corresponding equation of condition by adding the quantity $h - h_1$.

190. We may also apply the preceding method to the case where there are but three observations. The final equations are then nothing more than the three equations of condition themselves, from which the unknown quantities will be found by simple elimination. It will easily be seen that this elimination leads to the expressions for $d\varphi$ and $d\Delta T$ already given on p. 284, if we there exchange dT, dT', and dT'' for f, f', and f'' respectively. We can simplify the computation by assuming ΔT so as to make one of the quantities f, f', f'' zero. Thus, we shall have $f = 0$ if we determine ΔT by the formula

$$\Delta T = \alpha + t - [T + \delta T(T - T_0)] \tag{352}$$

then, finding f' and f'' with this value, and putting

$$k' = \frac{\sin \frac{1}{2} A' \cos \frac{1}{2} A'}{\sin \frac{1}{2}(A' - A) \sin \frac{1}{2}(A'' - A')} \cdot f'$$

$$k'' = \frac{\sin \frac{1}{2} A'' \cos \frac{1}{2} A''}{\sin \frac{1}{2}(A'' - A) \sin \frac{1}{2}(A'' - A')} \cdot f''$$

we shall have the following formulæ:

$$\left.\begin{aligned} d\Delta T &= -k' \sin \tfrac{1}{2}(A + A'') + k'' \sin \tfrac{1}{2}(A' + A) \\ \frac{d\varphi}{15 \cos \varphi} &= -k' \cos \tfrac{1}{2}(A + A'') + k'' \cos \tfrac{1}{2}(A' + A) \\ \frac{dh}{15 \cos \varphi} &= +k' \cos \tfrac{1}{2}(A'' - A) - k'' \cos \tfrac{1}{2}(A' - A) \end{aligned}\right\} \tag{353}$$

EXAMPLE.—Taking the three observations above employed, and assuming the approximate values

$$\Delta T = -11^m\ 0^s, \quad \varphi = 51°\ 32'\ 0'', \quad h = 52°\ 37'\ 0'',$$

we shall find, by (351),

$$\begin{array}{lll} t = -2^h\ 36^m\ 5^s.50 & t' = -3^h\ 19^m\ 55^s.65 & t'' = 3^h\ 23^m\ 58^s.25 \\ A = -66°\ 15'.2 & A' = -177°\ 50'.2 & A'' = 90°\ 18'.1 \end{array}$$

By (349), putting in this case $\delta T = 0$, we then have

$$f = -1^s.83 \qquad f' = +80^s.95 \qquad f'' = -6^s.21$$

and the equations of condition (350) become

$$\begin{array}{l} dh + 0.4027\, d\varphi - 8.5410\, d\Delta T + 15.63 = 0 \\ dh - 0.9993\, d\varphi - 0.3522\, d\Delta T - 28.51 = 0 \\ dh - 0.0053\, d\varphi + 9.3308\, d\Delta T - 57.94 = 0 \end{array}$$

whence

$$d\Delta T = +3^s.92 \qquad d\varphi = -8''.58 \qquad dh = +21''.31$$

and the true values of the required quantities are, therefore,

$$\Delta T = -10^m\ 56^s.08 \qquad \varphi = 51°\ 31'\ 51''.42 \qquad h = 52°\ 37'\ 21''.31$$

agreeing almost perfectly with the values before found.

Since in this example there are but three observations, we may also employ the formulæ (353), first assuming

$$\Delta T = -10^m\ 58^s.17$$

which is the value given by (352). With this we find

$$\begin{array}{ll} f' = +82^s.78 & f'' = -4^s.38 \\ \log k' = 0.4199 & \log k'' = n0.4932 \end{array}$$

and by (353) we shall find

$$d\Delta T = +2^s.09 \qquad d\varphi = -8''.58 \qquad dh = +21''.31$$

Hence the true clock correction is $-10^m\ 58^s.17 + 2^s.09 = -10^m\ 56^s.08$; and the values of the latitude and altitude also agree with the former values.

191. We may here observe that, theoretically, the latitude might be found also from three different altitudes of the same star and the differences of azimuth; for we should then have

$$\begin{aligned}
\sin\delta &= \sin\varphi\sin h \ + \cos\varphi\cos h \ \cos A \\
\sin\delta &= \sin\varphi\sin h' + \cos\varphi\cos h' \cos(A+\lambda) \\
\sin\delta &= \sin\varphi\sin h'' + \cos\varphi\cos h'' \cos(A+\lambda')
\end{aligned}$$

in which A is the azimuth of the star at the first observation, and the differences of azimuth λ and λ' are supposed to be given. The solution of Art. 187 may be applied to these equations by writing h for δ and A for t.

Again, there might be found from three different altitudes of the same star not only the latitude and time, but also the declination of the star; for we then have

$$\begin{aligned}
\sin h &= \sin\varphi\sin\delta + \cos\varphi\cos\delta\cos t \\
\sin h' &= \sin\varphi\sin\delta + \cos\varphi\cos\delta\cos(t+\lambda) \\
\sin h'' &= \sin\varphi\sin\delta + \cos\varphi\cos\delta\cos(t+\lambda')
\end{aligned}$$

from which we can readily deduce φ, t, and δ. But the method is of no practical value, as the errors of observation have too much influence upon the result.

NINTH METHOD.—BY THE TRANSITS OF STARS OVER VERTICAL CIRCLES.

192. We may observe the time of transit of a star over any vertical circle with a transit instrument (or with an altitude and azimuth instrument, or common theodolite); for when the rotation axis is horizontal, the collimation axis will, as the instrument revolves, describe the plane of a vertical circle. For any want of horizontality of the rotation axis, or other defects of adjustment, corrections must be applied to the observed time of transit over the instrument to reduce it to the time of transit over the assumed vertical circle. These corrections will be treated of in their proper places in Vol. II.; and I shall here assume that the observation has been corrected, and gives the clock time T of transit over some assumed vertical circle the azimuth of which is A. The clock correction ΔT being known, we have the star's hour angle by the formula

$$t = T + \Delta T - \alpha$$

and then, the declination of the star being given, we have the equation [from (14)]

$$\cos t \sin \varphi - \tan \delta \cos \varphi = \sin t \cot A \tag{354}$$

If, then, A is also known, the latitude φ can be found by this equation. Let us inquire under what conditions an accurate result is to be expected by this method. By differentiating the equation, we find [see (51)]

$$d\varphi = \frac{\cos q \cos \delta}{\cos \zeta \sin A}\, dt - \frac{\tan \zeta}{\sin A}\, dA + \frac{\sin q}{\cos \zeta \sin A}\, d\delta$$

from which it appears that $\sin A$ and $\cos \zeta$ must be as great as possible. The most favorable case is, therefore, that in which the assumed vertical circle is the *prime vertical*, and the star's declination differs but little from the latitude; for we then have $A = 90°$ and ζ small. Indeed, these conditions not only increase the denominator of the coefficient of dt, but also diminish its numerator, since, by (10), we have

$$\cos q \cos \delta = \sin \zeta \sin \varphi + \cos \zeta \cos \varphi \cos A$$

which vanishes wholly when the star passes through the zenith. Moreover, if the same star is observed at both its east and west transits over the prime vertical, we shall have at one transit sin $A = -1$, at the other $\sin A = +1$, and the mean of the two resulting values of the latitude will, therefore, be wholly free from the effect of a constant error in the clock times, that is, of an error in the clock correction. It is then necessary only that the *rate* should be known. This method, therefore, admits of a high degree of precision, and requires for its successful application only a transit instrument, of moderate dimensions, and a time-piece. Its advantages were first clearly demonstrated by Bessel* in the year 1824; but it appears that very early in the last century Römer had mounted a transit instrument in the prime vertical for the purpose of determining the declinations of stars from their transits, the latitude being given. The details of this important method will be given in Vol. II., under "Transit Instrument."

* *Astronom. Nach.*, Vol. III. p. 9.

193. It may sometimes be possible to observe transits only over some vertical circle the azimuth of which is undetermined. We must then observe either two stars, or the same star on opposite sides of the meridian. We shall then have the two equations

$$\cos t \,.\, \tan A \sin \varphi - \tan \delta \,.\, \tan A \cos \varphi = \sin t$$
$$\cos t' \,.\, \tan A \sin \varphi - \tan \delta' \,.\, \tan A \cos \varphi = \sin t'$$

from which the two unknown quantities A and φ can be determined. If the same star is observed, we shall only have to put $\delta' = \delta$. Regarding $\tan A \sin \varphi$ and $\tan A \cos \varphi$ as the unknown quantities, we have, by eliminating them in succession,

$$\tan A \sin \varphi = \frac{\sin t \sin \delta' \cos \delta - \sin t' \cos \delta' \sin \delta}{\cos t \sin \delta' \cos \delta - \cos t' \cos \delta' \sin \delta}$$

$$\tan A \cos \varphi = \frac{-\sin (t' - t) \cos \delta' \cos \delta}{\cos t \sin \delta' \cos \delta - \cos t' \cos \delta' \sin \delta}$$

If we introduce the auxiliaries m and M, such that

$$\left.\begin{aligned} m \sin M &= \sin (\delta' + \delta) \sin \tfrac{1}{2} (t' - t) \\ m \cos M &= \sin (\delta' - \delta) \cos \tfrac{1}{2} (t' - t) \end{aligned}\right\} \quad (355)$$

we shall easily find

$$m \sin [\tfrac{1}{2} (t' + t) - M] = \sin t \sin \delta' \cos \delta - \sin t' \cos \delta' \sin \delta$$
$$m \cos [\tfrac{1}{2} (t' + t) - M] = \cos t \sin \delta' \cos \delta - \cos t' \cos \delta' \sin \delta$$
$$m \sin [\tfrac{1}{2} (t' - t) - M] = - \sin (t' - t) \cos \delta' \sin \delta$$

and hence

$$\left.\begin{aligned} \tan A \sin \varphi &= \tan [\tfrac{1}{2} (t' + t) - M] \\ \tan A \cos \varphi &= \frac{\sin [\tfrac{1}{2} (t' - t) - M] \cot \delta}{\cos [\tfrac{1}{2} (t' + t) - M]} \end{aligned}\right\} \quad (356)$$

which determine A and φ by a simple logarithmic computation. The solution will be still more convenient in the following form:

$$\left.\begin{aligned} \tan M &= \tan \tfrac{1}{2} (t' - t) \frac{\sin (\delta' + \delta)}{\sin (\delta' - \delta)} \\ \tan \varphi &= \tan \delta \frac{\sin [\tfrac{1}{2} (t' + t) - M]}{\sin [\tfrac{1}{2} (t' - t) - M]} \\ \tan A &= \frac{\tan [\tfrac{1}{2} (t' + t) - M]}{\sin \varphi} \end{aligned}\right\} \quad (357)$$

If the same star is observed at each of its transits over the same vertical circle, we have $\delta' = \delta$, and hence $\tan M = \infty$, $M = 90°$, which gives

$$\tan\varphi = \tan\delta \frac{\cos\frac{1}{2}(t'+t)}{\cos\frac{1}{2}(t'-t)} \qquad \tan A = -\frac{\cot\frac{1}{2}(t'+t)}{\sin\varphi} \tag{358}$$

If the same star is observed twice on the prime vertical, we must have $t' + t = 0$, since $\tan A = \infty$; and then,

$$\tan\varphi = \frac{\tan\delta}{\cos\frac{1}{2}(t'-t)} = \frac{\tan\delta}{\cos t} \tag{359}$$

which follows also from (354) when $\cot A = 0$; or, geometrically, from the right triangle formed by the zenith, the pole, and the star, as in Art. 19.

If the latitude is given, we can find the time from the transits of two stars over any (undetermined) vertical circle by the second equation of (357), which gives

$$\sin\left[\tfrac{1}{2}(t'+t) - M\right] = \frac{\tan\varphi}{\tan\delta}\sin\left[\tfrac{1}{2}(t'-t) - M\right]$$

for the observation furnishes the elapsed time, and hence $t' - t$; and this equation determines $\frac{1}{2}(t'+t)$, and hence both t and t'.

If the latitude and time are given, we can find the declination of a star observed twice on the same vertical circle, by (358). When the observation is made in the prime vertical, this becomes one of the most perfect methods of determining declinations. See Vol. II., *Transit Instrument in the Prime Vertical.*

194. The following brief approximative methods of determining the latitude may be found useful in certain cases.

TENTH METHOD.—BY ALTITUDES NEAR THE MERIDIAN WHEN THE TIME IS NOT KNOWN.

195. (A.) *By two altitudes near the meridian and the chronometer times of the observations, when the rate of the chronometer is known, but not its correction.*

Let

h, h' = the true altitudes,

T, T' = the chronometer times,

$\tau = \frac{1}{2}(T' - T)$

then, t and t' being the (unknown) hour angles of the observations, we have, by (287), approximately,

$$h_1 = h + at^2$$
$$h_1 = h' + at'^2$$

in which h_1 is the meridian altitude, and

$$a = \frac{225 \sin 1''}{2} \cdot \frac{\cos \varphi \cos \delta}{\cos h_1}$$

The mean of these equations is

$$h_1 = \tfrac{1}{2}(h + h') + a\left[\left(\frac{t' - t}{2}\right)^2 + \left(\frac{t' + t}{2}\right)^2\right]$$

and their difference gives

$$h - h' = a\,(t' - t)\,(t' + t)$$

But we have

$$\tau = \tfrac{1}{2}(T' - T) = \tfrac{1}{2}(t' - t)$$

in which we suppose the interval $T' - T$ to be corrected for the rate of the chronometer. Hence

$$\frac{t' + t}{2} = \frac{\tfrac{1}{4}(h - h')}{a\tau}$$

which, substituted in the above expression for h_1, gives

$$h_1 = \tfrac{1}{2}(h + h') + a\tau^2 + \frac{[\tfrac{1}{4}(h - h')]^2}{a\tau^2} \qquad (360)$$

According to this formula, the mean of the two altitudes is reduced to the meridian by adding two corrections: 1st, the quantity $a\tau^2$, which is nothing more than the common "reduction to the meridian" computed with the half elapsed time as the hour angle; 2d, the square of one-fourth the difference of the altitudes divided by the first correction.

If we employ the form (285) for the reduction, we have

$$h_1 = \tfrac{1}{2}(h + h') + Am + \frac{[\tfrac{1}{4}(h - h')]^2}{Am} \qquad (361)$$

in which

$$A = \frac{\cos \varphi \cos \delta}{\cos h_1} \qquad m = \frac{2 \sin^2 \tfrac{1}{2} \tau}{\sin 1''}$$

and m is taken from Table V. or log m from Table VI.

EXAMPLE 1.—From the observations in the example of Art. 171, I select the following, which are very near the meridian.

Obsd. alts. ⊙		True alts. ⊙		Chronometer.
50° 5′ 42″.8	h' =	50° 21′ 7″.6		23ʰ 50ᵐ 46ˢ.5
50 7 25 .5	h =	50 22 50 .4		0 0 37 .5
	$\frac{1}{2}(h-h')$ =	25 .7	τ =	4 55 .5
	$\frac{1}{2}(h+h')$ =	50 21 59 .0		
	Am =	+ 59 .0	log m	1.6778
	2d corr. =	+ 11 .2	log A	0.0930
	h_1 =	50 23 9 .2	log Am	1.7708
	ζ_1 =	39 36 50 .8	log $[\frac{1}{2}(h-h')]^2$	2.8198
	δ_1 =	− 1 48 9 .2	log 2d corr.	1.0490
	φ =	37 48 41 .6		

EXAMPLE 2.—In the same example, the first and last observations, which are quite remote from the meridian, are as follows:

Obsd. alts. ⊙		True alts. ⊙		Chronometer.
49° 51′ 19″.3	h =	50° 6′ 43″.7		23ʰ 37ᵐ 35ˢ
49 50 24	h' =	50 5 48 .4		0 18 31
	$\frac{1}{2}(h-h')$ =	13 .8	τ =	20 28

which give $Am = 16' \ 58''$, and the 2d corr. $= 0''.2$, whence $\varphi = 37° \ 48' \ 37''$.

This simple approximative method may frequently be useful to the traveller, and especially at sea, where the meridian observation has been lost in consequence of flying clouds. At sea, however, the computation need not be carried out so minutely as the above, and the method becomes even more simple. See Art. 204.

M. V. CAILLET* gives a method for the same purpose, which is readily deduced from the above. Put

$$k = h' - h \qquad \tau' = T' - T = 2\tau$$

then (360) becomes

$$h_1 = h + \frac{k}{2} + \frac{a\tau'^2}{4} + \frac{k^2}{4a\tau'^2}$$
$$= h + \frac{(k + a\tau'^2)^2}{4a\tau'^2}$$

* *Traité de Navigation* (2d edition, Paris, 1857), p. 319.

or, putting

$$m = \frac{2 \sin^2 \frac{1}{2} \tau'}{\sin 1''} \qquad Am = a\tau'^2$$

$$h_1 = h + \frac{(k + Am)^2}{4Am} \tag{362}$$

in which h is the altitude farthest from the meridian. Although this reduces the two corrections of (361) to a single one, the computation is not quite so simple.

196. (B.) *By three altitudes near the meridian and the chronometer times of the observations, when neither the correction nor the rate of the chronometer is known.*—In this case we assume only that the chronometer goes *uniformly* during the time occupied by the observations. Let

h, h', h'' = the true altitudes,
T, T', T'' = the chronometer times,
T_1 = the chronometer time of the greatest altitude.

If we introduce the factor for rate $= k$, according to Art. 171, the formula for the reduction to the meridian by Gauss's method is, approximately,

$$h_1 = h + akt^2$$

in which t is the time reckoned from the greatest altitude. Denoting ak by α, we have then, from the three observations,

$$\left.\begin{aligned} h_1 &= h + \alpha\,(T - T_1)^2 \\ h_1 &= h' + \alpha\,(T' - T_1)^2 \\ h_1 &= h'' + \alpha\,(T'' - T_1)^2 \end{aligned}\right\} \tag{363}$$

which three equations suffice to determine the three unknown quantities α, T_1, and h_1. By subtracting the second from the first, and the third from the second, we obtain

$$\frac{h - h'}{T' - T} = \alpha\,(T' + T) - 2\alpha T_1$$

$$\frac{h' - h''}{T'' - T'} = \alpha\,(T'' + T') - 2\alpha T_1$$

and the difference of these is

$$\frac{h' - h''}{T'' - T'} - \frac{h - h'}{T' - T} = \alpha\,(T'' - T)$$

If, then, we put

$$b = \frac{h - h'}{T' - T} = \text{the mean change of altitude in one second of the chronometer from the first to the second observation,}$$

$$c = \frac{h' - h''}{T'' - T'} = \text{ditto from the second to the third observation,}$$

we have

$$\left.\begin{aligned} a &= \frac{c - b}{T'' - T} \\ T_1 &= \frac{T + T'}{2} - \frac{b}{2a} \quad \text{or } T_1 = \frac{T' + T''}{2} - \frac{c}{2a} \end{aligned}\right\} \quad (364)$$

Having thus found T_1, we can find h_1 from any one of the equations (363), all of which will give the same result if the computation is correct.*

EXAMPLE.—From the observations in the example of Art. 171 I select the following three observations:

Obsd. alts. ☉	True alts. ☉	Chronometer.
50° 5′ 42″.8	h = 50° 21′ 7″.6	T = 23ʰ 50ᵐ 46ˢ.5
50 7 27 .	h' = 50 22 51 .9	T' = 23 55 16 .
50 7 25 .5	h'' = 50 22 50 .4	T'' = 0 0 37 .5
$h - h'$ = − 104″.3	$T' - T$ = 269ˢ.5	b = − 0.3869
$h' - h''$ = + 1 .5	$T'' - T'$ = 321 .5	c = + 0.0047
	$T'' - T$ = 591 .	$c - b$ = + 0.3916

$\frac{1}{2}(T + T')$ = 23ʰ 53ᵐ 1ˢ.3

$-\frac{b}{2a}$ = + 4 52 .0

T_1 = 23 57 53 .3

$T - T_1$ = − 7ᵐ 6ˢ.8

log a = 6.8213

log $(T - T_1)^2$ = 5.2604

log $a(T - T_1)^2$ = 2.0817

h = 50° 21′ 7″.6

$a(T - T_1)^2$ = + 2 0 .7

h_1 = 50 23 8 .3

ζ_1 = 39 36 51 .7

δ_1 = − 1 48 9 .2

φ = 37 48 42 .5

The mean of the three values found from these altitudes in Art. 172 is 37° 48′ 42″.8.

---

* This method is essentially the same as that proposed by LITTROW (*Astronomie*, Vol. I. p. 171.) I have here rendered it applicable to the sun without considering the change of declination, by introducing GAUSS's form for the reduction to the meridian.

197. (C.) *By two altitudes or zenith distances near the meridian and the difference of the azimuths.*—If the observer has no chronometer, he may still obtain his latitude by circummeridian altitudes, if he observes the altitudes with a universal instrument, and reads the horizontal circle at each observation, taking care, of course, that the star is always observed at the middle vertical thread. As this instrument generally gives directly the zenith distances, we shall substitute ζ for $90° - h$. We have the equation

$$\begin{aligned}\sin\delta &= \sin\varphi\cos\zeta - \cos\varphi\sin\zeta\cos A \\ &= \sin(\varphi - \zeta) + 2\cos\varphi\sin\zeta\sin^2\tfrac{1}{2}A\end{aligned}$$

whence

$$\cos\tfrac{1}{2}(\varphi + \delta - \zeta)\sin\tfrac{1}{2}[\zeta - (\varphi - \delta)] = \cos\varphi\sin\zeta\sin^2\tfrac{1}{2}A$$

But

$$\varphi - \delta = \zeta_1 = \text{the meridian zenith distance;}$$

and hence

$$\sin\tfrac{1}{2}(\zeta - \zeta_1) = \frac{\cos\varphi\sin\zeta\sin^2\frac{1}{2}A}{\cos[\delta - \frac{1}{2}(\zeta - \zeta_1)]} \tag{365}$$

which expresses the reduction to the meridian $= \zeta - \zeta_1$ when the absolute azimuth A is given. If the observation is very near the meridian, we may neglect $\frac{1}{2}(\zeta - \zeta_1)$ in the denominator of the second member, and take

$$\zeta - \zeta_1 = \frac{\cos\varphi\sin\zeta_1}{\cos\delta}\cdot\frac{2\sin^2\frac{1}{2}A}{\sin 1''}$$

or, putting

$$a = \frac{\cos\varphi\sin\zeta_1}{\cos\delta}\cdot\frac{\sin 1''}{2} \tag{366}$$

$$\zeta - \zeta_1 = aA^2 \tag{367}$$

from which it follows that near the meridian the reduction of the zenith distance varies as the square of the azimuth.

Now, when we have taken two observations, we have

$$\begin{aligned}\zeta_1 &= \zeta - aA^2 \\ \zeta_1 &= \zeta' - aA'^2\end{aligned}$$

whence, putting

$$\tau = \tfrac{1}{2}(A' - A)$$

we deduce the following equation, analogous to (360),

$$\zeta_1 = \tfrac{1}{2}(\zeta + \zeta') - a\tau^2 - \frac{[\tfrac{1}{2}(\zeta - \zeta')]^2}{a\tau^2} \tag{368}$$

Here τ is equal to one-half the difference of the readings of the horizontal circle, and is therefore known; and the computation is entirely similar to that of the formula (360).

198. (D.) *By three altitudes or zenith distances near the meridian and the differences of azimuths.*

Supposing the observations taken with a universal instrument, let

$$\zeta, \zeta', \zeta'' = \text{the true zenith distances,}$$
$$A, A', A'' = \text{the readings of the horizontal circle,}$$

we shall have, by the preceding article,

$$\left.\begin{aligned} \zeta_1 &= \zeta \;\; - \alpha\,(A \;\; - A_1)^2 \\ \zeta_1 &= \zeta' \; - \alpha\,(A' \; - A_1)^2 \\ \zeta_1 &= \zeta'' - \alpha\,(A'' - A_1)^2 \end{aligned}\right\} \tag{369}$$

in which A_1 is the (unknown) circle reading in the meridian, and α is the (unknown) change of zenith distance for 1″ of azimuth. These equations are solved in the same manner as (363); and hence we have the formulæ

$$\left.\begin{aligned} b = \frac{\zeta' - \zeta}{A' - A} \qquad & c = \frac{\zeta'' - \zeta'}{A'' - A'} \\ \alpha = & \frac{c - b}{A'' - A} \\ A_1 = \frac{A + A'}{2} - \frac{b}{2\alpha} \quad & \text{or } A_1 = \frac{A' + A''}{2} - \frac{c}{2\alpha} \end{aligned}\right\} \tag{370}$$

which determine α and A_1, after which ζ_1 is found by any one of the equations (369).*

* In this connection, see an article by LITTROW in ZACH's *Monatliche Correspondenz*, Vol. X. (1824).

ELEVENTH METHOD.—BY THE RATE OF CHANGE OF ALTITUDES NEAR THE PRIME VERTICAL.*

199. We have, Art. 149,

$$\frac{d\zeta}{15\,dt} = \cos\varphi \sin A$$

If then we observe two altitudes near the prime vertical in quick succession, noting the times by a stop-watch with as great precision as possible, and denote the difference of the altitudes, or of the zenith distances, by $d\zeta$, and the difference of the times by dt, we shall have

$$\cos\varphi = \frac{d\zeta}{15\,dt} \operatorname{cosec} A \tag{371}$$

The observation being made near the prime vertical, an error in the supposed azimuth A will have but small influence upon the result. If the observation is exactly in the prime vertical, or within a few minutes of it, we may put

$$\cos\varphi = \frac{d\zeta}{15\,dt} \tag{372}$$

This exceedingly simple method, though not susceptible of great precision, may be very useful to the navigator, as it is available when the sun is exactly east or west, and, consequently, when no other method is practicable, and, moreover, requires no previous knowledge of the time or the approximate latitude, or of the star's declination.†

EXAMPLE.—1853 July 3, PRESTEL observed, near the prime vertical, the time required by the sun to change its altitude by a quantity equal to its apparent diameter, by observing with a sextant first the contact of the lower limb with its image in an artificial horizon, and then the contact of the upper limb with

---

* PRESTEL, in *Astron. Nach.*, Vol XXXVII. p. 281.

† Since the star's declination is not required, this method has the additional advantage (which may at times be of great importance to the traveller) of being practicable *without the use of the Ephemeris.* This feature entitles this method to a prominent place in works on navigation.

its image, the sextant reading being the same at both observations, namely, 30° 15′ 0″. He found

	Chronometer.	
Contact of lower limb,	$4^h\ 43^m\ 34^s.$	P.M.
" upper "	$4\ \ 47\ \ \ 5.5$	
	$3\ \ 31.5$	

The sun's diameter was 31′ 32″. Hence we have

$$
\begin{array}{lll}
d\zeta = 31'\ 32'' = 1892'' & \log & 3.2769 \\
dt = 3^m\ 31^s.5 = 211^s.5 & \text{ar. co. log} & 7.6747 \\
 & \log \frac{1}{15} & 8.8239 \\
\varphi = 53^\circ\ 23'.5 & \log \cos \varphi & 9.7755
\end{array}
$$

The azimuth, however, was not exactly 90°, but about 88° 20′ Hence we shall have, more exactly,

$$
\begin{array}{lll}
 & & 9.7755 \\
A = 88^\circ\ 20' & \log \operatorname{cosec} A & 0.0002 \\
\varphi = 53\ \ 22.3 & \log \cos \varphi & 9.7757
\end{array}
$$

It is evident that the method will be more precise in high latitudes than in low ones.

FINDING THE LATITUDE AT SEA.

First Method.—By Meridian Altitudes.

200. This is the most common, as well as the simplest and most reliable, of the methods used by the navigator. The altitude is observed with the sextant (or quadrant) from the sea horizon, and, in addition to the corrections used on shore, the dip of the horizon is to be applied. The true altitude being deduced, the latitude is found by (277) or (278), Art. 161.

At sea the time is seldom so well known as to enable the navigator to take the star at the precise instant of its meridian passage. But the meridian altitude of a star is distinguished as the *greatest*, to secure which the observer commences to measure the star's altitude some minutes before the approximately computed time of passage, and continues to observe it until he perceives it to be falling. The greatest of all his measures is then assumed as the meridian altitude.

But, as before, we shall neglect the insensible term $\sin n \sin c$, and put $\cos n = 1$, and then the first and third of these equations will suffice to determine δ'. Moreover, since in the case of the moon τ will not exceed 1^m, the neglect of m will cause no sensible error in $\cos(\tau - m)$. Hence we take

$$\sin \delta' = \cos c \sin (\delta_1 \mp d)$$
$$\cos \delta' \cos \tau = \cos c \cos (\delta_1 \mp d)$$

or, developing the second members,

$$\sin \delta' = \cos c \cos d \sin \delta_1 \mp \sin s' \cos \delta_1$$
$$\cos \delta' \cos \tau = \cos c \cos d \cos \delta_1 \pm \sin s' \sin \delta_1$$

whence, by eliminating $\cos c \cos d$, we find

$$\mp \sin s' = \sin \delta' \cos \delta_1 - \cos \delta' \sin \delta_1 \cos \tau \qquad (195)$$

If now we put

$\delta =$ the moon's geocentric declination,
$s =$ " " semidiameter,
$\pi =$ " eq. hor. parallax,
$\varphi' =$ the geocentric or reduced latitude of the place of observation,
$\rho =$ the earth's radius for the latitude φ,
$\Delta, \Delta' =$ the moon's distance from the centre of the earth and from the place of observation, respectively, the equatorial radius of the earth being unity,

we have, by the formulæ of Art. 98, Vol. I.,

$$\Delta' \sin \delta' = \Delta \sin \delta - \rho \sin \varphi'$$
$$\Delta' \cos \delta' = \Delta \cos \delta - \rho \cos \varphi' \cos \tau$$

this last being equivalent to the more rigorous one in (133) of Vol. I., when the moon is near the meridian; and by Art. 128, Vol. I., we also have

$$\Delta' \sin s' = \Delta \sin s$$

Substituting these expressions in (195), after multiplying it by Δ', we find

$$\mp \Delta \sin s = \Delta \sin (\delta - \delta_1) + 2 \Delta \cos \delta \sin \delta_1 \sin^2 \tfrac{1}{2} \tau$$
$$- \rho \sin (\varphi' - \delta_1) - \rho \cos \varphi' \sin \delta_1 \sin^2 \tau$$

Dividing by $\Delta = \frac{1}{\sin \pi}$, this becomes

$$\mp \sin s = \sin (\delta - \delta_1) + 2 \cos \delta \sin \delta_1 \sin^2 \tfrac{1}{2} \tau$$
$$- \rho \sin \pi \sin (\varphi' - \delta_1) - \rho \sin \pi \cos \varphi' \sin \delta_1 \sin^2 \tau$$

where the last term is evidently insensible. If then we put

$$\sin p = \rho \sin \pi \sin (\varphi' - \delta_1) \tag{196}$$

we have

$$\sin (\delta - \delta_1) = \sin p \mp \sin s - 2 \cos \delta \sin \delta_1 \sin^2 \tfrac{1}{2} \tau$$

The last term (which is the reduction to the meridian) will seldom exceed 1″, and may be put under the form

$$\sin R = \left(\frac{15}{2}\right)^2 \sin^2 1'' . \sin 2\delta . \tau^2$$

The quantity τ is here the true hour angle of the moon, to find which, let

μ_1 = the sidereal time of the observation,
μ = " " moon's transit,
λ = the increase of the moon's right ascension in one sidereal second;

then

$$\tau = (1 - \lambda)(\mu - \mu_1)$$

and hence

$$R = \frac{225}{4} \sin 1'' \sin 2\delta (1 - \lambda)^2 (\mu - \mu_1)^2 \tag{197}$$

The first two terms of the value of $\sin (\delta - \delta_1)$ differ but little from $\sin (p \mp s)$. To find their exact value, we have

$$\begin{aligned}\sin p \mp \sin s &= \sin (p \mp s) + \sin p (1 - \cos s) \mp \sin s (1 - \cos p)\\ &= \sin (p \mp s) + 2 \sin p \sin^2 \tfrac{1}{2} s \mp 2 \sin s \sin^2 \tfrac{1}{2} p\end{aligned}$$

The last two terms of this will seldom amount to a tenth of a second, and therefore the formula may be regarded as perfectly accurate under the form

$$\sin p \mp \sin s = \sin (p \mp s) \mp \tfrac{1}{2} (p \mp s) \sin 1'' \sin p \sin s$$

Now, since $\delta - \delta_1$ and $p \mp s$ differ by so small a quantity, the ratio of the sine to the arc will be the same for both of them: hence we shall have, with the utmost precision,

$$\delta = \delta_1 + p \mp s \mp \tfrac{1}{2} (p \mp s) \sin p \sin s - R \tag{198}$$

as given by BESSEL.* The upper or lower sign is to be used according as the north or the south limb is observed.

The declination thus found is reduced to the time μ_1 of the observation. But if we wish its value at the time of the meridian passage, we must add to it the correction $(\mu - \mu_1)\lambda'$, in which λ' is the increase of the declination in one sidereal second, or

$$\lambda' = \frac{\Delta\delta}{60.1643}$$

where $\Delta\delta$ = the increase of declination in one minute of mean time, as now given in the American Ephemeris. The value of $1 - \lambda$ is found as in Art. 154: namely, taking $\Delta\alpha$ = the increase of the moon's right ascension in one minute of mean time, we have

$$\lambda = \frac{\Delta\alpha}{60.1643}$$

so that, putting

$$1 - \lambda = \frac{1}{B}$$

we shall have

$$\log(1 - \lambda) = \text{ar. co.} \log B$$

and log B may be taken from the table on page 179.

In practice, it will generally be most convenient to apply the several reductions directly to the observed zenith distance, as in the following example.

EXAMPLE.—The declination of the moon was observed with the meridian circle of the Washington Observatory, 1850, September 17. The nadir point was first observed as follows:

	Circle Microscopes.					
	A	B	C	D	Means.	
Nadir point at $20^h.5$	0″.9	1″.9	2″.2	1″.4	1″.60	Micrometer thread in coincidence with its image: mean of 10 readings = $38^r.934$.
	0 .7	1 .4	2 .0	1 .6	1 .42	
Means	0 .80	1 .65	2 .10	1 .50	1 .51	

The value of one revolution of the micrometer = 34″.356, or

* *Tabulæ Regiomontanæ*, Introd. p. LV.

$1'' = 0^r.0291$; and hence, by the method of Art. 197, the micrometer zero (or reading of the micrometer when the circle reading was $0° \ 0' \ 0''$) was

$$(M) = 38^r.934 + 0^r.0291 \times 1.51 = 38^r.978$$

The observation of the moon was as follows, S.L. denoting south limb:

	Circle Microscopes.					Clock $= \mu_1$	Micrometer $= M$.
	A	B	C	D	Mean.		
Moon, S. L.	55° 52′ 45″.7	42″.8	45″.2	46″.1	44″.95	$21^h 17^m 21^s$	$39^r.956$
	Barom. 30in.114 Att. Therm. 64°. Ext. Therm. 52°.8					32	39 .904
						43	39 .875

The circle was *west*, in which position the readings are zenith distances towards the south. The correction for runs was $-0''.75$ for $3'$, and since the excess of the reading over a multiple of $3'$ is $1' \ 44''.95$, the proportional correction for runs is $-0''.43$.

The clock time of transit of the moon's centre over the meridian was $\mu = 21^h \ 17^m \ 16^s.80$.

The latitude of the observatory is $\varphi = 38° \ 53' \ 39''.25$, and therefore $\varphi - \varphi' = 11' \ 14''.54$, $\log \rho = 9.9994302$. The longitude is $5^h \ 8^m \ 12^s$ west of Greenwich.

For the date of the observation, we take from the Nautical Almanac

$$\delta = -16° \ 1'.7$$
$$\Delta\delta = +6''.377 \quad \text{in } 1^m \text{ mean time,} \qquad \pi = 54' \ 9''.64$$
$$\Delta\alpha = 2^s.0150 \quad " \quad " \quad " \quad " \qquad s = 14' \ 45''.49$$

whence $\log(1 - \lambda) = 9.98521$ and $\lambda' = +0''.1060$

The correction for the micrometer, or $M - (M)$, converted into seconds, is additive to the circle reading. The reduction to the meridian, or R, found by (197), is also algebraically additive to the circle reading, attention being paid to the sign of δ; and the correction for change of declination to be added to the circle reading will be $-(\mu - \mu_1)\lambda'$. Since the sum of these three corrections should be the same for each micrometer observation, the precision of the observations will be shown by computing this sum for each. Thus, we find

$\mu - \mu_1$	$M-(M)$	R	$-(\mu-\mu_1)\lambda'$	Sums.
$-$ 4^s.2	33″.60	$-$ 0″.00	$+$ 0″.44	34″.04
$-$ 15 .2	31 .82	$-$ .03	$+$ 1 .61	33 .40
$-$ 26 .2	30 .82	$-$ .09	$+$ 2 .78	33 .51
			Mean =	33 .65

Hence we have

	Circle reading =	55° 52′ 44″.95
	Corr. for runs =	$-$ 0 .43
	Mean corr. for microm., &c. =	$+$ 33 .65
	Apparent zenith distance =	55 53 18 .17
	By Table II. Refraction =	$+$ 1 25 .60
$\varphi' - \delta_1 = \varphi - \delta_1 - (\varphi - \varphi')$	$\varphi - \delta_1 =$	55 54 43 .77
$= 55° \, 43' \, 29''$	$-(p+s) =$	$-$ 59 27 .24
By (196), $p = 44' \, 41''.75$	$-\frac{1}{2}(p+s)\sin p \sin s =$	$-$ 0 .10
	$\varphi - \delta =$	54 55 16 .43
	$\varphi =$	38 53 39 .25
	$\delta =$	$-$ 16 1 37 .18

206. *Observations of the declination of a planet, or the sun.*—The larger planets are observed in the same manner as the moon, that is, by making the micrometer thread tangent to the limb, and when the planet is treated as a spherical body the observation is also reduced in the same manner.

In the case of the sun, both limbs may be observed. The reduction to the meridian may be facilitated by a table giving the logarithm of the factor

$$b = \frac{225}{4} \sin 1'' \, (1 - \lambda)^2 \sin 2\delta$$

for each day of the fictitious year (Vol. I. Art. 406), such as Bessel's Table XII. of the *Tabulæ Regiomontanæ.* This table also gives for each day of the year the value of

a = increase of the sun's declination in 100 sidereal seconds,

so that the reduction of the observed declination to the meridian, including the correction for the change of declination in the interval τ, is

$$\frac{a\tau}{100} + b\tau\tau$$

The correction for parallax may be put under the form

$$p = \frac{8''.57116}{r} \rho \sin(\varphi' - \delta)$$

in which $r =$ sun's distance from the earth, the mean distance being unity, and in each observatory this quantity may be computed for the latitude, and for each day of the year, and also inserted in the table. In order to embrace every thing necessary for the complete reduction of the observed declination, the table contains also the sun's semidiameter for each day of the fictitious year.

207. *Correction of the observed declination of a planet's or the moon's limb for spheroidal figure and defective illumination.*—Let us consider the most general case of a spheroidal planet partially illuminated. The correction to reduce the observed declination of the limb to that of the centre is equal to the perpendicular distance from the centre to the micrometer thread, which is tangent to the limb and perpendicular to the meridian. The formulæ for computing this perpendicular in general are (Vol. I. p. 580)

$$\tan \vartheta' = \frac{\tan \vartheta}{c} \qquad \sin \chi = \sin \vartheta' \sin V$$

$$s'' = \frac{s \sin \vartheta \cos \chi}{\sin \vartheta'}$$

in which s'' is the required perpendicular, ϑ the angle which it makes with the axis of the planet (reckoning from the north point of the disc towards the east), c is a constant depending upon the eccentricity of the planet's meridian, V the angular distance of the earth and sun as seen from the planet, and s is the equatorial radius of the disc, or greatest apparent semidiameter at the time of the observation. The perpendicular here coincides with the declination circle, and consequently we have at once $\vartheta = -p$, or $180° - p$, according as the north or the south limb is observed; p denoting, as in the article referred to, the position angle of the axis of the planet. From the discussion in Vol I. Art. 354, it follows that (putting $-p$ for ϑ) the north limb will be full (and, consequently, the south limb gibbous) when $\sin p$ and $\sin V$ have the same sign. We shall, therefore, here change the sign of $\sin \chi$, and take

$$\left.\begin{array}{ll} \tan p' = \dfrac{\tan p}{c} & \sin\chi = \sin p' \sin V \\ & s'' = \dfrac{s_0}{r'}\cdot\dfrac{\sin p}{\sin p'}\cos\chi \end{array}\right\} \quad (199)$$

in which s_0 = the greatest apparent semidiameter at the mean distance of the sun from the earth, and r' = the planet's geocentric distance. We then have the rule: *the north or the south limb is the full limb according as sin χ is positive or negative.* The formulæ for computing p, V, and c are given in Vol. I. Arts. 348 et seq., and s_0 is given on p. 578.

The gibbosity of Saturn, however, is wholly insensible, and even that of Jupiter at the north and south points of the limb cannot exceed 0″.05, which is so much less than the usual errors of declination observations that it may be disregarded. Hence, for Saturn and Jupiter the correction will depend only upon the figure of the planet, and will be computed by the equations

$$\tan p' = \frac{\tan p}{c} \qquad s'' = \frac{s_0}{r'}\cdot\frac{\sin p}{\sin p'} = \frac{cs_0}{r'}\cdot\frac{\cos p}{\cos p'}$$

in which for Jupiter we take $\log c = 9.9672$, and for Saturn $c = \sqrt{(1 - ee\cos^2 l)} = \sqrt{(1 - [9.2706]\cos^2 l)}$, l and p being taken directly from the tables for Saturn's Ring given in the Ephemeris.

A further simplification may be permitted in the case of Saturn; for, on account of the small values of p, the ratio $\dfrac{\cos p}{\cos p'}$ will be very nearly unity, and if we take $s'' = \dfrac{cs_0}{r'}$ we shall have the true value of s'' within less than 0″.05.

It is hardly necessary to remark that when we neglect the gibbosity of Jupiter or Saturn, the mean of the observed declinations of the north and south limbs gives at once the declination of the centre.

For Mars, Venus, and Mercury the correction will be only for defective illumination; but in this case we can avoid the separate computation of p and V, as follows. Substituting in the equation for $\sin\chi$ (199) the values of $\sin p$ and $\sin V$ given in Vol. I. p. 577, and moreover observing that, since these bodies are regarded as spherical, we have $c = 1$, and, consequently, $p' = p$, there results

$$\sin\chi = \frac{R}{R'}\left[\cos\delta'\sin D - \sin\delta'\cos D\cos(\alpha' - A)\right] \quad (200)$$

in which

α', δ' = the planet's right ascension and declination,
A, D = the sun's " " "
R, R' = the earth's and the planet's distances from the sun;

and a positive value of $\sin\chi$ will here also indicate that the north limb is full and the south limb gibbous, and a negative value the reverse. Adapting this formula for logarithms, we have, therefore,

$$\left.\begin{aligned}\tan F &= \tan D \sec(\alpha' - A)\\ \sin\chi &= \frac{R}{R'}\cdot\frac{\sin(F-\delta')\sin D}{\sin F}\end{aligned}\right\} \quad (201)$$

or, more conveniently, perhaps,

$$\left.\begin{aligned}\tan E &= \tan\delta' \cos(\alpha' - A)\\ \sin\chi &= \frac{R}{R'}\cdot\frac{\sin(D-E)\cos\delta'}{\cos E}\end{aligned}\right\} \quad (201^*)$$

E being taken less than 90°, with the sign of its tangent. Then we find the reduction to the centre of the planet by the formula

$$s'' = \frac{s_0}{r'}\cos\chi \quad (202)$$

If the declination of a *cusp* of Venus or Mercury has been observed, we must find p by the formula (Vol. I. p. 577)

$$\tan p = \cot(\alpha' - A)\sin(F - \delta')\sec F \quad (203)$$

in which F has the same value as above, and then the reduction to the centre of the planet will be

$$s'' = \frac{s_0}{r'}\cos p$$

For the moon, when the gibbous limb has been observed, the formulæ (201) may be used for computing χ; but on account of the small difference of R and R', we may put their quotient $=1$. Since the declination of the gibbous limb will not be observed except when the moon is nearly full, it will be best to reduce the observations as if the observed limb were full, according to Art. 205, and then to apply a small correction for gibbosity.

This correction will be $\Delta s = s - s\cos\chi = s$ versin χ. Hence the formulæ for the moon will be

$$\left.\begin{aligned} \tan E &= \tan\delta'\cos(\alpha' - A) \\ \sin\chi &= \frac{\sin(D - E)\cos\delta'}{\cos E} \\ \Delta s &= s \operatorname{versin}\chi \end{aligned}\right\} \quad (204)$$

Example 1.—The apparent declination of the southern cusp of Venus, at its transit over the meridian of Greenwich, July 16, 1852, observed with the transit circle, was

$$\delta' = 15^\circ\ 0'\ 45''.60$$

From the Nautical Almanac, we have

$$\begin{array}{ll} \alpha' = 8^h\ 11^m\ \ 1^s.46 & \log r' = 9.4675 \\ A = 7\ \ 43\ \ 42.80 & D\ \ = 21^\circ\ 19'\ 8'' \end{array}$$

and from Vol. I. p. 578,

$$s_0 = 8''.55$$

Hence, by (203), we find log tan p = 0.0031, and, consequently,

$$s'' = \frac{s_0}{r'}\cos p = 20''.53$$

and the apparent declination of the planet's centre was, therefore,

$$\delta = 15^\circ\ 1'\ 6''.13$$

Example 2.—The apparent declinations of Jupiter's north and south limbs, observed at Greenwich, March 18, 1852, were—

$$\begin{array}{ll} \text{N.L.} & \delta' = -\ 17^\circ\ 21'\ 57''.36 \\ \text{S.L.} & \delta' = -\ 17\ \ \ 22\ \ 37\ .61 \end{array}$$

To illustrate the complete formulæ, let us take the gibbosity of the planet into account. For this purpose, we take from the Nautical Almanac

$$\begin{array}{lll} \alpha' = \ \ 230^\circ\ 56'.4 & A = 224^\circ\ 25'.0 & \\ \delta' = -\ \ 17\ \ \ 22\ .2 & \epsilon\ = \ \ 23\ \ \ 27\ .5 & \log r' = 0.6783 \end{array}$$

and from Vol. I. p. 574,

$$n = 357^\circ\ 56'.5 \qquad i = 25^\circ\ 25'.8$$

Hence, by the formulæ (619), Vol. I.,

$$F = 201° \; 23'.5 \qquad \lambda = 234° \; 52'\,3$$
$$V = \Lambda - \lambda = -10° \; 27'.7$$
$$F' = -20° \; 47'.5 \qquad \log \tan p = 9.4281$$

Then, by (199), taking $\log c = 9.9672$, we have

$$\log \sin \chi = n8.7025$$

from which it follows that the south limb was full. Hence, taking $s_0 = 99''.70$, we find

$$\text{For full limb} \qquad (s'') = \frac{s_0}{r'} \cdot \frac{\sin p}{\sin p'} = 19''.50$$
$$\text{For gibbous limb} \quad s'' = (s'') \cos \chi \quad = 19\;.47$$

The declination of the centre was, therefore, according to these observations,

$$\text{From N.L.} \quad \delta = -17° \; 22' \; 16''.83$$
$$\text{" S.L.} \qquad \text{" " } 18\;.11$$

Considering the difference of these results, which is by no means as great as often occurs in the Greenwich observations of Jupiter, it appears that the practice there followed of always applying the *polar* semidiameter (which is the one given in the Nautical Almanac) is quite accurate enough *for these observations.* Our more exact method will not be without application, however, in cases where greater refinement both in observation and reduction are attained.

EXAMPLE 3.—At Greenwich, Feb. 6, 1852, the declination of the moon's centre deduced from an observation of the north limb, on the assumption that this limb was full, was

$$\delta' = +13° \; 17' \; 0''.58$$

For the time of the moon's transit on this date, we have

$$\alpha' = 158° \; 18'.6 \qquad A = 319° \; 56'.1$$
$$s = 16' \; 31'' \qquad D = -15 \;\; 36.3$$

whence, by (204),

$$\chi = -2° \; 58'$$

which shows that the north limb was gibbous. The correction was

$$\Delta s = s \text{ versin } \chi = 1''.33$$

and the true declination was, therefore,

$$\delta = + 13° \; 17' \; 1''.91$$

CHAPTER VII.

THE ALTITUDE AND AZIMUTH INSTRUMENT.

208. This instrument may be regarded as a transit instrument combined with both a vertical and a horizontal circle, by means of which both the altitude and the azimuth of a star may be observed at the instant of its transit through the vertical plane described by the telescope. This combination is not often used for the higher purposes of astronomical research, as every additional movement introduced into an instrument diminishes its stability and increases the risk of error. However, at Greenwich, a regular series of extra-meridian observations of the moon is carried on with such an instrument, for the sake of comparison with meridian observations. The instrument has there received the name of the *altazimuth*. In other places, it has been called the *astronomical theodolite;* and, in fact, the general theory of the instrument, which will be given hereafter, will be found to be directly applicable to the common theodolite employed in geodetic measurement.

Still another name is the *universal instrument*, so called on account of its numerous applications; but this name is usually given only to the portable instruments of this class. The small universal instruments of Ertel are well known.

209. Sometimes the horizontal circle is reduced to small dimensions, and designed simply as a finder, or to set the instru-

ment approximately at a given azimuth; while the vertical circle is made of unusually large dimensions, and is intended for the most refined astronomical measurement. The instrument is then known simply as a *vertical circle*. Such is the ERTEL Vertical Circle of the Pulkowa Observatory, the telescope of which has a focal length of 77 inches, and its vertical circle a diameter of 43 inches.*

This instrument is permanently mounted upon a solid granite pier G, Plates X. and XI., which is insulated from the walls and floor of the building. It stands upon a tripod which is adjusted by foot screws. The three feet are so placed that two of them are in the east and west line: hence, but one of these two is seen in Plate X., which is a projection of the instrument upon the plane of the meridian, while all three are seen in Plate XI., which is a projection upon the plane of the prime vertical. The meridional foot screw ω carries a small circle γ graduated into 360°, the index of which is attached to the foot. One revolution of this circle changes the inclination of the instrument in the plane of the meridian 318″: consequently, one division corresponds to 0″.88.

The centre of the instrument is held in place by the support a attached to the pier.

The vertical stand consists of a hollow cone of brass, in which turns the steel axis b. The lower extremity of this axis is convex and smoothly finished, and is supported by a system of three counterpoises c, suspended upon levers which relieve the pressure upon the bearing points of the vertical axis, and thus diminish the friction. At the top of the conical stand is a 13 inch azimuth circle, the verniers of which are attached to the axis. This is provided with a clamp and tangent screw which is moved by the rod d in giving the upper portion of the instrument a small motion in azimuth.

The upper extremity of the vertical steel axis carries the strong oblong bar e, which may be called the bed of the instrument. On this bed rests the adjustable frame $vfgv$, which supports the horizontal axis i in the Vs at vv. This axis should be perpendicular to the vertical axis, and its adjustment in this respect is effected by means of two opposing screws at h.

The axis i has two equal cylindrical pivots of steel at vv. It is hollow, to admit light from the lamp $\varkappa$, which is reflected upon

* See *Description de l'obser. cent.*, &c., p. 130.

the threads of the reticule of the telescope by a mirror in the interior of the tube at *u*. The telescope and principal vertical circle *o* are firmly and invariably attached to one extremity of this axis. At the opposite end of the axis is a smaller vertical circle *m*, which serves as a finder. From the centre of this finding circle radiate four conical arms terminating in ivory balls *n*. The telescope is swept in the vertical plane solely by means of these balls, never by touching the telescope or principal vertical circle. When the telescope is approximately pointed and clamped, fine vertical motion is given to the tangent screw by the rod *k*. The instrument is swept in azimuth by means of an ivory ball at *l*, the fine azimuthal motion being given by the rod *d*.

The circle is read off by four microscopes attached to a square frame α, which is fixed to the frame *vfgv*. The level β attached to this frame indicates its inclination with respect to the horizon. The circle is divided to 2′, and the microscopes read directly to single seconds, and by estimation to 0″.1, or even less. The probable error of reading of a single microscope is given by PETERS as only 0″.090 in observations by day, and 0″.098 in observations by night.

The friction of the horizontal axis in the Vs is diminished by the single counterpoise *p*, which, by means of a lever, the fulcrum of which is at *q*, supports the principal part of the weight of the telescope, vertical circles, and horizontal axis, by exerting an upward pressure at *r*. The point *r* being at suitable distances from the two Vs respectively (nearer to the principal circle than to the finder), the friction in both Vs is equally relieved; while the whole weight of the movable portion of the instrument is transferred to a point *q*, very near to the vertical axis of rotation.

The striding level *s* rests upon the pivots of the horizontal axis, and, by reversal in the usual manner, serves to measure the inclination of this axis to the horizon.

The reticule at *t* is composed of three horizontal threads, two of which are close parallel threads (the clear space between them being only 6″), which serve for the observation of objects which present sensible discs, or of those which are too faint to be observed by bisection (see Art. 198). The third thread is 18″ from the others, and is used in observing stars by bisection. The unequal distances prevent mistakes in the choice of threads. These horizontal threads are crossed by two vertical ones, the

distance of which is 1′ of arc. The middle point between these determines the optical centre of the instrument, and all observations are made as nearly as possible at this point.

The extreme accuracy attainable in the observation of zenith distances with this instrument may be inferred from the following values of the *zenith point Z* (see Art. 219) of the circle, as cited by Struve, from observations by Peters upon *Polaris* at its upper and lower culminations:

1843.	Upper transit. Z	Diff. from mean.		Lower transit. Z	Diff. from mean.
April 13	0° 0′ 33″.13	− 0″.32	April 14	0° 0′ 33″.64	− 0″.08
14	33 .26	− 0 .19	16	33 .32	− 0 .40
17	33 .82	+ 0 .37	20	33 .45	− 0 .27
19	33 .27	− 0 .18	21	33 .94	+ 0 .22
20	33 .75	+ 0 .30	22	33 .48	− 0 .24
22	33 .17	− 0 .28	24	33 .50	− 0 .22
24	33 .45	0 .00	25	33 .94	+ 0 .22
25	33 .68	+ 0 .23	26	33 .98	+ 0 .26
26	33 .29	− 0 .16	27	33 .82	+ 0 .10
27	33 .68	+ 0 .23	28	34 .12	+ 0 .40
Mean	0 0 33 .45		Mean	0 0 33 .72	

Hence, assuming that the zenith point of the circle was constant, the probable error of an observed value of Z was, for either series, $= 0''.22$. This error, however, is the combined effect of error of observation and variability of Z. But the probable error of observation was obtained from the discrepancies between the several values of the latitude deduced from these same observations, and was $= 0''.17$: so that the probable error of Z arising from variation in the instrument was $= \sqrt{[(0''.22)^2 - (0''.17)^2]} = 0''.14$. The means for the two transits differ by $0''.27$, which results from the use of different divisions of the circle and different parts of the micrometers. To compare them justly, it would be necessary first to eliminate especially the division errors.

In order to eliminate the effects of flexure, the objective and ocular are made interchangeable (see Art. 204).

The dimensions of the various parts of the instrument may be

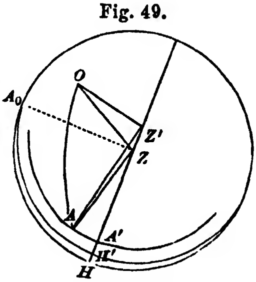
taken from the plates, which are accurately drawn upon a scale of $\frac{1}{20}$.*

210. The portable universal instruments are usually so arranged that the vertical circle may be removed altogether from the instrument when horizontal angles only are to be measured. One of these instruments is represented in Plate XII. In Fig. 1, the instrument is arranged for measuring horizontal angles exclusively. In Fig. 2, the telescope of Fig. 1 is replaced by another which is connected with a vertical circle and (unlike the azimuth telescope) is at the end of the horizontal axis. The weight of the telescope and vertical circle is counterpoised by a weight at the opposite end of the axis. The focal length of the telescope in instruments of this kind seldom exceeds 24 inches.

The following discussion of the theory of these instruments will apply to any of the forms above mentioned, as I shall consider their two applications—to azimuths and to altitudes—independently of each other.

211. *Azimuths.*—Let A_0H, Fig. 49, represent the true horizon, Z the zenith. Let us suppose the vertical axis of the instrument to be inclined to the true vertical line, so that when produced it meets the celestial sphere in Z'. Let A_0H' be the great circle of which Z' is the pole. The plane of this circle is that of the graduated horizontal circle of the instrument. Let us suppose, further, that the horizontal rotation axis, which should be at right angles to the vertical axis, and, consequently, parallel to the horizontal circle, makes a small angle with this circle. As the instrument revolves about its vertical axis, this rotation axis will describe a conical surface, and the prolongation of this axis to the celestial sphere will describe a small circle AA' parallel to A_0H'. Let A be the point in which this axis produced through the circle end meets the sphere at the time of an observation, and O the position of a star observed on any given vertical thread

Fig. 49.



* For all the particulars of the use of this instrument in the determination of the declination of a circumpolar star, consult the memoir of Dr. C. A. F. Peters, *Astron. Nach.*, Vol. XXII., *Resultate aus Beobachtungen des Polarsterns am Ertelschen Verticalkreise der Pulkowaer Sternwarte.*

in the field. As the telescope revolves upon the horizontal axis, its axis of collimation describes a great circle of which A is the pole, and the given thread describes a small circle parallel to this great circle. Let

c = the distance of the thread from the collimation axis, positive when the thread is on the same side of the collimation axis as the vertical circle,
b = the elevation of A above the horizon as given by the spirit level applied to the horizontal axis, positive when the circle end of this axis is too high,
i = the inclination of the vertical axis to the true vertical line,
i' = the inclination of the horizontal axis to the azimuth circle,
$a = AZH$,
$a' = AZ'H$,
A = the azimuth of the star O, reckoned from A_0 as the origin,
z = the zenith distance of the star;

then, in the triangle AZZ', we have $AZ = 90° - b$, $ZZ' = i$, $AZ' = 90° - i'$, $AZZ' = 180° - a$, $AZ'Z = a'$, and hence, by Sph. Trig.,

$$\begin{aligned}\sin b &= \cos a' \cos i' \sin i + \sin i' \cos i \\ \cos b \cos a &= \cos a' \cos i' \cos i - \sin i' \sin i \\ \cos b \sin a &= \sin a' \cos i'\end{aligned}$$

But, i, i', and b being always so small that we can neglect their squares, these equations may be reduced to the following

$$\left.\begin{aligned}a &= a' \\ b &= i \cos a' + i' = i \cos a + i'\end{aligned}\right\} \quad (205)$$

In the triangle AZO, we have the angle $AZO = A_0ZO + A_0ZA = A + 90° - a$, and the sides $AO = 90° + c$, $AZ = 90° - b$, $ZO = z$; and hence

$$-\sin c = \sin b \cos z - \cos b \sin z \sin(A - a)$$

or, since c and b are small,

$$\sin(A - a) = \frac{b}{\tan z} + \frac{c}{\sin z}$$

Hence $\sin(A - a)$ is also a small quantity, and the angle $A - a$

	$\Delta' T$
$F.$	$-4^h\ 40^m\ 59^s.20$
$M.$	$-4\ \ 9\ \ 55.53$
$P.$	$-5\ \ 18\ \ \ 3.24$

And finally, at Carthagena, observations on the 25th and 29th of June gave the corrections and rates at the mean epoch June $27^d.0$ as follows:

	$\Delta' T$	$\delta' T$
$F.$	$-5^h\ \ 7^m\ 23^s.55$	$+0^s.85$
$M.$	$-4\ \ 37\ \ 47.98$	-5.90
$P.$	$-5\ \ 44\ \ 34.42$	$+0.30$

Employing the rates found at La Guayra, the corrections of the chronometers on June $5^d.870$ at Porto Cabello (for which we have $t = 11^d.985$), and the resulting difference of longitude, are, by formula (383), are as follows:

	$\Delta T + t.\delta T$	P. Cabello—La Guayra.
$F.$	$-4^h\ 32^m\ 58^s.57$	$+4^m\ 17^s.23$
$M.$	$-4\ \ 1\ \ 11.81$	19.47
$P.$	$-5\ \ 10\ \ \ 1.32$	12.06
		Mean $+4\ \ 16.25$

With the same rates, we have on June 12.890 at Curaçoa (for $t = 19^d.005$) the corrections and the corresponding difference of longitude, as follows:

	$\Delta T + t.\delta T$	Curaçoa—La Guayra.
$F.$	$-4^h\ 32^m\ 53^s.17$	$+8^m\ \ 6^s.03$
$M.$	$-4\ \ 1\ \ 43.68$	$8\ \ 11.85$
$P.$	$-4\ \ 10\ \ 11.64$	$7\ \ 51.60$
		Mean $+8\ \ \ 3.16$

With the same rates, we have on June 27^d at Carthagena (for $t = 33^d.115$) the corrections and the corresponding difference of longitude, as follows:

	$\Delta T + t.\delta T$	Carthagena—La Guayra.
F	$-4^h\ 32^m\ 42^s.30$	$+34^m\ 41^s.25$
$M.$	$-4\ \ 2\ \ 47.74$	$35\ \ \ 0.24$
$P.$	$-5\ \ 10\ \ 32.38$	$34\ \ \ 2.04$
		Mean $+34\ \ 34.51$

Now, to correct these results for the changes in the rates of the chronometers, we have, in the interval $n = 33.115$,

	$\delta' T - \delta T$
$F.$	$+ 0^s.08$
$M.$	$- 1.36$
$P.$	$+ 1.77$
	$s = + 0.49$

and, consequently,

$$q = \frac{+ 0^s.49}{2 \times 3 \times 33.115} = + 0^s.002466$$

Applying the correction $t^2 q$ to the several results, the true differences of longitude from La Guayra are found as follows:

	Approx. diff. long.	$t^2 . q$	Corrected diff. long.
P. Cabello	$+ 4^m\ 16^s.25$	$+ 0^s.35$	$+ 4^m\ 16^s.60$
Curaçoa	$+ 8\ \ 3.16$	$+ 0.89$	$+ 8\ \ 4.05$
Carthagena	$+ 34\ \ 34.51$	$+ 2.70$	$+ 34\ \ 37.21$

But it is usually preferable to carry out the result by each chronometer separately, in order to judge of the weight to be attached to the final mean by the agreement of the several individual values. For this purpose we have here, by the formula (384), for $n = 33.115$,

	$\frac{1}{2} x$
$F.$	$+ 0.00121$
$M.$	$- 0.02054$
$P.$	$+ 0.02673$

and hence the correction $\frac{1}{2} t^2 . x$ is, for the several cases, as follows:

	P. Cabello.	Curaçoa.	Carthagena.
$F.$	$+ 0^s.17$	$+ 0^s.44$	$+ 1^s.32$
$M.$	$- 2.95$	$- 7.41$	$- 22.52$
$P.$	$+ 3.84$	$+ 9.65$	$+ 29.31$

Applying these corrections severally to the above approximate results, we have, for the differences of longitude from La Guayra,

	P. Cabello.	Curaçoa.	Carthagena.
$F.$	$+ 4^m\ 17^s.40$	$+ 8^m\ 6^s.47$	$+ 34^m\ 42^s.57$
$M.$	16.52	4.44	37.72
$P.$	15.90	1.25	31.35
Means	$+ 4\ \ 16.61$	$+ 8\ \ 4.05$	$+ 34\ \ 37.21$

agreeing precisely with the corrected means found above.

If the chronometers have been exposed to considerable changes of temperature, the proper correction may be introduced by the method of Art. 223.

216. *Chronometric expeditions between two points.*—Where a difference of longitude is to be determined with the greatest possible precision, a large number of chronometers are transported back and forth between the extreme points. There are two classes of errors of chronometers which are to be eliminated: 1st, the *accidental* errors, or variations of rate which follow no law, and may be either positive or negative; 2d, the *constant* errors, or variations of rate which, for any given chronometer, appear with the same sign and of the same amount when the chronometer is transported from place to place; in other words, a constant acceleration, or a constant retardation, as compared with the rates found when the chronometer is at rest. The accidental errors are eliminated in a great degree by employing a large number of chronometers, the probability being that such errors will have different signs for different chronometers. The constant errors cannot be determined by comparing the rates at the two extreme points, since these rates are found only when the chronometer is at rest; but if the chronometers are transported in both directions, from east to west and from west to east, a constant error in their *travelling* rates will affect the difference of longitude with opposite signs in the two journeys, and will disappear when the mean is taken. These considerations have given rise to extensive expeditions, of which probably the most thoroughly executed was that carried out by Struve, in 1843, between Pulkova and Altona.* In this expedition sixty-eight chronometers were transported eight times from Pulkova to Altona and back, making sixteen voyages in all, giving the difference of longitude between the centre of the Pulkova Observatory and the Altona Observatory $1^h\ 21^m\ 32^s.527$, with a probable error of only $0^s.039$.

Chronometric expeditions between Liverpool (England) and

* *Expédition chronométrique exécutée par ordre de Sa Majesté L'Empereur Nicolas I. pour la détermination de la longitude géographique rélative de l'observatoire central de Russie.* St. Petersburg, 1844.

For an account of the carefully executed expedition under Professor Airy to determine the longitude of Valentia in Ireland, see the Appendix to the Greenwich Observations of 1845.

Cambridge (U. S.) were instituted in the years 1849, '50, '51, and '55 by the U. S. Coast Survey, under the superintendence of Professor A. D. BACHE. The results of the expeditions of 1849, '50, and '51, discussed by Mr. G. P. BOND,* proved the necessity of introducing a correction for the temperature to which the chronometers were exposed during the voyages, and particular attention was therefore paid to this point in the expedition of 1855, the details of which were arranged by Mr. W. C. BOND. The results of six voyages,—three in each direction,—according to the discussion of Mr. G. P. BOND,† were as follows:

	Longitude.
Voyages from Liverpool to Cambridge	$4^h\ 32^m\ 31^s.92$
" " Cambridge to Liverpool	4 32 31 .75
Mean	4 32 31 .84

with a probable error of $0^s.19$. In this expedition fifty chronometers were used. The greater probable error of the result, as compared with STRUVE'S, is sufficiently explained by the greater length of the voyages and their smaller number.

217. The following is essentially STRUVE'S method of conducting the expeditions and discussing the results.

Before embarking the chronometers at the first station (A), they are carefully compared with a standard clock the correction of which on the time at that station has been obtained with the greatest precision by transits of well-determined stars. (See Vol. II., "Transit Instrument.") Upon their arrival at the second station (B), they are compared with the standard clock at that station.‡ From these two comparisons the chronometer corrections at the two stations become known, and, if the rates are known, a value of the longitude is found by each chronometer by (383). But here it is to be observed that the rate of a chronometer is rarely the same when in motion as when at rest. It is necessary, therefore, to find its *travelling rate* (or *sea rate*, as it is called when the chronometer is transported by sea). This might be effected by finding—*first*, the correction of the chrono-

* Report of the Superintendent of the U. S. Coast Survey for 1854, Appendix No. 42.

† Report of the Superintendent of the U. S. Coast Survey for 1856, p. 182.

‡ For the method of comparing chronometers and clocks with the greatest precision, see Vol. II.

meter at the station A immediately before starting; *secondly*, its correction at B immediately upon its arrival there; and *thirdly*, having, without any delay at B, returned directly to A, finding again its correction there immediately upon arriving. The difference between the two corrections at A is the whole travelling rate during the elapsed time, and this rate would be used in making the comparison with the correction obtained at B, and in deducing the longitude by (383).

But, since the chronometer cannot generally be *immediately* returned from B, its correction for that station should be found both upon its arrival there and again just before leaving, and the travelling rate inferred only from the time the instrument is in motion. For this purpose, let us suppose that we have found

at the times	t,	t',	t'',	t''',
the chron. corrections	a,	b,	b',	a',

the correction a at the station A before leaving; b upon arriving at B; b' before leaving B; and a' upon the return to A. The times t, t', t'', t''', being all reckoned at the same meridian, if we now put

$m =$ the mean travelling rate of the chronometer in a unit of time,
$\lambda =$ the longitude of B west of A,

we shall have, upon the supposition that the mean travelling rate is the same for both the east and west voyages,

$$\lambda = a + m\,(t' - t\,) - b$$
$$\lambda = a' - m\,(t''' - t'') - b'$$

From these two equations the two unknown quantities m and λ become known. Putting

$$\tau = t' - t \qquad\qquad \tau'' = t''' - t''$$

we find, first,

$$m = \frac{(a' - a) - (b' - b)}{\tau + \tau''} \tag{387}$$

in which the numerator evidently expresses the whole travelling rate, and the denominator the whole travelling time. Then, putting

we have
$$\left.\begin{aligned}(a) &= a + m\tau\\ \lambda &= (a) - b\end{aligned}\right\}\quad(388)$$

in which (a) is the interpolated value of the chronometer correction on the time at A, for the same absolute instant t' to which the correction b on the time at B corresponds.

EXAMPLE.—In the first two voyages of STRUVE'S expedition between Pulkova and Altona in 1843, the corrections of the chronometer "Hauth 31" were found, by comparison with the standard clocks at the two stations, as below. The dates are all in Pulkova time, as shown by one of the chronometers employed in the comparison:

At Pulkova (A), t	= May	19, $21^h.54$	$a = +\ 0^h$	6^m	$38^s.10$
" Altona (B), t'	= "	24, 22 .66	$b = -\ 1$	14	39 .92
" Altona (B), t''	= "	26, 10 .72	$b' = -\ 1$	14	36 .77
" Pulkova (A), t'''	= "	31, 0 .00	$a' = +\ 0$	7	9 .58

Hence

$$\tau = 5^d\ 1^h.12 = 5^d.047, \qquad a' - a = +\ 31^s.48$$
$$\tau'' = 4\ 13\ .28 = 4\ .553, \qquad b' - b = +\ 3\ .15$$

$$m = \frac{31^s.48 - 3^s.15}{5.047 + 4.553} = \frac{28^s.33}{9.6} = +\ 2^s.951$$

$$\begin{aligned}a &= +\ 0^h\ \ 6^m\ 38^s.10\\ m\tau &= \qquad\quad +\ 14\ .89\\ (a) &= +\ 0\ \ 6\ 52\ .99\\ b &= -\ 1\ 14\ 39\ .92\\ \lambda = (a) - b &= +\ 1\ 21\ 32\ .91\end{aligned}$$

218. In the above, the rate of the chronometer is assumed to be constant, and the problem is treated as one of simple interpolation. But most chronometers exhibit more or less acceleration or retardation in successive voyages, and a strict interpolation requires that we should have regard to second differences. If we always start from the station A, as in the above example, using only simple interpolation, we commit a small error, which always affects the longitude in the same way so long as the variation of the chronometer's rate preserves the same sign. But if we commence the next computation with the station B,

so that the two chronometer corrections at A are intermediate between the two at B, then the error in the longitude will have a different sign, and the mean of the two values of the longitude will be, partially at least, freed from the influence of the acceleration or retardation. To show this more clearly under an algebraic form, let us suppose that we have, omitting the intervals of rest at the two stations,

at the times	t,		t',		t'',		t''',
the chron. corrections	a,		b,		a',		b',
intervals		τ,		τ',		τ'',	

and that

μ = daily rate of the chronometer at the time t,
2β = the daily acceleration of the rate μ after the time t,

the true values of the four corrections, observing that b and b' refer to the meridian of B, will be, according to the law of uniformly accelerating motion,

$$\begin{aligned} a &= a \\ b &= a + \mu\tau + \beta\tau^2 - \lambda \\ a' &= a + \mu(\tau + \tau') + \beta(\tau + \tau')^2 \\ b' &= a + \mu(\tau + \tau' + \tau'') + \beta(\tau + \tau' + \tau'')^2 - \lambda \end{aligned}$$

If now we find the value of (a) corresponding to b (that is, for the time t') by simple interpolation between the values of a and a', we have

$$\begin{aligned} (a) &= a + \left(\frac{a' - a}{\tau + \tau'}\right)\tau \\ &= a + \mu\tau + \beta.\tau(\tau + \tau') \end{aligned}$$

from which we obtain the erroneous longitude

$$\lambda' = (a) - b = \lambda + \beta\tau\tau'$$

Hence the error in the longitude, by simple interpolation and commencing with the station A, is $d\lambda' = \beta\tau\tau'$.

In the next place, if we commence at the station B, with the correction b, employing simple interpolation between b and b', to find the correction (b) for the time t'' corresponding to a', we have

$$(b) = b + \left(\frac{b' - b}{\tau' + \tau''}\right)\tau'$$

$$= a + \mu(\tau + \tau') + \beta(\tau^2 + 2\tau\tau' + \tau'^2 + \tau'\tau'') - \lambda$$

and we find the erroneous longitude

$$\lambda'' = a' - (b) = \lambda - \beta\tau'\tau''$$

Hence the error by simple interpolation, commencing with the station B, is $d\lambda'' = -\beta\tau'\tau''$; and the error in the mean of the two longitudes is

$$\tfrac{1}{2}(d\lambda' + d\lambda'') = \tfrac{1}{2}\beta\tau'(\tau - \tau'')$$

an error which disappears altogether when the intervals τ and τ'' are equal. Since the voyages are of very nearly equal duration, it follows that by computing the longitude, as proposed by STRUVE, commencing alternately at the two stations, the final result will be free from the effect of any regular acceleration or retardation of the chronometers.

EXAMPLE.—From the "Expédition Chronométrique" we take the following values for the chronometer "Hauth 31," being the combination next following after that given in the example of the preceding article, commencing now with the station B, or Altona:

At Altona (B), t	= May 26, $10^h.72$	b =	$-\ 1^h\ 14^m\ 36^s.77$
" Pulkova (A), t'	= " 31, 0 .00	a =	$+\ 0\ \ 7\ \ 9.58$
" Pulkova (A), t''	= June 3, 5 .62	a' =	$+\ 0\ \ 7\ \ 19.36$
" Altona (B), t'''	= " 7, 20 .52	b' =	$-\ 1\ \ 14\ \ 0.35$

Here

$$\tau = 4^d\ 13^h.28 = 4^d.553 \qquad b' - b = +\ 36^s.42$$
$$\tau'' = 4\ \ 14\ .90 = 4\ .621 \qquad a' - a = +\ \ 9\ .78$$

$$m = \frac{36^s.42 - 9^s.78}{4.553 + 4.621} = \frac{26^s.64}{9.174} = +\ 2^s.904$$

$$\begin{aligned}
b &= -\ 1^h\ 14^m\ 36^s.77 \\
m\tau &= \qquad\quad +\ 13\ .22 \\
(b) &= -\ 1\ \ 14\ \ 23\ .55 \\
a &= +\ 0\ \ \ 7\ \ \ 9\ .58 \\
\lambda = a - (b) &= +\ 1\ \ 21\ \ 33\ .13
\end{aligned}$$

The mean of this result and that of Art. 217 is $\lambda = 1^h\ 21^m\ 33^s.02$.

219. *Relative weight of the longitudes determined in different voyages by the same chronometer.*—From the above it appears that the problem of finding the longitude by chronometers is one of interpolation. If the irregularities of the chronometer are regarded as accidental, the mean error of an interpolated value of the correction may be expressed by the formula*

$$\mp \varepsilon \sqrt{\frac{\tau\tau'}{\tau + \tau'}}$$

where τ and τ' have the same signification as in the preceding article, and ε is the mean (accidental) error in a unit of time. The weight of such an interpolated value of the correction, and, therefore, also the weight of a value of the longitude deduced from it, is inversely proportional to the square of this error, and may, therefore, be expressed under the form

$$p = k \cdot \frac{\tau + \tau'}{\tau\tau'}$$

where k is a constant arbitrarily taken for the whole expedition, so as to give p convenient values, since it is only the *relative* weights of the different voyages which are in question.

But if the chronometer variations are no longer accidental, but follow some law though unknown, a special investigation may serve to give empirically a more suitable expression of the weight than the above. Thus, according to STRUVE'S investigations in the case of certain clocks, the weight of an interpolated value of the correction for these clocks could be well expressed by the formula†

$$p = k\left(\frac{\tau + \tau'}{\tau\tau'}\right)^2$$

But even this expression he found could not be generally applied; and he finally adopted the following form for the chronometric expedition:

$$p = \frac{K}{T\sqrt{\tau\tau'}} \tag{389}$$

in which T is the duration of an entire voyage, including the

* See Vol. II., "Chronometer."

† *Expédition Chron.*, p. 102.

time of rest at one of the stations, τ, τ'' are the travelling times of the voyage to and from a station, and K is an arbitrary constant.

Although this is but an empirical formula, it represents well the several conditions of the problem. For, *first*, the weight of a resulting longitude must decrease as the length of the voyage increases; and, *second*, it must become greater as the difference between the two travelling times τ, τ'' decreases, since (as is shown in Vol. II., "Chronometer") an interpolated value of a clock correction is probably most in error for the middle time between the two instants at which its corrections are given.

220. *Combination of results obtained by the same chronometer, according to their weights.*—Let λ', λ'', λ'''..... be the several values of the longitude found by the same chronometer, according to the method of Arts. 217 and 218; and p', p'', p'''...... their weights by formula (389) (or any other formula which may be found to represent the actual condition of the voyages); then, according to the method of least squares, the most probable value of the longitude by this chronometer is

$$L = \frac{p'\lambda' + p''\lambda'' + p'''\lambda''' + \ldots\ldots}{p' + p'' + p''' + \ldots\ldots} \tag{390}$$

and if the difference between this value and each particular value be found, putting

$$\lambda' - L = v', \qquad \lambda'' - L = v'', \qquad \lambda''' - L = v''', \text{ \&c.}$$

n = the number of values of λ,
ϵ = the mean error of L,
r = the probable error of L,

then we shall have

$$\epsilon = \sqrt{\frac{[pvv]}{(n-1)[p]}} \qquad r = 0.6745\,\epsilon \tag{391}$$

where $[p]$ denotes the sum of p', p'', &c., and $[pvv]$ the sum of $p'v'v'$, $p''v''v''$, &c.

221. *Combination of the results obtained by different chronometers, according to their weights.*—The weights of the results by different

chronometers are inversely proportional to the squares of their mean errors. The weight P of a longitude L will, therefore, be expressed generally by

$$P = \frac{k}{\varepsilon\varepsilon}$$

in which k is arbitrary. For simplicity, we may assume $k = 1$, and then by the above value of ε we shall have

$$P = \frac{(n-1)[p]}{[pvv]} \tag{392}$$

If, then, L', L'', L'''..... are the values found by the several chronometers by (390), P', P'', P'''..... their weights by (392), the most probable final value of the longitude is

$$L_0 = \frac{P'L' + P''L'' + P'''L''' + \ldots\ldots}{P' + P'' + P''' + \ldots\ldots} \tag{393}$$

Then, putting

$$L' - L_0 = V', \qquad L'' - L_0 = V'', \qquad L''' - L_0 = V''' \quad \text{\&c.}$$

$N =$ the number of values of L,
$E =$ the mean error of L_0,
$R =$ the probable error of L_0,

we have

$$E = \sqrt{\frac{[PVV]}{(N-1)[P]}} \qquad R = 0.6745\ E \tag{394}$$

222. I propose to illustrate the preceding formulæ by applying them to two chronometers of STRUVE's expedition, namely, "Dent 1774" and "Hauth 31." In the following table the longitudes found by beginning at Pulkova are marked P, those found by beginning at Altona are marked A, and the numeral accent denotes the number of the voyage. The weights p in the second column are as given by STRUVE, who computed them by the formula (389), taking $K = 34560$ (the intervals T, τ, τ'' being in hours), which is a convenient value, as it makes the weight of a voyage of nearly mean duration equal to unity; namely, for $T = 288^h$, $\tau = \tau' = 120^h$. If we express T, τ, τ'', in days, we take

$$K = \frac{34560}{(24)^2} = 60$$

and we shall have STRUVE's values of p by the formula

$$p = \frac{60}{T\sqrt{\tau\tau''}} \tag{395}$$

Thus, for the first voyage, we have, from the data in the example of Art. 217,

$$T = t''' - t = 11^d\ 2^h.46 = 11^d.103$$
$$\tau = 5^d.047 \qquad \tau'' = 4^d.553$$

whence, by (395),

$$p = \frac{60}{11.103\sqrt{(5.047 \times 4.553)}} = 1.13$$

The values of L' and L'' are found by (390). In applying this formula, it is not necessary to multiply the entire longitudes by their weights, but only those figures which differ in the several values. Thus, by "Dent 1774" we have

$$L' = 1^h\ 21^m\ 30^s + \frac{2^s.51 \times 1.10 + 2^s.83 \times 1.02 + 2^s.09 \times 1.14 + \&c.}{1.10 + 1.02 + 1.14 + \&c.}$$
$$= 1^h\ 21^m\ 30^s + 2^s.46$$

	Weight. p	Longitudes by Chronometer *Dent* 1774.	v	pvv	Longitudes by Chronometer *Hauth* 31	v	pvv
P^{i}	1.13				$1^h\ 21^m\ 32^s.91$	$+ 0^s.30$	0.102
A^{i}	1.06				33 .13	+ 0 .52	0.287
P^{ii}	1.10	$1^h\ 21^m\ 32^s.51$	$+ 0^s.05$	0.003	33 .36	+ 0 .75	0.619
A^{ii}	1.02	32 .83	+ 0 .37	0.140	33 .12	+ 0 .51	0.265
P^{iii}	1.14	32 .09	− 0 .37	0.156	32 .55	− 0 .06	0.004
A^{iii}	1.05	32 .25	− 0 .21	0.046	31 .56	− 1 .05	1.158
P^{iv}	1.19	31 .69	− 0 .77	0.706	32 .70	+ 0 .09	0.010
A^{iv}	0.96	32 .77	+ 0 .31	0.092	34 .16	+ 1 .55	2.306
P^{v}	1.09	32 .79	+ 0 .33	0.119	32 .23	− 0 .38	0.157
A^{v}	0.80	32 .54	+ 0 .08	0.005	31 .65	− 0 .96	0.737
P^{vi}	1.00	32 .94	+ 0 .48	0.230	33 .38	+ 0 .77	0.593
A^{vi}	1.10	31 .93	− 0 .53	0.309	31 .97	− 0 .64	0.451
P^{vii}	1.20	32 .34	− 0 .12	0.017	33 .16	+ 0 .55	0.363
A^{vii}	1.09	32 .95	+ 0 .49	0.262	31 .78	− 0 .83	0.751
P^{viii}	0.76	31 .86	− 0 .60	0.274	30 .92	− 1 .69	2.171
A^{viii}	0.41	33 .77	+ 1 .31	0.704			

$L' = 1^h\ 21^m\ 32^s.46 \qquad [pvv] = 3.063$

$n = 14 \qquad [p] = 18.91$

$$P' = \frac{13 \times 18.91}{3.063} = 59.04$$

$$r' = \pm\frac{.6745}{\sqrt{P'}} = \pm\ 0^s.09$$

$L'' = 1^h\ 21^m\ 32^s.61 \qquad [pvv] = 9.974$

$n = 15 \qquad [p] = 15.69$

$$P'' = \frac{14 \times 15.69}{9.974} = 22.02$$

$$r'' = \pm\frac{.6745}{\sqrt{P''}} = \pm\ 0^s.14$$

Combining these two results, we have, by (393),

$$L_0 = 1^h\,21^m\,32^s + \frac{0^s.46 \times 59 + 0^s.61 \times 22}{59 + 22} = 1^h\,21^m\,32^s.501$$

with the probable error, by (394),

$$R = \pm\, 0^s.067$$

This agrees very nearly with the final result from the sixty-eight chronometers.

223. In the preceding method, the sea rate is inferred from two comparisons of the chronometer made at the same place before and after the voyages to and from the second place; and the correction of the chronometer on the time of the first place at the instant when it is compared with the time of the second place is interpolated upon the theory that the rate has changed uniformly. This theory is insufficient when the temperature to which the chronometer is exposed is not constant during the two voyages, or nearly so. I shall, therefore, add the method of introducing the correction for temperature in cases where circumstances may seem to require it.

According to the experience of M. Lieusson, the rate m of a chronometer at a given temperature ϑ may be expressed by the formula (see Vol. II., "Chronometer")

$$m = m_0 + k(\vartheta - \vartheta_0)^2 - k't \tag{396}$$

in which ϑ_0 is the temperature for which the balance is compensated, m_0 the rate determined at that temperature at the epoch $t = 0$, t being the time from this epoch for which the rate m is required, k the constant coefficient of temperature, and k' that of acceleration of the chronometer resulting from thickening of the oil or other gradual changes which are supposed to be proportional to the time.

It is evident that, since every change of temperature produces an increase of m, the term $k(\vartheta - \vartheta_0)^2$ will not disappear even when the *mean* value of ϑ is the same as ϑ_0. It is necessary, therefore, to determine the sum of the effects of all the changes. Let us, therefore, determine the accumulated rate for a given period of time τ. Let m_0 be the rate at the middle of this period, in which case we have in the formula $t = 0$. A strict theory requires that

we should know the temperature at every instant; but, in default of this, let us assume that the period τ is divided into sufficiently small intervals, and that the temperature is observed in each. Let us suppose n equal intervals whose sum is τ, and denote the observed values of ϑ by $\vartheta^{(1)}$, $\vartheta^{(2)}$, $\vartheta^{(3)} \dots \vartheta^{(n)}$. The rate

in the 1st interval is $[m_0 + k(\vartheta^{(1)} - \vartheta_0)^2] \times \frac{\tau}{n}$

" 2d " $[m_0 + k(\vartheta^{(2)} - \vartheta_0)^2] \times \frac{\tau}{n}$

&c. &c.

in the nth interval is $[m_0 + k(\vartheta^{(n)} - \vartheta_0)^2] \times \frac{\tau}{n}$

and the accumulated rate in the time τ is the sum of these quantities,

$$= m_0\tau + k\,\Sigma_n(\vartheta - \vartheta_0)^2\,\frac{\tau}{n}$$

where $\Sigma_n(\vartheta - \vartheta_0)^2$ denotes the sum of the n values of $(\vartheta - \vartheta_0)^2$. To make this expression exact, we should have an infinite number of infinitesimal intervals, or we must put $\frac{\tau}{n} = d\tau$, and substitute the integral sign $\int$ for the summation symbol Σ: thus, the exact expression for the whole rate in the time τ is

$$m_0\tau + k\int_0^\tau (\vartheta - \vartheta_0)^2\,d\tau \tag{397}$$

This integral cannot be found in general terms, since ϑ cannot be expressed as a function of τ; but we can obtain an approximate expression for it, as follows. Let ϑ_1 be the mean of all the observed values of ϑ; then we have

$$\begin{aligned}\Sigma_n(\vartheta - \vartheta_0)^2 &= \Sigma_n[(\vartheta_1 - \vartheta_0) + (\vartheta - \vartheta_1)]^2 \\ &= \Sigma_n(\vartheta_1 - \vartheta_0)^2 + \Sigma_n 2(\vartheta_1 - \vartheta_0)(\vartheta - \vartheta_1) + \Sigma_n(\vartheta - \vartheta_1)^2\end{aligned}$$

in which $\vartheta_1 - \vartheta_0$ is constant, and, therefore, for n values we have $\Sigma_n(\vartheta_1 - \vartheta_0)^2 = n(\vartheta_1 - \vartheta_0)^2$. Moreover, since ϑ_1 is the mean of all the values of ϑ, we have $\Sigma_n(\vartheta - \vartheta_1) = 0$, and, consequently, also $\Sigma_n 2(\vartheta_1 - \vartheta_0)(\vartheta - \vartheta_1) = 2(\vartheta_1 - \vartheta_0)\,\Sigma_n(\vartheta - \vartheta_1) = 0$; and the above expression becomes

$$\Sigma_n(\vartheta - \vartheta_0)^2 = n(\vartheta_1 - \vartheta_0)^2 + \Sigma_n(\vartheta - \vartheta_1)^2$$

Hence, also,

$$\Sigma_n (\vartheta - \vartheta_0)^2 \frac{\tau}{n} = \tau (\vartheta_1 - \vartheta_0)^2 + \Sigma_n (\vartheta - \vartheta_1)^2 \frac{\tau}{n}$$

or, for an infinite value of n,

$$\int_0^\tau (\vartheta - \vartheta_0)^2 d\tau = \tau (\vartheta_1 - \vartheta_0)^2 + \int_0^\tau (\vartheta - \vartheta_1)^2 d\tau$$

Thus, the required integral depends upon the integral $\int_0^\tau (\vartheta - \vartheta_1)^2 d\tau$, which may be approximately found from the observed values of ϑ by the theory of least squares. For, if we treat the values of $\vartheta - \vartheta_1$ as the errors of the observed values of ϑ, and denote the mean error (according to the received acceptation of that term in the method of least squares) by e, we have

$$e^2 = \frac{\Sigma_n (\vartheta - \vartheta_1)^2}{n - 1} \tag{398}$$

in which n is the actual number of observed values of ϑ. If we assume that a more extended series of values, or indeed an infinite series, would exhibit the same mean error (which will be the more nearly true the greater the number n), we assume the general relation

$$\Sigma_N (\vartheta - \vartheta_1)^2 = (N - 1) e^2$$

in which N is any number. Hence, also,

$$\Sigma_N (\vartheta - \vartheta_1)^2 \frac{\tau}{N} = \tau e^2 \frac{N - 1}{N}$$

and, making N infinite,

$$\int_0^\tau (\vartheta - \vartheta_1)^2 d\tau = \tau e^2 \tag{399}$$

Substituting this value, the formula (397) becomes

$$m_0 \tau + k\tau (\vartheta_1 - \vartheta_0)^2 + k\tau e^2$$

or

$$[m_0 + k (\vartheta_1 - \vartheta_0)^2 + ke^2]\tau \tag{400}$$

from which it appears that $m_0 + k (\vartheta_1 - \vartheta_0)^2 + ke^2$ is the mean rate in a unit of time for the interval τ, m_0 being the rate at the middle of the interval for a temperature $\vartheta = \vartheta_0$. For any subsequent interval τ', we must, according to (396), replace m_0 by $m_0 - k't$, t being the interval from the middle of τ to the middle of τ'.

Now, let us suppose that the chronometer correction is obtained by astronomical observations at the station A, at the times T_1 and T_2, before starting upon the voyage, and again after reaching the station B, at the times T_3 and T_4, these times being all reckoned at the same meridian. Let a_1, a_2, a_3, a_4, be the observed corrections, and put

$$T_2 - T_1 = \tau, \qquad T_3 - T_2 = \tau', \qquad T_4 - T_3 = \tau''$$

so that τ and τ'' are the shore intervals and τ' the sea interval. Let the adopted epoch of the rate m_0 be the middle of the sea interval τ'; then, by (400), with the correction $k't$, the accumulated rates in the three intervals are

$$\left.\begin{aligned} a_2 - a_1 &= \left[m_0 + k'\left(\frac{\tau+\tau'}{2}\right) + k(\vartheta_1 - \vartheta_0)^2 + ke^2\right]\tau \\ \lambda + a_3 - a_2 &= \left[m_0 + k(\vartheta_1' - \vartheta_0)^2 + ke'^2\right]\tau' \\ a_4 - a_3 &= \left[m_0 - k'\left(\frac{\tau''+\tau'}{2}\right) + k(\vartheta_1'' - \vartheta_0)^2 + ke''^2\right]\tau'' \end{aligned}\right\} \tag{401}$$

in which ϑ_1, ϑ_1', ϑ_1'' are the mean temperatures in the intervals τ, τ', τ'', and e, e', e'' are found by the formula (398). These three equations determine the three unknown quantities m_0, k', and λ. If we put

$$f = \frac{a_2 - a_1}{\tau} - k(\vartheta_1 - \vartheta_0)^2 - ke^2$$

$$f'' = \frac{a_4 - a_3}{\tau''} - k(\vartheta_1'' - \vartheta_0)^2 - ke''^2$$

we have, from the first and third equations,

$$k' = \frac{f - f''}{\tau' + \frac{1}{2}(\tau + \tau'')}$$

$$m_0 = \frac{f + f''}{2} + \tfrac{1}{4}k'(\tau'' - \tau)$$

which substituted in the second equation gives λ. If, however, we prefer to compute the approximate longitude without considering the temperatures, and afterwards to correct for temperature, we shall have

$$
\left.\begin{aligned}
(\lambda) &= -(a_3 - a_2) + \left(\frac{a_2 - a_1}{\tau} + \frac{a_4 - a_3}{\tau''}\right)\frac{\tau'}{2} + \tfrac{1}{4}k'(\tau'' - \tau)\,\tau' \\
\Delta\lambda &= k\left((\vartheta_1' - \vartheta_0)^2 - \frac{(\vartheta_1 - \vartheta_0)^2 + (\vartheta_1'' - \vartheta_0)^2}{2} + e'^2 - \frac{e^2 + e''^2}{2}\right) \\
\lambda &= (\lambda) + \Delta\lambda
\end{aligned}\right\} \quad (402)
$$

These formulæ apply to a voyage in either direction; but in the case of a voyage from west to east they give λ with the negative sign.

The term $\frac{1}{4}k'(\tau'' - \tau)\,\tau'$ in the first equation of (402) will not be rigorously obtained if the temperatures are neglected; but it is usually an insensible term in practice, as τ'' and τ are made as nearly equal as possible, and k' is always very small.

In combining the results of different chronometers employed in the same voyage, the weight of each may be assigned according to the regularity of the chronometer as determined from its observed rates from day to day.*

SECOND METHOD.—BY SIGNALS.

224. *Terrestrial Signals.*—If the two stations are so near to each other that a signal made at either, or at an intermediate station, can be observed at both, the time may be noted simultaneously by the clocks of the two stations, and the difference of longitude at once inferred. The signals may be the sudden disappearance or reappearance of a fixed light, or flashes of gunpowder, &c.

If the places are remote, they may be connected by intermediate signals. For example: suppose four stations, A, B, C, D, chosen from east to west, the first and last being the principal stations whose difference of longitude is required. At the intermediate stations B, C let observers be stationed with good chronometers whose rates are known. Let signals be made at three points intermediate between A and B, B and C, C and D, respectively. The signals must, by a preconcerted arrangement, be made successively, and so that the observers at the intermediate stations may have their attention properly directed upon the appearance of the signal. If, then, at the first signal the observers at A and B have noted the times a and b; at the

* Besides the papers already referred to, see the Report of the Superintendent of the U. S. Coast Survey for 1857, p. 314.

second signal the observers at B and C the times b' and c; at the third signal the observers at C and D the times c' and d; it is evident that the time at A when the third signal is made is $a + (b' - b) + (c' - c)$, at which instant the time at D is d: hence the difference of longitude of A and D is

$$\lambda = a + (b' - b) + (c' - c) - d \tag{403}$$

and so on for any number of intermediate stations. It is required of the intermediate chronometers only that they should give correctly the differences $b' - b$, $c' - c$, for which purpose only their rates must be accurately known. The daily rates are obtained by a comparison of the instants of the signals on successive days. Small errors in the rates will be eliminated by making the signals both from west to east and from east to west, and taking the mean of the results.

The intervals given by the intermediate chronometers should, of course, be reduced to sidereal intervals, if the clocks at the extreme stations are regulated to sidereal time.

Example.—From the *Description Géométrique de la France* (Puissant). On the 25th of August, 1824, signals were observed between *Paris* and *Strasburg* as follows:

Paris.	Intermediate Stations.		Strasburg.
A	B	C	D
$19^h\ 6^m\ 20^s.3$	$8^h\ 49^m\ 48^s.2$		
	8 54 10 .8	$9^h\ 16^m\ 0^s.2$	
		9 30 37 .8	$19^h\ 46^m\ 51^s.4$

The correction of the Paris clock on Paris sidereal time was $-36^s.2$; that of the Strasburg clock on Strasburg sidereal time was $-27^s.7$. The chronometers at B and C were regulated to mean time, and their daily rates were so small as not to be sensible in the short intervals which occurred.

We have

$$\begin{array}{lrr}
b' - b = & 4^m & 22^s.6 \\
c' - c = & 14 & 37\ .6 \\
\hline
\text{Mean interval} = & 19 & 0\ .2 \\
\text{Red. to sid. int.} = & & +\ 3\ .1 \\
\hline
\text{Sid. interval} = & 19 & 3\ .3
\end{array}$$

Paris clock	$19^h\ 6^m\ 20^s.3$	Strasburg clock	$19^h\ 46^m\ 51^s.4$
Correction	$-\ 36\ .2$	Correction	$-\ 27\ .7$
Paris sid. time	$19\ \ 5\ \ 44\ .1$	Strasburg sid. time	$19\ \ 46\ \ 23\ .7$
Sid. interval	$+\ 19\ \ 3\ .3$		
Paris sid. time of the last signal	$19\ \ 24\ \ 47\ .4$		
Strasburg do.	$19\ \ 46\ \ 23\ .7$		
$\lambda =$	$0^h\ 21^m\ 36^s.3$		

In the survey of the boundary between the United States and Mexico, Major W. H. Emory, in 1852, employed flashes of gunpowder as signals in determining the diff. of long. of Frontera and San Elciario.*

The signals may be given by the *heliotrope* of Gauss, by which an image of the sun is reflected constantly in a given direction towards the distant observer. Either the sudden eclipse of the light, or its reappearance, may be taken as the signal; the eclipse is usually preferred.

Among the methods by terrestrial signals may be included that in which the signal is given by means of an electro-telegraphic wire connecting the two stations; but this important and exceedingly accurate method will be separately considered below.

225. *Celestial Signals.*—Certain celestial phenomena which are visible *at the same absolute instant* by observers in various parts of the globe, may be used instead of the terrestrial signals of the preceding article: among these we may note—

a. The bursting of a meteor, and the appearance or disappearance of a shooting star.—The difficulty of identifying these objects at remote stations prevents the extended use of this method.

b. The instant of beginning or ending of an eclipse of the moon.—This instant, however, cannot be accurately observed, on account of the imperfect definition of the earth's shadow. A rude approximation to the difference of longitude is all that can be expected by this method.

c. The *eclipses* of Jupiter's satellites by the shadow of that planet.—The Greenwich times of the disappearance of each

---

* Proceedings of 8th Meeting of Am. Association, p. 64.

satellite, and of its reappearance, are accurately given in the Ephemeris: so that an observer who has noted one of these phenomena has only to take the difference between this observed local time of its occurrence and the Greenwich time given in the Ephemeris, to have his absolute longitude. With telescopes of different powers, however, the instant of a satellite's disappearance must evidently vary, since the eclipse of the satellite takes place gradually, and the more powerful the telescope the longer will it continue to show the satellite. If the disappearance and reappearance are both observed with the same telescope, the mean of the results obtained will be nearly free from this error. The first satellite is to be preferred, as its eclipses occur more frequently and also more suddenly. Observers who wish to deduce their difference of longitude by these eclipses should use telescopes of the same power, and observe under the same atmospheric conditions, as nearly as possible. But in no case can extreme precision be attained by this method.

d. The *occultations* of Jupiter's satellites by the body of the planet.—The *approximate* Greenwich times of the disappearance behind the disc, and the reappearance of each satellite, are given in the Ephemeris. These predicted times serve only to enable the observers to direct their attention to the phenomenon at the proper moment.

e. The *transits* of the satellites over Jupiter's disc.—The approximate Greenwich times of "ingress" and "egress," or the first and last instants when the satellite appears projected on the planet's disc, are given in the Ephemeris.

f. The *transits of the shadows* of the satellites over Jupiter's disc.—The Greenwich times of "ingress" and "egress" of the shadow are also approximately given in the Ephemeris.

Among the celestial signals we may include also eclipses of the sun, or occultations of stars and planets by the moon, or, in general, the arrival of the moon at any given position in the heavens; but, in consequence of the moon's parallax, these eclipses and occultations do not occur at the same absolute instant for all observers, and, in general, the moon's apparent position in the heavens is affected by both parallax and refraction. The methods of employing these phenomena as signals, therefore, involve special computations, and will be hereafter treated of. See the general theory of eclipses, and the method of lunar distances

THIRD METHOD.—BY THE ELECTRIC TELEGRAPH.

226. It is evident that the clocks at two stations, A and B, may be compared by means of signals communicated through an electro-telegraphic wire which connects the stations. Suppose at a time T by the clock at A, a signal is made which is perceived at B at the time T' by the clock at that station. Let ΔT and $\Delta T'$ be the clock corrections on the times at these stations respectively (both being solar or both sidereal). Let x be the time required by the electric current to pass over the wire; then, A being the more easterly station, we have the difference of longitude λ by the formula

$$\lambda = (T + \Delta T) - (T' + \Delta T') + x = \lambda_1 + x$$

Since x is unknown, we must endeavor to eliminate it. For this purpose, let a signal be made at B at the clock time T'', which is perceived at A at the clock time T'''; then we have

$$\lambda = (T''' + \Delta T''') - (T'' + \Delta T'') - x = \lambda_2 - x$$

In these formulæ λ_1 and λ_2 denote the approximate values of the difference of longitude, found by signals east-west and west-east respectively, when the transmission time x is disregarded; and the true value is

$$\lambda = \tfrac{1}{2}(\lambda_1 + \lambda_2)$$

Such is the simple and obvious application of the telegraph to the determination of longitudes; but the degree of accuracy of the result depends greatly—more than at first appears—upon the manner in which the signals are communicated and received.

Suppose the observer at A taps upon a signal key* at an exact second by his clock, thereby producing an audible click of the armature of the electro-magnet at B. The observer at B may not only determine the nearest second by his clock when he hears this click, but may also estimate the fraction of a second; and it would seem that we ought in this way to be able to determine a longitude within one-tenth of a second. But, before even this degree of accuracy can be secured, we have yet to eliminate, or reduce to a minimum, the following sources of error:

* See Vol. II., "Chronograph," for the details of the apparatus here alluded to.

1st. The personal error of the observer who gives the signal;

2d. The personal error of the observer who receives the signal and estimates the fraction of a second by the ear;

3d. The small fraction of time required to complete the galvanic circuit after the finger touches the signal key;

4th. The *armature time*, or the time required by the armature at the station where the signal is received, to move through the space in which it plays, and to give the audible click;

5th. The errors of the supposed clock corrections, which involve errors of observation, and errors in the right ascensions of the stars employed.

For the means of contending successfully with these sources of error we are indebted to our Coast Survey, which, under the superintendence of Prof. Bache, not only called into existence the chronographic instruments, but has given us the most efficient method of using them. The "method of star signals," as it is called, was originally suggested by the distinguished astronomer Mr. S. C. Walker, but its full development in the form now employed in the Coast Survey is due to Dr. B. A. Gould.

227. *Method of Star Signals.*—The difference of longitude between the two stations is merely the time required by a star to pass from one meridian to the other, and this interval may be measured by means of a single clock placed at either station,* but in the main galvanic circuit extending from one station to the other. Two chronographs, one at each station, are also in the circuit, and, when the wires are suitably connected, the clock seconds are recorded upon both. A good transit instrument is carefully mounted at each station.

When the star enters the field of the transit instrument at A (the eastern station), the observer, by a preconcerted signal with his signal key, gives notice to the assistants at both A and B, who at once set the chronographs in motion, and the clock then records its seconds upon both. The instants of the star's transits over the several threads of the reticule are also recorded upon both chronographs by the taps of the observer upon his signal key. When the star has passed all the threads, the ob-

* The clock may, indeed, be at any place which is in telegraphic connection with the two stations whose difference of longitude is to be found.

server indicates it by another preconcerted signal, the chronographs are stopped, and the record is suitably marked with date, name of the star, and place of observation, to be subsequently identified and read off accurately by a scale. When the star arrives at the meridian of B, the transit is recorded in the same manner upon both chronographs.

Suitable observations having been made by each observer to determine the errors of his transit instrument and the rate of the clock, let us put

T_1 = the mean of the clock times of the eastern transit of the star over all the threads, as read from the chronograph at A,
T_2 = the same, as read from the chronograph at B,
T_1' = the mean of the clock times of the western transit of the star over all the threads, as read from the chronograph at A,
T_2' = the same as read from the chronograph at B,
e, e' = the personal equations of the observers at A and B respectively,
τ, τ' = the corrections of T_1 and T_1' (or of T_2 and T_2') for the state of the transit instruments at A and B, or the respective "reductions to the meridian" (Vol. II., Transit Inst.),
δT = the correction for clock rate in the interval $T_1' - T_1$,
x = the transmission time of the electric current between A and B,
λ = the difference of longitude;

then it is easily seen that we have, from the chronographic records at A,

$$\lambda = T_1' + \delta T + \tau' + e' - x - (T_1 + \tau + e)$$

and from the chronographic records at B,

$$\lambda = T_2' + \delta T + \tau' + e' + x - (T_2 + \tau + e)$$

and the mean of these values is

$$\lambda = [\tfrac{1}{2}(T_1' + T_2') + \tau'] - [\tfrac{1}{2}(T_1 + T_2) + \tau] + \delta T + e' - e \quad (404)$$

which we may briefly express thus:

$$\lambda = \lambda_1 + e' - e$$

in which

$\lambda_1 =$ the approximate difference of longitude found by the exchange of star signals, when the personal equations of the observers are neglected.

This equation would be final if $e' - e$, or the relative personal equation of the observers, were known: however, if the observers now exchange stations and repeat the above process, we shall have, provided the relative personal equation is constant,

$$\lambda = \lambda_2 + e - e'$$

in which λ_2 is the approximate difference of longitude found as before; and hence the final value is

$$\lambda = \tfrac{1}{2}(\lambda_1 + \lambda_2)$$

I have not here introduced any consideration of the armature time, because it affects clock signals and star signals in the same manner; and therefore the time read from the chronographic fillet or sheet is the same as if the armature acted instantaneously.* It is necessary, however, that this time should be constant from the first observation at the first station to the last observation at the second, and therefore it is important that no changes should be made in the adjustments of the apparatus during the interval.

As the observer has only to tap the transits of the star over the threads, the latter may be placed very close together. The reticules prepared by Mr. W. Würdemann for the Coast Survey have generally contained twenty-five threads, in groups or "tallies" of five, the equatorial intervals between the threads, of a group being $2^s.5$, and those between the groups 5^s; with an additional thread on each side at the distance of 10^s for use in observations by "eye and ear." Except when clouds intervene and render it necessary to take whatever threads may be available, only the three middle tallies, or fifteen threads, are used. The use of more has been found to add less to the accuracy of a

* Dr. B. A. Gould thinks that the armature time varies with the strength of the battery and the distance (and consequent weakness) of the signal; being thus liable to be confounded with the transmission time. The effect upon the difference of longitude will be inappreciable if the batteries are maintained at nearly the same strength

determination than is lost in consequence of the greater fatigue from concentrating the attention for nearly twice as long.

A large number of stars may thus be observed on the same night; and it will be well to record half of them by the clock at one station, and the other half by the clock at the other station, upon the general principle of varying the circumstances under which several determinations are made, whenever practicable, without a sacrifice of the integrity of the method. For this reason, also, the transit instruments should be reversed during a night's work at least once, an equal number of stars being observed in each position, whereby the results will be freed from any undetermined errors of collimation and inequality of pivots. Before and after the exchange of the star signals, each observer should take at least two circumpolar stars to determine the instrumental constants upon which τ and τ' depend. This part of the work must be carried out with the greatest precision, employing only standard stars, as the errors of τ and τ' come directly into the difference of longitude. The right ascensions of the "signal stars" do not enter into the computation, and the result is, therefore, wholly free from any error in their tabular places: hence any of the stars of the larger catalogues may be used as signal stars, and it will always be possible to select a sufficient number which culminate at moderate zenith distances at both stations, (unless the difference of latitude is unusually great), so that instrumental errors will have the minimum effect.

A single night's work, however, is not to be regarded as conclusive, although a large number of stars may have been observed and the results appear very accordant; for experience shows that there are always errors which are constant, or nearly so, for the same night, and which do not appear to be represented in the corrections computed and applied. Their existence is proved when the mean results of different nights are compared. Moreover, it is necessary to interchange the observers in order to eliminate their personal equations. The rule of the Coast Survey has been that when fifty stars have been exchanged on not less than three nights, the observers exchange stations, and fifty stars are again exchanged on not less than three nights. The observers should also meet and determine their relative personal equation, if possible, before and after each series, as it may prove that this equation is not absolutely constant.

Before entering upon a series of star signals, each observer will be provided with a list of the stars to be employed. The preparation of this list requires a knowledge of the approximate difference of longitude in order that the stars may be so selected that transits at the two stations may not occur simultaneously.

EXAMPLE.—For the purpose of finding the difference of longitude between the Seaton Station of the U. S. Coast Survey and Raleigh, a list of stars was prepared, from which I extract the following for illustration. The latitudes are

$$\begin{array}{lll} \text{Seaton Station (Washington)} & \varphi = + 38^\circ\ 53'.4 \\ \text{Raleigh } \quad \text{" (North Carolina)} & \varphi = + 35\ \ 47.0 \end{array}$$

and Raleigh is assumed to be west from Washington $6^m\ 30^s$.

Star.	Mag.	α	δ	Seaton sidereal time of Raleigh transit.
No. 5036 B.A.C.	3	$15^h\ 9^m\ 36^s$	$+ 33^\circ\ 52'$	$15^h\ 16^m\ 6^s$
5084	4.3	18 58	37 54	25 28
5131	$4\frac{1}{2}$	27 2	31 51	33 32
5192	5	36 35	26 46	43 5
5259	5	45 43	36 7	52 13
5322	$4\frac{1}{2}$	55 59	23 12	16 2 29
5388	5	16 4 9	45 19	10 39
5463	3.4	15 21	46 40	21 51

The following table contains the observations made on one of these stars at the above-named stations by the U. S. Coast Survey telegraphic party in 1853, April 28, under the direction of Dr. B. A. GOULD.

In this table "Lamp W." expresses the position of the rotation axes of the transit instruments. The 1st column contains the symbols by which the fifteen threads of the three middle tallies were denoted; the 2d column, the times of transit of the star over each thread at Seaton, as read from the chronographs at Seaton; the 3d column, the times of these transits as read from the chronographs at Raleigh; the 4th column, the mean of the 2d and 3d columns; the 5th column, the reduction of each thread to the mean of all, computed from the known equatorial intervals of the threads; the 6th column, the time of the star's transit over

the mean of the threads, being the algebraic sum of the numbers in the 4th and 5th columns; and the remaining columns, the Raleigh observations similarly recorded and reduced.

SEATON—RALEIGH, 1853 April 28.						Star No. 5259 B A. C.				
	Seaton Obs. Lamp W.					Raleigh Obs. Lamp W.				
Thread.	T_1	T_2	Mean.	Red.	$\frac{T_1+T_2}{2}$	T_1'	T_2'	Mean	Red.	$\frac{T_1'+T_2'}{2}$
C_1	37ˢ 97	38ˢ.00	37ˢ 98	+ 25ˢ 49	3ˢ 47	....	11ˢ.00	11ˢ.00	+ 25ˢ.45	36ˢ.45
C_2	41 37	41 .34	41 .36	22 .21	3 .57	14ˢ.58	14 .50	14 .54	22 .25	36 .79
C_3	44 .03	44 .21	44 12	19 .06	3 .18	17 .60	17 .55	17 .58	19 .05	36 .63
C_4	47 .81	47 74	47 .78	15 .71	3 .49	20 .88	20 .79	20 .84	15 .85	36 .69
C_5	50 .76	50 .70	50 .73	12 .71	3 .44	23 .90	23 .87	23 .89	12 .70	36 .59
D_1	56 .96	57 .10	57 .03	6 .21	3 ,24	30 .19	30 .05	30 .12	6 .32	36 .44
D_2	0 .06	0 .04	0 .05	3 .25	3 30	33 34	33 .25	33 30	3 .18	36 .48
D_3	15ʰ 46ᵐ 3 .40	3 .38	3 .39	+ 0 05	3 .44	15ʰ 52ᵐ 36 .40	36 .30	36 .35	+ 0 .07	36 .42
D_4	6 .70	6 .70	6 .70	— 3 .03	[3 .67]	39 .61	39 .53	39 .57	— 3 .16	36 .41
D_5	9 .58	9 .58	9 .58	6 .28	3 .30	43 .00	43 .00	43 .00	6 .36	36 .64
E_1	16 .03	15 .93	15 .98	12 .54	3 .44	49 .04	48 .81	48 .92	12 .75	[36 .17]
E_2	19 .26	19 .30	19 .28	15 .83	3 .45	52 .30	52 .33	52 32	15 .90	36 .42
E_3	22 .47	22 .45	22 46	18 .99	3 .47	55 50	55 .41	55 .46	19 .10	36 .36
E_4	25 .60	25 .60	25 .60	22 .23	3 .38	58 .73	58 .60	58 .67	22 20	36 .47
E_5	28 .60	28 .70	28 .65	25 .33	3 32	2 .06	2 .08	2 08	25 .38	36 .70
				Mean =	3 .392				Mean =	36 .535

The numbers in the last column for each station would be equal if the observations and chronographic apparatus were perfect; and by carrying them out thus individually we can estimate their accuracy. The numbers [3.67] at Seaton and [36.17] at Raleigh are rejected by the application of PEIRCE's Criterion (see Appendix, Method of Least Squares), and the given means are found from the remaining numbers.

The corrections of the transit instruments for this star ($\delta = + 36° 6'.9$) were

for the Seaton instrument, $\tau = - 0^s.028$
" " Raleigh " $\tau' = - 0.193$

The rate of the clock was insensible in the brief interval $T_1' - T$. Hence, neglecting the personal equations of the observers, the difference of longitude is found as follows:

$$\begin{aligned} \tfrac{1}{2}(T_1' + T_2') + \tau' &= 15^h\ 52^m\ 36^s.342 \\ \tfrac{1}{2}(T_1 + T_2) + \tau &= 15\ \ 46\ \ \ 3.364 \\ \lambda_1 &= \qquad\ 6\ \ 32.978 \end{aligned}$$

In this manner seven other stars were observed on the same night, and the results were as follows:

Star	λ_1	Diff. from mean $= v$
5036 B. A. C.	$6^m\ 33^s.03$	$+\ 0^s.04$
5084 "	33 .09	+ 0 .10
5131 "	32 .91	− 0 .08
5192 "	33 .00	+ 0 .01
5259 "	32 .98	− 0 .01
5322 "	33 .00	+ 0 .01
5388 "	33 .02	+ 0 .03
5463 "	32 .91	− 0 .08

Mean $\lambda_1 = 6\ 32.99$

From the residuals v, we deduce the mean error of a single determination by one star,

$$\varepsilon = \sqrt{\left(\frac{[vv]}{m-1}\right)} = \sqrt{\left(\frac{.0256}{7}\right)} = \pm\ 0^s.06$$

and hence the mean error of the value $6^m\ 32^s.99$ is

$$\varepsilon_0 = \pm\ \frac{0^s.06}{\sqrt{8}} = \pm\ 0^s.02$$

But this error will be somewhat increased by those errors of the instruments which are constant for the night, and not represented in τ and τ', and by the errors of the personal equations yet to be applied. Moreover, a greater number of determinations should be compared, in order to arrive at a just evaluation of the mean error.

228. *Velocity of the galvanic current.*—Recurring to the equations of p. 343, we find, by taking the difference between the values of λ given by the chronographic records at the two stations,

$$x = \tfrac{1}{2}(T_1' - T_2') + \tfrac{1}{2}(T_2 - T_1)$$

If the clock is at the eastern station (A), the time T_2 will not differ from T_1, except in consequence of irregularities in the chronographs and errors in reading them, and therefore we should find x solely from the times T_1' and T_2', or

$$x = \tfrac{1}{2}(T_1' - T_2') \qquad (405)$$

In like manner, if the clock is at the western station, we find x by the formula

$$x = \tfrac{1}{2}(T_2 - T_1)$$

Thus, in general, the transmission time will be deduced by comparing the records of the star signals made at one station when the clock is at the other station.

In the above example, the clock was at Washington, and hence, from the record of the transit at Raleigh, we have fourteen values of $T_1' - T_2' = 2x$, as follows:

$+\ 0^s.08$	$+\ 0^s.08$
$+\ .05$	$+\ 00$
$+\ .09$	$+\ .23$
$+\ .03$	$-\ .03$
$+\ .14$	$+\ .09$
$+\ .09$	$+\ .13$
$+\ .10$	$+\ .00$

That these are not merely accidental residuals is shown by the permanence of sign, with the single exception in the case of the eleventh observation. The discrepancies between them indicate accidental variations in the chronographs, combined with errors in reading off the record. Taking the mean, as eliminating to a certain extent these errors, we have

$$2x = 0^s.077 \qquad x = 0^s.0385$$

From this value of x and the distance of the stations we can deduce the velocity per second of the galvanic current. In the present instance, the length of the wire was very nearly 300 miles, and, if the above single observation could be depended upon, we should have, velocity per second $= \dfrac{300}{0.0385} = 7792$ miles, which is doubtless too small.

The velocity thus found, however, appears to depend upon the intensity of the current,* as has been shown by varying the battery power on different nights. It has also been found that the velocities determined from signals made at the east and west stations differed, and that this difference was apparently depend-

* It depends also upon the sectional area, molecular structure, and, of course, material, of the wire.

ent upon the strength of the batteries; the velocities from signals east-west and signals west-east coming out more and more nearly equal as the strength of the batteries was increased. See Dr. GOULD's Report on telegraphic determinations of differences of longitude, in the Report of the Superintendent of the U. S. Coast Survey for 1857, Appendix No. 27.

FOURTH METHOD.—BY MOON CULMINATIONS.

229. The moon's motion in right ascension is so rapid that the change in this element while the moon is passing from one meridian to another may be used to determine the difference of longitude. Its right ascension at the instant of its meridian transit is most accurately found by means of the interval of sidereal time between this transit and that of a neighboring well-known star. For this purpose, therefore, the Ephemerides contain a list of *moon-culminating stars*, which are selected for each day so that at least four of them are given, the mean of whose declinations is nearly the same as that of the moon on that day, and, generally, so that two precede and two follow the moon. The Ephemerides also contain the right ascension of the moon's bright limb for each culmination, both upper and lower, and the variation of this right ascension in one hour of longitude, —*i.e.* the variation during the interval between the moon's transits over two meridians whose difference of longitude is one hour. This variation is not uniform, and its value is given for the instant of the passage over the meridian of the Ephemeris. These quantities facilitate the reduction of corresponding observations, as will be seen below.

230. As to the observation, let

ϑ, ϑ' = the sidereal times of the culmination of the moon's limb and the star, respectively, corrected for all the known errors of the transit instrument, and for clock rate,

α, α' = the right ascensions of the moon's limb and the star at the instants of transit;

then we evidently have

$$\alpha = \alpha' + \vartheta - \vartheta' \tag{406}$$

The star and the moon being nearly in the same parallel, the instrumental errors which affect ϑ also affect ϑ' by *nearly* the same quantity. We should not, however, for this reason omit to apply all the corrections for *known* instrumental errors, since by this omission we should introduce an error in the longitude precisely equal to the uncorrected error of the instrument. For if the instrumental error produces the error z in the time of the star's transit, the effect is the same as if the instrument were *perfectly* mounted in a meridian whose longitude west of the place of observation is equal to z; but the sidereal time required by the moon to describe this interval z is equal to z + the increase of the moon's right ascension in this interval. Hence the longitude found, by the methods hereafter given, would be in error by the quantity z.

231. If the lunar tables were perfectly accurate, the true longitude given by the observation would be found at once by comparing the observed right ascension with that of the Ephemeris. There are two methods of avoiding or eliminating the errors of the Ephemeris. In the first, which has heretofore been exclusively followed, the observation is compared with a corresponding one on the same day at the first meridian, or at some meridian the longitude of which is well established. In this method the increase of the right ascension in passing from one meridian to the other is directly observed, and the error of the Ephemeris on the day of observation is consequently avoided; but observations at the unknown meridian are frequently rendered useless by a failure to obtain the corresponding observation at the first meridian.

In the second method, proposed by Professor PEIRCE, the Ephemeris is first corrected by means of all the observations taken at the fixed observatories during the semi-lunation within which the observation for longitude falls. The corrected Ephemeris then takes the place of the corresponding observation, and is even better than the single corresponding observation, since it has been corrected by means of *all* the observations at the fixed observatories during the semi-lunation.

I shall consider first the method of reducing corresponding observations.

232. *Corresponding observations at places whose difference of longitude is less than two hours.*—At each place the true sidereal times of transit of the moon-culminating stars and of the moon's bright limb are to be obtained with all possible precision: from these, according to the formula (406), will follow the right ascension of the moon's limb at the instants of transit over the two meridians, taking in each case the mean value found from all the stars observed. Put

$L_1, L_2 =$ the approximate or assumed longitudes,
$\lambda =$ the true difference of longitude,
$\alpha_1, \alpha_2 =$ the observed right ascensions of the moon's bright limb at L_1 and L_2 respectively,
$H_0 =$ the variation of the R. A. of the moon's limb for 1^h of longitude while passing from L_1 to L_2;

then we have

$$\lambda = \frac{\alpha_2 - \alpha_1}{H_0} \tag{407}$$

in which, $\alpha_2 - \alpha_1$ and H_0 being both expressed in seconds, λ will be in hours and decimal parts.

When the difference of longitude is less than two hours, it is found to be sufficiently accurate to regard H_0 as constant, provided we employ its value for the middle longitude $L_0 = \frac{1}{2}(L_1 + L_2)$, found by interpolation from the values in the Ephemeris, having regard to second differences.

EXAMPLE.—The following observations were made, May 15, 1851, at Santiago, Chili, by the U. S. Astronomical Expedition under Lieut. GILLISS, and at Philadelphia, by Prof. KENDALL:

Object.	Santiago sid. time.	Philad'a sid. time.
ϑ *Librae*	$15^h\ 46^m\ 3^s.37$	$15^h\ 45^m\ 22^s.33$
Moon II Limb	16 21 36 .84	16 21 39 .11
B. A. C. 5579	16 33 40 .12	16 32 58 .96

We shall assume the longitudes from Greenwich to be,

$$\text{Philadelphia, } L_1 = 5^h\ 0^m\ 39^s.85$$
$$\text{Santiago, } L_2 = 4\ \ 42\ \ 19\ .$$

the longitude of Philadelphia being that which results from the last chronometric expeditions of the U. S. Coast Survey, and that of Santiago the value which Lieut. GILLISS at first assumed.

The apparent right ascensions of the stars on May 15, by the moon-culminating list in the Nautical Almanac, were

	α'
ϑ *Librae*	$15^h\ 45^m\ 22^s.59$
B. A. C. 5579	16 32 59 .20

We have then at Philadelphia, by (406),

	$\vartheta - \vartheta'$	$\alpha' + \vartheta - \vartheta'$
ϑ *Librae*	$+ 36^m\ 16^s.78$	$16^h\ 21^m\ 39^s.37$
B. A. C. 5579	$-$ 11 19 .85	16 21 39 .35
	Mean α_1 =	16 21 39 .36

and at Santiago:

ϑ *Librae*	+ 35 33 .47	16 20 56 .06
B. A. C. 5579	$-$ 12 3 .28	16 20 55 .92
	Mean α_2 =	16 20 55 .99

Hence

$$\alpha_2 - \alpha_1 = - 43^s.37$$

We shall find H_0 for the mean longitude $L_0 = \frac{1}{2}(L_1 + L_2) = 4^h.86$, by the interpolation formula (72), or

$$H_0 = H + A a' + B b_0$$

in which, if we put $n = \frac{4^h.86}{12}$, we have

$$A = n = 0.405 \qquad B = \frac{n(n-1)}{2} = -0.120$$

and a' and b_0 are found from the values of H in the Ephemeris as follows:

		1st diff.	2d diff.
May 15, L. C.	$142^s.56$	$+ 0^s.92$	
" 15, U. C.	143 .48		$- 0^s.28$
		+ 0 .64	[$-$ 0 .35]
" 16, L. C.	144 .12	+ 0.23	$-$ 0 .41
" 16, U. C.	144 .35		

whence

$$H = 143^s.48 \qquad a' = 0^s.64 \qquad b_0 = \tfrac{1}{2}(-0^s.28 - 0^s.41) = -0^s.35$$

$$H_0 = 143^s.48 + 0^s.259 + 0^s.042 = 143^s.781$$

$$\lambda = \frac{-43.37}{143.781} = -0^h.30164 = -18^m\ 5^s.90$$

which is the longitude of Santiago from Philadelphia. Hence, if the longitude of Philadelphia is correct, we have

$$\text{Long. of Santiago} = 4^h\ 42^m\ 33^s.95 \text{ from Greenwich.}$$

233. *Corresponding observations at places whose difference of longitude is greater than two hours.*—Having found α_1 and α_2 as in the preceding case, we employ in this case an indirect method of solution. For each assumed longitude we interpolate the right ascension of the moon's limb from the Moon Culminations in the Ephemeris to fourth differences. Let

A_1, A_2 = the interpolated right ascensions of the moon's limb for the assumed longitudes L_1 and L_2 respectively,

If the correction of the Ephemeris on the given day is e, the true values of the right ascension for L_1 and L_2 are $A_1 + e$ and $A_2 + e$, the error of the Ephemeris being supposed to be sensibly constant for a few hours; but their difference is

$$(A_2 + e) - (A_1 + e) = A_2 - A_1$$

so that the computed difference of right ascension is the same as if the Ephemeris were correct. If now the observed difference $\alpha_2 - \alpha_1$ is the same as this computed difference, the assumed difference of longitude, or $L_2 - L_1$, is correct;* but, if this is not the case, put

$$\gamma = (\alpha_2 - \alpha_1) - (A_2 - A_1) \tag{408}$$

and

ΔL = the correction of the uncertain longitude, which we will suppose to be L_2

then γ is the change of the right ascension while the moon is describing the small arc of longitude ΔL; and for this small difference we may apply the solution of the preceding article, so that we have at once

$$\Delta L = \frac{\gamma}{H} \text{ (in hours)} \tag{409}$$

or

$$\Delta L = \gamma \times \frac{3600}{H} \text{ (in seconds)} \tag{409*}$$

---

* It should be observed, however, that one of the assumed longitudes must be nearly correct, for it is evident that the same difference of right ascension will not exactly correspond to the same difference of longitude if we increase or decrease both longitudes by the same quantity.

in which the value of H must be that which belongs to the uncertain meridian L_2, or, more strictly, H must be taken for the mean longitude between L_2 and $L_2 + \Delta L$; but, as ΔL is generally very small, great precision in H is here superfluous. However, if in any case ΔL is large, we can first find H for the meridian L_2, and with this value an approximate value of ΔL; then, interpolating H for the meridian $L_2 + \frac{1}{2}\Delta L$, a more correct value of ΔL will be found.*

EXAMPLE.—The following observations were made May 15, 1851, at Santiago and Greenwich:

Object.	Santiago.	Greenwich.
ϑ *Librae*	$15^h\ 46^m\ 3^s.37$	$15^h\ 45^m\ 22^s.37$
Moon II Limb	16 21 36 .84	16 9 39 .41
B. A. C. 5579	16 33 40 .12	16 32 59 .17

We assume here, as in the preceding example, for Santiago $L_2 = 4^h\ 42^m\ 19^s$, and for Greenwich we have $L_1 = 0$. The places of the stars being as in the preceding article, we find for

$$\begin{aligned} \text{Greenwich,}\ \alpha_1 &= 16^h\ \ 9^m\ 39^s.54 \\ \text{Santiago,}\ \alpha_2 &= 16\ \ 20\ \ 55\ .99 \\ \alpha_2 - \alpha_1 &= \qquad 11\ \ 16\ .45 \end{aligned}$$

The computed right ascension for Greenwich is in this case simply that given in the Ephemeris for May 15; the increase to the meridian $4^h\ 42^m\ 19^s.0$ has been found in our example of interpolation, Art. 71, to be

$$A_2 - A_1 = 11^m\ 15^s.84$$

and hence

$$\gamma = +\ 0^s.61$$

We find, moreover, for the longitude $4^h\ 42^m\ 19^s$,

$$H = 143^s.77$$

whence

$$\Delta L = +\ 0^s.61 \times \frac{3600}{143.77} = +\ 15^s.28$$

By these observations we have, therefore,

$$\text{Longitude of Santiago} = 4^h\ 42^m\ 34^s.28$$

* This method of reducing moon culminations was developed by WALKER, *Transactions of the American Philosophical Society*, new series, Vol. V.

234. *Reduction of moon culminations by the hourly Ephemeris.*— The method of reduction given in the preceding article is perfectly exact; but the interpolation of the moon's place to fourth differences is laborious. The hourly Ephemeris, however, requires the use of second differences only. The sidereal time of the transit of the moon's centre at the meridian L_1 is = the observed right ascension of the centre = α_1. If then we put

T_1 = the mean Greenwich time corresponding to α, as found by the hourly Ephemeris,
Θ_1 = the Greenwich sidereal time corresponding to T_1,

we have at once, if the Ephemeris is correct,

$$L_1 = \Theta_1 - \alpha_1 \tag{410}$$

This, indeed, was one of the earliest methods proposed, but was abandoned on account of the imperfection of the Ephemeris. The substitution of corresponding observations, however, does not require a departure from this simple process; for we shall have in the same manner, from the observations made at another meridian (which may be the meridian of the Ephemeris),

$$L_2 = \Theta_2 - \alpha_2$$

and hence

$$\lambda = L_1 - L_2 = (\Theta_1 - \Theta_2) - (\alpha_1 - \alpha_2) \tag{411}$$

and it is evident that the difference $(\Theta_1 - \Theta_0)$ of the Greenwich times will be correct, although the absolute right ascension of the Ephemeris is in error, provided the hourly motion is correct. The correctness of the hourly motion must be assumed in all methods of reducing moon culminations; and in the present state of the lunar theory there can be no error in it which can be sensible in the time required by the moon to pass from one meridian to another.

In this method α is the right ascension of the moon's centre at the instant of the transit of the centre; which may be deduced from the time of transit of the limb by adding or subtracting the "sidereal time of semidiameter passing the meridian," given in the table of moon culminations in the Ephemeris.*

To find T_1 corresponding to α_1, we may proceed as in Art. 64,

---

* If we wish to be altogether independent of the moon-culminating table, we can compute the sidereal time of semidiameter passing the meridian by the formula (see Vol. II., Transit Instrument),

or as follows: Let T_0 and $T_0 + 1^h$ be the two Greenwich hours between which α_1 falls, and put

$\Delta\alpha$ = the increase of right ascension in 1^m of mean time at the time T_0,
$\delta\alpha$ = the increase of $\Delta\alpha$ in 1^h,
α_0 = the right ascension of the Ephemeris at the hour T_0,

then, by the method of interpolation by second differences, we have

$$\alpha_1 = \alpha_0 + \left[\Delta\alpha + \frac{\delta\alpha}{2} \cdot \frac{T_1 - T_0}{3600}\right]\left(\frac{T_1 - T_0}{60}\right)$$

in which the interval $T_1 - T_0$ is supposed to be expressed in seconds. This gives

$$T_1 - T_0 = \frac{60\,(\alpha_1 - \alpha_0)}{\Delta\alpha + \frac{\delta\alpha}{2} \cdot \frac{T_1 - T_0}{3600}}$$

and in the second member an approximate value of T_1 may be used, deduced from the local time of the observation and an approximate longitude. A still more convenient form, which dispenses with finding an approximate value of T_1, is obtained as follows: Put

$$T_1 = T_0 + x$$

then we have

$$\frac{S}{15\,(1 - \lambda) \cos \delta}$$

in which S = the moon's semidiameter, λ = the increase of the moon's right ascension in one sidereal second, and δ = the moon's declination, which are to be taken for the Greenwich time of the observation, approximately known from the local time and the approximate longitude.

Or we may apply to the sidereal time ($= \vartheta_1$) of the transit of the limb the quantity

$$\frac{S}{15 \cos \delta}$$

and the resulting $\alpha_1 = \vartheta_1 \pm \frac{1}{15} S \sec \delta$ will be the right ascension of the moon's centre at the local sidereal time ϑ_1 We then find the Greenwich time Θ_1 corresponding to α_1 as in the text, and we have

$$L_1 = \Theta_1 - \vartheta_1$$

$$x = \frac{60\,(\alpha_1 - \alpha_0)}{\Delta\alpha\left(1 + \frac{x}{7200}\cdot\frac{\delta\alpha}{\Delta\alpha}\right)}$$

$$= \frac{60\,(\alpha_1 - \alpha_0)}{\Delta\alpha}\left(1 + \frac{x}{7200}\cdot\frac{\delta\alpha}{\Delta\alpha}\right)^{-1}$$

or, with sufficient accuracy,

$$x = \frac{60\,(\alpha_1 - \alpha_0)}{\Delta\alpha}\left(1 - \frac{x}{7200}\cdot\frac{\delta\alpha}{\Delta\alpha}\right)$$

Putting then

$$x' = \frac{60\,(\alpha_1 - \alpha_0)}{\Delta\alpha} \qquad x'' = -\frac{x'^2}{7200}\cdot\frac{\delta\alpha}{\Delta\alpha} \tag{412}$$

we have, very nearly,

$$x = x' + x'' \tag{413}$$

As a practical rule for the computer, we may observe that x'' will be a positive quantity when $\Delta\alpha$ is decreasing, and negative when $\Delta\alpha$ is increasing.

The method of this article will be found particularly convenient when the observation is compared directly with the Ephemeris, the latter being corrected by the following process. See page 362.

235. *Peirce's method of correcting the Ephemeris.**—The accuracy of the longitude found by a moon culmination depends upon that of the observed difference of right ascension. When this difference is obtained from two corresponding observations, if the probable errors of the observed right ascensions at the two meridians are ε_1 and ε_2, the probable error of the difference will be $= \sqrt{(\varepsilon_1^2 + \varepsilon_2^2)}$. [Appendix]. But if instead of an actual observation at L_2 we had a perfect Ephemeris, or $\varepsilon_2 = 0$, the probable error of the observed difference would be reduced to ε_1; and if we have an Ephemeris the probable error of which is less than that of an observation, the error of the observed difference is reduced. At the same time, we shall gain the additional advantage that every observation taken at the meridian whose longitude is required will become available, even when no corresponding observation has been taken on the same day; and

* Report of the Superintendent of the U. S Coast Survey for 1854, Appendix, p. 115*.

experience has shown that, when we depend on corresponding observations alone, about one-third of the observations are lost.

The defects of the lunar theory, according to PEIRCE, are involved in several terms which for each lunation may be principally combined into two, of which one is constant and the other has a period of about half a lunation, and he finds that for all practical purposes we may put the correction of the Ephemeris for each semi-lunation under the form

$$X = A + Bt + Ct^2 \tag{414}$$

in which A, B, and C are constants to be determined from the observations made at the principal observatories during the semi-lunation, and t denotes the time reckoned from any assumed epoch, which it will be convenient to take near the mean of the observations. The value of t is expressed in days; and small fractions of a day may be neglected. Let

$\alpha_1, \alpha_2, \alpha_3$, &c. = the right ascension observed at any observatory at the dates t_1, t_2, t_3, &c., from the assumed epoch,

$\alpha_1', \alpha_2', \alpha_3'$, &c. = the right ascension at the same instant found from the Ephemeris,

and put

$$n_1 = \alpha_1 - \alpha_1', \qquad n_2 = \alpha_2 - \alpha_2', \qquad n_3 = \alpha_3 - \alpha_3', \text{ \&c.}$$

then n_1, n_2, n_3, &c. are the corrections which (according to the observations) the Ephemeris requires on the given dates, and hence we have the equations of condition

$$\begin{aligned} A + Bt_1 + Ct_1^2 - n_1 &= 0 \\ A + Bt_2 + Ct_2^2 - n_2 &= 0 \\ A + Bt_3 + Ct_3^2 - n_3 &= 0 \\ \text{\&c.} \end{aligned}$$

In order to eliminate constant errors peculiar to any observatory, when the observation is not made at Greenwich, the observed right ascension is to be increased by the average excess for the year (determined by simultaneous observations) of the right ascensions of the moon's limb made at Greenwich above those made at the actual place of observation.

If now we put

m = the number of observations = the number of equations of condition,
T = the algebraic sum of the values of t,
T_2 = the sum of the squares of t,
T_3 = the algebraic sum of the third powers of t,
T_4 = the sum of the fourth powers of t,
N = the algebraic sum of the values of n,
N_1 = the algebraic sum of the products of n multiplied by t,
N_2 = the algebraic sum of the products of n multiplied by t^2,

the normal equations, according to the method of least squares, will be

$$\left.\begin{aligned} mA + TB + T_2C - N &= 0 \\ TA + T_2B + T_3C - N_1 &= 0 \\ T_2A + T_3B + T_4C - N_2 &= 0 \end{aligned}\right\} \quad (415)$$

The solution of these equations by the method of successive substitution, according to the forms given in the Appendix, may be expressed as follows:

$$T_2' = T_2 - \frac{T^2}{m} \qquad\qquad N_1' = N_1 - \frac{TN}{m}$$

$$T_3' = T_3 - \frac{TT_2}{m} \qquad\qquad N_2' = N_2 - \frac{T_2N}{m}$$

$$T_4' = T_4 - \frac{T_2^{\,2}}{m} \qquad\qquad N_2'' = N_2' - \frac{T_3'N_1'}{T_2'}$$

$$T_4'' = T_4' - \frac{(T_3')^2}{T_2'}$$

$$C = \frac{N_2''}{T_4''} \qquad B = \frac{N_1' - T_3'C}{T_2'} \qquad A = \frac{N - T_2C - TB}{m} \quad (416)$$

Then, to find the mean error of the corrected Ephemeris, we observe that this error is simply that of the function X, which is to be found by the method of the Appendix, according to which we first find the coefficients k_0, k_1, k_2 by the following formulæ:

$$\begin{aligned} mk_0 &= 1 \\ mk_0 + T_2'k_1 &= t \\ mk_0 + T_3'k_1 + T_4''k_2 &= t^2 \end{aligned}$$

and then, putting

$$M = \sqrt{(k_0^2 m + k_1^2 T_2' + k_2^2 T_4'')}$$

we have

$$(\varepsilon X) = M\varepsilon \tag{417}$$

in which ε denotes the mean error of a single observation and (εX) the mean error of the corrected Ephemeris; or, if ε denotes the probable error of an observation, (εX) denotes the probable error of the corrected Ephemeris. (Appendix.)

If the values of k_0, k_1, and k_2 are substituted in M, we shall have

$$M = \sqrt{\left[\frac{1}{m} + \frac{(t-1)^2}{T_2'} + \frac{(t-1)^2}{T_4''}\left(t + 1 - \frac{T_3'}{T_2'}\right)^2\right]} \tag{418}$$

It will generally happen, where a sufficient number of observations are combined, that $\frac{T_3'}{T_2'}$ is a small fraction which may be neglected without sensibly affecting the estimation of a probable error, and we may then take

$$M = \sqrt{\left[\frac{1}{m} + \frac{(t-1)^2}{T_2'} + \frac{(t^2-1)^2}{T_4''}\right]} \tag{418*}$$

According to PEIRCE, the probable error of a *standard* observation of the moon's transit is $0^s.104$ (found from the discussion of a large number of Greenwich, Cambridge, Edinburgh, and Washington observations); so that the probable error of the corrected Ephemeris will be equal to $M.(0^s.104)$.

EXAMPLE.—At the Washington Observatory, the following right ascensions of the moon were obtained from the transits over twenty-five threads, observed with the electro-chronograph:

Approx. Green Mean Time.	R. A. of ☽ II Limb.	Sid. time semid. passing merid.	R. A of ☽ centre $= \alpha_1$.
1859, Aug. 16, 19^h	$0^h\ 8^m\ 53^s.40$	$62^s.06$	$0^h\ 7^m\ 51^s.34$
" 17, 20	0 54 33 .57	63 .54	0 53 30 .03
" 18, 21	1 42 48 .53	65 .77	1 41 42 .76

The sidereal time of the semidiameter passing the meridian is here taken from the British Almanac, as we propose to reduce the observations by means of the Greenwich observations which are reduced by this almanac. We thus avoid any error in the semidiameter.

During the semi-lunation from Aug. 13 to Aug. 27, the Greenwich observations, also made with the electro-chronograph,

gave the following corrections ($= n$) of the Nautical Almanac right ascensions of the moon:

Approx. Greenwich Mean Time.	n	t
1859. Aug. 14, 13^h	$-0^s.39$	$-3.$
" 15, 14	-0.26	-1.9
" 16, 14	-0.49	-0.9
" 18, 16	-0.63	$+1.2$
" 19, 17	-1.04	$+2.2$
" 20, 17	-1.08	$+3.2$

Let us employ these observations to determine by Peirce's method the most probable correction of the Ephemeris on the dates of the Washington observations. Adopting as the epoch Aug. 17th 12^h or $17^d.5$, the values of t are approximately as above given. The correction of the Ephemeris being sensibly constant for at least one hour, these values are sufficiently exact. We find then

$T = 0.8$	$T_2 = 29.94$	$T_3 = 10.556$	$T_4 = 225.045$	
	$T_2' = 29.83$	$T_3' = 6.564$	$T_4' = 75.644$	$T_4'' = 74.200$
$m = 6$	$N = -3^s.89$	$N_1 = -4^s.41$	$N_2 = -21^s.85$	
		$N_1' = -3.89$	$N_2' = -2.44$	$N_2'' = -1^s.58$

and hence, by (416),

$$C = -0^s.02135 \qquad B = -0^s.1257 \qquad A = -0^s.525$$

The correction of the Ephemeris for any given date t, reckoning from Aug. 17.5, is, therefore,

$$X = -0^s.525 - 0^s.1257\,t - 0^s.02135\,t^2$$

Consequently, for the dates of the Washington observations, the correction and the probable error ($M\varepsilon$) of the correction, found by (418) or (418*), are as follows:

Aug. 16, 19^h	$t = -0.7$	$X = -0^s.45$	$M\varepsilon = 0^s.05$
17, 20	$t = +0.3$	$X = -0.56$	$M\varepsilon = 0.04$
18, 21	$t = +1.4$	$X = -0.74$	$M\varepsilon = 0.04$

The longitude of the Washington Observatory may now be found by the hourly Ephemeris (after applying these corrections), by the method of Art. 234. Taking the observation of Aug. 16, we have

$$
\begin{array}{lrl}
\text{Aug. 16, } T_0 = 19^h, \text{ R. A. of Ephemeris} & = & 0^h\ 6^m\ 47^s.56 \\
X & = & \quad\ -\ 0\ .45 \\
\Delta\alpha = 1.8122 \quad \delta\alpha = +\ 0.0023 \quad \alpha_0 & = & 0\ \ 6\ \ 47\ .11 \\
\alpha_1 & = & 0\ \ 7\ \ 51\ .34 \\
\alpha_1 - \alpha_0 & = & \quad 1\ \ \ 4\ .23
\end{array}
$$

$$
\begin{array}{rl}
\log (\alpha_1 - \alpha_0) & 1.80774 \\
\text{ar. co. } \log \Delta\alpha & 9.74179 \\
\log 60 & 1.77815 \\
\hline
\log x' & 3.32768 \\
x' & = 35^m\ 26^s.57 \\
x'' & = -\quad 0\ .80 \\
\hline
x & = 35\ \ 25\ .77
\end{array}
\qquad
\begin{array}{rl}
\log x'^2 & 6.6554 \\
\log \delta\alpha & 7.3617 \\
\text{ar. co. } \log \Delta\alpha & 9.7418 \\
\log \frac{1}{7200} & 6.1427 \\
\hline
\log x'' & n9.9016
\end{array}
$$

$$
\begin{array}{lrl}
\text{Hence, Greenwich mean time} = T_0 + x & = & 19^h\ 35^m\ 25^s.77 \\
\quad \text{Sidereal time mean noon} & = & \ \ 9\ \ 37\ \ 24\ .18 \\
\quad \text{Correction for } 19^h\ 35^m\ 25^s.77 & = & \qquad\ \ 3\ \ 13\ .09 \\
\hline
\quad \text{Greenwich sidereal time} & = & \ \ 5\ \ 16\ \ \ \ 3\ .04 \\
\quad \text{Local sidereal time} = \alpha_1 & = & \ \ 0\ \ \ \ 7\ \ 51\ .34 \\
\hline
\text{Longitude} & = & \ \ 5\ \ \ \ 8\ \ 11\ .70
\end{array}
$$

The observations of the 17th and 18th being reduced in th same manner, the three results are

		Probable error.*	Weight.
Aug. 16,	$5^h\ 8^m\ 11^s.70$	$3^s.5$	1.
" 17,	12 .50	3 .1	1.3
" 18,	11 .10	2 .9	1.5
Mean by weights =	5 8 11 .74	1 .8	

236. *Combination of moon culminations by weights.*—When some of the transits either of the moon or of the comparison stars are incomplete, one or more of the threads being lost, such observa tions should evidently have less weight than complete ones, if we wish to combine them strictly according to the theory of probabilities. Besides, other things being equal, a determination of the longitude will have more or less weight according to the greater or less rapidity of the moon's motion in right ascension.

* For the computation of the probable error and weight, see the following article.

If the weight of a transit either of the moon or a star were simply proportional to the number of observed threads, as has been assumed by those who have heretofore treated of this subject,* the methods which they have given, and which are obvious applications of the method of least squares, would be quite sufficient. But the subject, strictly considered, is by no means so simple.

Let us first consider the formula

$$\alpha_1 = \alpha' + \vartheta_1 - \vartheta'$$

or, rather

$$\alpha_1 = \vartheta_1 + (\alpha' - \vartheta')$$

in which ϑ_1 and ϑ' are the observed sidereal times of the transit of the moon and star, respectively; α' is the tabular right ascension of the star, and α_1 is the deduced right ascension of the moon. The probable error of α_1 is composed of the probable errors of ϑ_1 and of $\alpha' - \vartheta'$, which belong respectively to the moon and the star. We may here disregard the clock errors, as well as the unknown instrumental errors, since they affect ϑ_1 and ϑ' in the same manner, very nearly, and are sensibly eliminated in the difference $\vartheta_1 - \vartheta'$. The probable error of the quantity $\alpha' - \vartheta'$ is composed of the errors of α' and ϑ'. The probable error of the tabular right ascension of the moon-culminating stars is not only very small, but in the case of corresponding observations is wholly eliminated; and even when we use a corrected Ephemeris it will have but little effect, since the observed right ascension of the moon at the principal observatories always depends (or at least should depend) chiefly upon these stars. We may, therefore, consider the error of $\alpha' - \vartheta'$ as simply the error of ϑ'. We have here to deal with those errors only which do not necessarily affect ϑ' and ϑ_1 in the same manner, and of these the chief and only ones that need be considered here are—1st, the *culmination error* produced by the peculiar conditions of the atmosphere at the time of the star's transit, which are constant, or nearly so, during the transit, but are different for different stars and on different days; and, 2d, the *accidental error of observation*. It is only the latter which can be diminished

* NICOLAI, in the *Astronomische Nachrichten*, No. 26; and S. C. WALKER, Transactions of the American Philosophical Society, Vol. VI. p. 258.

by increasing the number of threads. In Vol. II. (Transit Instrument) I shall show that the probable error of a single determination of the right ascension of an equatorial star (and this may embrace the moon-culminating stars) at the Greenwich Observatory is $0^s.06$, whereas, if the culmination error did not exist it would be only $0^s.03$, the probable error of a single thread being $= 0^s.08$, and the number of threads $= 7$. Hence, putting

$c =$ the probable culmination error for a star,

we deduce*

$$c = \sqrt{(0.06)^2 - (0.03)^2} = 0^s.052$$

If, then, we put

$\varepsilon =$ the probable accidental error of the transit of a star over a single thread,
$n =$ the number of threads on which the star is observed,

the probable error of ϑ', and, consequently, also of $\alpha' - \vartheta'$, is

$$= \sqrt{c^2 + \frac{\varepsilon^2}{n}}$$

and the weight of $\alpha' - \vartheta'$ for each star may be found by the formula

$$p = \frac{E^2}{c^2 + \frac{\varepsilon^2}{n}}$$

in which E is the probable error of an observation of the weight unity, which is, of course, arbitrary. If we make $p = 1$ when $n = 7$, we have $E = 0^s.06$. Substituting this value, and also $c = 0^s.052$, $\varepsilon = 0^s.08$, the formula may be reduced to the following:

$$p = \frac{134}{100 + \frac{238}{n}} \tag{419}$$

The value of α_1 is to be deduced by adding to ϑ_1 the mean

* The value of c thus found involves other errors besides the culmination error proper, such as unknown irregularities of the clock and transit instrument, &c. These cannot readily be separated from c, nor is it necessary for our present purpose

ccording to weights of all the values of $\alpha_1 - \vartheta_1$ given by the everal stars, or

$$\alpha_1 = \vartheta_1 + \frac{[p(\alpha' - \vartheta')]}{[p]} \tag{420}$$

vhere the rectangular brackets are employed to express the sum f all the quantities of the same form. The probable error of he last term will be

$$= \frac{E}{\sqrt{[p]}} = \frac{0^s.06}{\sqrt{[p]}}$$

If now we put

$\varepsilon_1 =$ the probable error of α_1,
$c_1 =$ the culmination error for the moon,
$k\varepsilon =$ the probable accidental error of the transit of the moon's limb over a single thread,
$n_1 =$ the number of threads on which the moon is observed,

he probable error of ϑ_1 will be $= \sqrt{c_1^2 + \frac{(k\varepsilon)^2}{n_1}}$, and hence

$$\varepsilon_1^2 = c_1^2 + \frac{(k\varepsilon)^2}{n_1} + \frac{E^2}{[p]} \tag{421}$$

To determine c_1 I shall employ the values of the other quantities n this equation which have been found from the Greenwich bservations. Professor Peirce gives $\varepsilon_1 = 0^s.104$, and in the ases which I examined I found the mean value $k = 1.3$. Asuming $[p] = 4$ as the average number of stars upon which α_1 lepends in the Greenwich series, we have

$$(0.104)^2 = c_1^2 + \frac{(0.104)^2}{7} + \frac{(0.06)^2}{4}$$

vhence

$$c_1 = 0^s.091$$

nd the formula for the probable error of α_1 observed at the neridian L_1 is

$$\varepsilon_1^2 = (0.091)^2 + \frac{(0.104)^2}{n_1} + \frac{(0.06)^2}{[p]} \tag{422}$$

n the case of corresponding observations at a second meridian L_2, the probable error ε_2 is also to be found by this formula, and hen the probable error of the deduced difference of right ascenion will be

$$= \sqrt{\varepsilon_1^2 + \varepsilon_2^2}$$

and the probable error of the deduced longitude will be.

$$= h\sqrt{\varepsilon_1^2 + \varepsilon_2^2} \tag{423}$$

where, H being the increase of the moon's right ascension in 1^h of longitude, we have

$$h = \frac{3600}{H} \tag{424}$$

But if the observation at the meridian L_1 is compared with a corrected Ephemeris (Art. 235) the probable error of which is $M(0^s.104)$, the probable error of the deduced longitude will be

$$= h\sqrt{\varepsilon_1^2 + M^2(0.104)^2} \tag{425}$$

Finally, all the different values of the longitude will be combined by giving them weights reciprocally proportional to the squares of their probable errors.

The preponderating influence of the constant error represented by the first term of (422) is such that a very precise evaluation of the other terms is quite unimportant. It is also evident that we shall add very little to the accuracy of an observation by increasing the number of threads of the reticule beyond five or seven. For example, suppose, as in the Washington observations used in Art. 235, that twenty-five threads are taken, and that four stars are compared with the moon; we have for each star, by (419),

$$p = \frac{134}{100 + \frac{238}{25}} = 1.22$$

and hence

$$\varepsilon_1 = \sqrt{\left[(0.091)^2 + \frac{(0.104)^2}{25} + \frac{(0.06)^2}{4.88}\right]} = 0^s.097$$

whereas for seven threads we have $\varepsilon_1 = 0^s.104$, and therefore the increase of the number of threads has not diminished the probable error by so much as $0^s.01$.

For the observations of 1859 August 16, 17, 18, Art. 235, the values of h are respectively

32.1 30.8 and 28.8

and, taking $M\varepsilon = M(0^s.104)$ as given in that article, namely,

$0^s.05$ $0^s.04$ and $0^s.04$

with the value of $\varepsilon_1 = 0^s.097$ above found, we deduce the probable errors of the three values of the longitude, by (425),

$$3^s.5 \qquad 3^s.1 \qquad \text{and} \qquad 2^s.9$$

The reciprocals of the squares of these errors are very nearly in the proportion of the numbers 1, 1.3, 1.5, which were used as the weights in combining the three values.

237. The advantage of employing a corrected Ephemeris instead of corresponding observations can now be determined by the above equations. If the observations are all *standard* observations (represented by $n_1 = 7$ and $[p] = 4$), we shall have $\varepsilon_1 = \varepsilon_2 = 0^s.104$, and the probable error of the longitude will be

$$\text{by corresponding observations} = h\varepsilon_1 \sqrt{2}$$
$$\text{by the corrected Ephemeris} \quad = h\varepsilon_1 \sqrt{1 + M^2}$$

The latter will, therefore, be preferable when $M < 1$, which will always be the case except when very few observations have been taken at the principal observatories.

But experience has shown that when we depend wholly on corresponding observations we lose about one-third of the observations, and, consequently, the probable error of the final longitude from a series of observations is greater than it would be were all available in the ratio of $\sqrt{3} : \sqrt{2}$. Hence the probable errors of the final results obtained by corresponding observations exclusively, and by employing the corrected Ephemeris by which all the observations are rendered available, are in the ratio $\sqrt{3} : \sqrt{1 + M^2}$, and, the average value of M being about 0.6, this is as 1 : 0.67.

If, however, on the date of any given observation at the meridian to be determined, we can find corresponding observations at *two* principal observatories, the probable error of the longitude found by comparing their mean with the given observation will be only $h\varepsilon_1 \sqrt{1.5}$, which is so little greater than the average error in the use of the corrected Ephemeris, that it will hardly be worth while to incur the labor attending the latter. If there should be *three* corresponding observations, the error will be reduced to $h\varepsilon_1 \sqrt{1.33}$, and, therefore, less than the average error of the corrected Ephemeris.

The advantage of the new method will, therefore, be felt chiefly in cases where either no corresponding observation, or but one, has been taken at any of the principal observatories.

238. The mean value of h is about $= 27$, and therefore a probable error of $0^s.1$ in the observed right ascension, supposing the Ephemeris perfect, will produce a mean probable error of $2''.7$ in the longitude. If the probable error diminished without limit in proportion to the square root of the number of observations, as is assumed in the theory of least squares, we should only have to accumulate observations to obtain a result of any given degree of accuracy. But all experience proves the fallacy of this law when it is extended to minute errors which must wholly escape the most delicate observation. The remarks of Professor PEIRCE on this point, in the report above cited, are of the highest importance. He says: "If the law of error embodied in the method of least squares were the sole law to which human error is subject, it would happen that by a sufficient accumulation of observations any imagined degree of accuracy would be attainable in the determination of a constant; and the evanescent influence of minute increments of error would have the effect of exalting man's power of exact observation to an unlimited extent. I believe that the careful examination of observations reveals another law of error, which is involved in the popular statement that 'man cannot measure what he cannot see.' The small errors which are beyond the limits of human perception are not distributed according to the mode recognized by the method of least squares, but either with the uniformity which is the ordinary characteristic of matters of chance, or more frequently in some arbitrary form dependent upon individual peculiarities,—such, for instance, as an habitual inclination to the use of certain numbers. On this account, it is in vain to attempt the comparison of the distribution of errors with the law of least squares to too great a degree of minuteness; and on this account, *there is in every species of observation an ultimate limit of accuracy beyond which no mass of accumulated observations can ever penetrate.* A wise observer, when he perceives that he is approaching this limit, will apply his powers to improving the methods, rather than to increasing the number of observations. This principle will thus serve to stimulate, and not to paralyze, effort; and its

vivifying influence will prevent science from stagnating into mere mechanical drudgery.

"In approaching the ultimate limit of accuracy, the probable error ceases to diminish proportionably to the increase of the number of observations, so that the accuracy of the mean of several determinations does not surpass that of the single determinations as much as it should do in conformity with the law of least squares: thus it appears that the probable error of the mean of the determinations of the longitude of the Harvard Observatory, deduced from the moon-culminating observations of 1845, 1846, and 1847, is $1^s.28$ instead of being $1^s.00$, to which it should have been reduced conformably to the accuracy of the separate determinations of those years.

"One of the fundamental principles of the doctrine of probabilities is, that the probability of an hypothesis is proportionate to its agreement with observation. But any supposed computed lunar epoch may be changed by several hundredths of a second without perceptibly affecting the comparison with observation, provided the comparison is restricted within its legitimate limits of tenths of a second. Observation, therefore, gives no information which is opposed to such a change."

The ultimate limit of accuracy in the determination of a longitude by moon culminations, according to the same distinguished authority, is *not less than one second of time.* This limit can probably be reached by the observations of two or three years, if all the possible ones are taken; and a longer continuance of them would be a waste of time and labor.

From these considerations it follows that the method of moon culminations, when the transits of the limb are employed, cannot come into competition with the methods by chronometers and occultations where the latter are practicable.*

* In consequence of the uncertainty attending the observation of the transit of the moon's limb, it has been proposed by MAEDLER (*Astron. Nach.* No. 337) to substitute the transit of a well-defined lunar spot. The only attempt to carry out this suggestion, I think, is that of the U. S. Coast Survey, a report upon which by Mr. PETERS will be found in the Report of the Superintendent for 1856, p. 198. The varying character of a spot as seen in telescopes of different powers presents, it seems to me, a very formidable obstacle to the successful application of this method.

FIFTH METHOD.—BY AZIMUTHS OF THE MOON, OR TRANSITS OF THE MOON AND A STAR OVER THE SAME VERTICAL CIRCLE.

239. The travelling observer, pressed for time, will not unfrequently find it expedient to mount his transit instrument in the vertical circle of a circumpolar star, without waiting for the meridian passage of such a star. The methods of determining the local time and the instrumental constants in this case are given in Vol. II. He may then also observe the transit of the moon and a neighboring star, and hence deduce the right ascension of the moon, which may be used for determining his longitude precisely as the culminations are used in Art. 234.

240. But if the local time is previously determined, we may dispense with all observations except those of the moon and the neighboring star, and then we can repeat the observation several times on the same night by setting the instrument successively in different azimuths on each side of the meridian. It will not be advisable to extend the observations to azimuths of more than 15° on either side.

The altitude and azimuth instrument is peculiarly adapted for such observations, as its horizontal circle enables us to set it at any assumed azimuth when the direction of the meridian is approximately known. The zenith telescope will also answer the same purpose. But as the horizontal circle reading is not required further than for setting the instrument, it is not indispensable, and therefore the ordinary portable transit instrument may be employed, though it will not be so easy to identify the comparison star.

The comparison star should be one of the well-determined moon-culminating stars, as nearly as possible in the same parallel with the moon, and not far distant in right ascension, either preceding or following.

The chronometer correction and rate must be determined, with all possible precision, by observations either before or after the moon observations, or both. An approximate value of the correction should be known before commencing the observations, as it will be expedient to compute the hour angles and zenith distances of the two objects for the several azimuths at which it is proposed to observe, in order to point the instrument properly and thus avoid observing the wrong star.

To secure the greatest degree of accuracy, the observations should be conducted substantially as follows:—

1st. The instrument being supposed to have a horizontal circle, let the telescope be directed to some terrestrial object, the azimuth of which is known (or to a circumpolar star in the meridian), and read the circle. The reading for an object in the meridian will then be known; denote it by a.

2d. The first assumed azimuth at which the transits are to be observed being A, set the horizontal circle to the reading $A + a$, and the vertical circle to the computed zenith distance of the moon or the star (whichever precedes). This must be done a few minutes before the computed time of the first transit.

3d. Observe the inclination of the horizontal axis with the spirit level.

4th. Observe the transit of the first object over the several threads.

5th. If there is time, observe the inclination of the horizontal axis.

6th. Set the vertical circle for the zenith distance of the second object, and observe its transit.

7th. Observe the inclination of the horizontal axis with the spirit level.

The instrument must not be disturbed in azimuth during these operations, which constitute one complete observation.

Now set upon a new azimuth, sufficiently greater to bring the instrument in advance of the preceding object, and repeat the observation. It will often be possible to obtain in this way four or six observations, two or three on each side of the meridian, but the value of the result will not be much increased by taking more than one observation on each side of the meridian.

The collimation constant is supposed to be known; but, in order to eliminate any error in it, as well as inequality of pivots, one-half the observations should be taken in each position of the rotation axis.

The azimuth of the instrument at each observation is only known from the local time, and hence the following indirect method of computation will be found more convenient than the usual method of reducing extra-meridian transits; but the reader will find it easy to adapt the methods given in Vol. II. for such purpose to the present case.

We shall make use of the following notation:

T, T' = the mean of the chronometer times of transit of the moon's limb and the star, respectively, over the several threads,*

ΔT, $\Delta T'$ = the corresponding chronometer corrections,

b, b' = the inclinations of the horizontal axis at the times T and T',

c = the collimation constant for the mean of the threads,

α, α' = the moon's and the star's right ascensions,

δ, δ' = " " " declinations,

t, t' = " " " hour angles,

ζ, ζ' = " " " true zenith distances,

q, q' = " " " parallactic angles,

A, A' = " " " azimuths,

$\Delta\alpha$ = the increase of the moon's right ascension in one minute of mean time,

$\Delta\delta$ = the increase (positive towards the north) of the moon's declination in one minute of mean time,

π = the moon's equatorial horizontal parallax,

S = the moon's geocentric semidiameter,

φ = the observer's latitude,

L' = the assumed longitude,

ΔL = the required correction of this longitude,

L = the true longitude $= L' + \Delta L$

The moon's α, δ, π, and S are to be taken from the Ephemeris for the Greenwich time $T + \Delta T + L'$(expressed in mean time). The changes $\Delta\alpha$, $\Delta\delta$ are also to be reduced to this time. The right ascension and declination must be accurately interpolated, from the hourly Ephemeris, with second differences.

The quantities A, ζ, q are now to be computed for the chronometer time T, and A', ζ', q' for the time T'. Since A and A'

* The chronometer time of passage over the mean of the threads will be obtained rigorously by reducing each thread separately to the mean of all by the general formula given for the purpose in Vol. II. If, however, the same threads are employed for both moon and star, and c denotes the equatorial distance of the mean of the actually observed threads from the collimation axis, it will suffice (unless the observations are extended greatly beyond the limits recommended in the text) to take the means of the observed times at the times of passage over the fictitious thread the collimation of which is $= c$. The slight theoretical error which this procedure involves will be eliminated if the observations are arranged symmetrically with respect to the meridian.

are required with all possible precision, logarithms of at least six decimal places are to be employed in their computation; but for ζ, q, ζ', q', four decimal places will suffice. The following formulæ for this purpose result from a combination of (16) and (20):

For the moon.		For the star.
$t = T + \Delta T - \alpha$		$t' = T' + \Delta T' - \alpha'$
$\tan M = \tan\delta \sec t$ $\tan A = \dfrac{\tan t \cos M}{\sin(\varphi - M)}$	with six decimals;	$\tan M' = \tan\delta' \sec t'$ $\tan A' = \dfrac{\tan t' \cos M'}{\sin(\varphi - M')}$
$\tan N = \cot\varphi \cos t$ $\tan q = \dfrac{\tan t \sin N}{\cos(\delta + N)}$ $\tan\zeta = \dfrac{\cot(\delta + N)}{\cos q}$	with four decimals;	$\tan N' = \cot\varphi \cos t'$ $\tan q' = \dfrac{\tan t' \sin N'}{\cos(\delta' + N')}$ $\tan\zeta' = \dfrac{\cot(\delta' + N')}{\cos q'}$

(426)

in which A and q are to be so taken that $\sin A$ and $\sin q$ shall have the same sign as $\sin t$.

The true azimuth of the moon's limb will be found by applying to the azimuth of the centre the correction

$$\pm \frac{S}{\sin\zeta} \begin{bmatrix} \text{upper sign for 1st limb} \\ \text{lower " " 2d "} \end{bmatrix}$$

If we assume the parallax of the limb to be the same as that of the centre (which involves but an insensible error in this case), we next find the apparent azimuth of the limb by applying the correction given by (116), or

$$\rho\pi(\varphi - \varphi') \sin 1'' \sin A' \operatorname{cosec} \zeta$$

in which $\varphi - \varphi'$ is the reduction of the latitude, and ρ is the terrestrial radius for the latitude φ. In this expression we employ A', which is the computed azimuth of the star, for the apparent azimuth of the moon's limb, since by the nature of the observation they are very nearly equal.

To correct strictly for the collimation and level of the instrument, we must have the moon's and star's apparent zenith distances, which will be found with more than sufficient accuracy for the purpose by the formulæ

$$\begin{aligned} &\text{moon's app. zen. dist.} = \zeta_1 = \zeta + \pi \sin\zeta - \text{refraction} \\ &\text{star's " " "} = \zeta_1' = \zeta' - \text{refraction} \end{aligned}$$

and then the reduction of the true azimuth to the instrumental azimuth (see Vol. II., *Altitude and Azimuth Instrument*) is

$$\text{for the moon,} \quad \mp \frac{c}{\sin \zeta_1} \mp \frac{b}{\tan \zeta_1}$$

$$\text{for the star,} \quad \mp \frac{c}{\sin \zeta_1'} \mp \frac{b'}{\tan \zeta_1'}$$

the upper or lower sign being used according as the vertical circle is on the left or the right of the observer. The computed instrumental azimuths are, therefore,

$$\left.\begin{aligned} (\text{moon})\ A_1 &= A \pm \frac{S}{\sin \zeta} + \frac{\rho\pi(\varphi - \varphi')\sin 1''\sin A'}{\sin \zeta} \mp \frac{c}{\sin \zeta_1} \mp \frac{b}{\tan \zeta_1} \\ (\text{star})\ A_1' &= A' \mp \frac{c}{\sin \zeta_1'} \mp \frac{b'}{\tan \zeta_1'} \end{aligned}\right\} \quad (427)$$

If now the longitude and other elements of the computation are correct, we shall find A_1 and A_1' to be equal: otherwise, put

$$x = A_1 - A_1' \qquad (428)$$

then we are to find how the required correction ΔL depends on x, supposing here that all the elements which do not involve the longitude are correct. Now, we have taken α and δ from the Ephemeris for the Greenwich sidereal time $T + \Delta T + L'$, when they should be taken for the time $T + \Delta T + L' + \Delta L$. Hence, if λ and β denote the increments of the moon's right ascension and declination in one sidereal second, both expressed in seconds of arc,

$$\left.\begin{aligned} \lambda &= \frac{15\,\Delta\alpha}{60.164} = [9.39675]\,\Delta\alpha \\ \beta &= \frac{\Delta\delta}{60.164} = [8.22066]\,\Delta\delta \end{aligned}\right\} \quad (429)$$

we find that

α	requires the correction	$\lambda \,.\, \Delta L$
δ	" "	$\beta \,.\, \Delta L$
t	" "	$-\lambda \,.\, \Delta L$

and these corrections must produce the correction $-x$ in the moon's azimuth. The relations between the corrections of the azimuth, the hour angle, and the declination, where these are so small as to be treated as differentials, is, by (51),

$$dA = \frac{\cos\delta\cos q}{\sin\zeta}\,dt + \frac{\sin q}{\sin\zeta}\,d\delta$$

that is,

$$-x = -\frac{\cos\delta\cos q}{\sin\zeta}\,\lambda.\Delta L + \frac{\sin q}{\sin\zeta}\,\beta.\Delta L$$

Hence, if we put

$$a = \lambda.\frac{\cos\delta\cos q}{\sin\zeta} - \beta.\frac{\sin q}{\sin\zeta} \tag{430}$$

we have

$$\Delta L = \frac{x}{a} \tag{431}$$

and hence, finally, the true longitude $L' + \Delta L$.

241. In order to determine the relative advantages of this method and that of meridian transits, let us investigate a formula which shall exhibit the effect of every source of error. Let

$\delta\alpha, \delta\delta, \delta\pi, \delta S$ = the corrections of the elements taken from the Ephemeris of the moon,

$\delta\alpha', \delta\delta'$ = the corrections of the star's place,

$\delta T, \delta T'$ = the corrections for error in the obs'd time,

$\delta\Delta T$ = the correction of ΔT,

$\delta\varphi$ = the correction of φ.

If, when the corrected values of all the elements—that of the longitude included—are substituted in the above computation, A_1 and A_1' become $A_1 + dA_1$ and $A_1' + dA_1'$, we ought to find, rigorously,

$$A_1 + dA_1 = A_1' + dA_1'$$

which compared with (428) gives

$$x = -dA_1 + dA_1' \tag{432}$$

We have, therefore, to find expressions for dA_1 and dA_1' in terms of the above corrections and of ΔL. We have, first, by differentiating (427),

$$dA_1 = dA \pm \frac{dS}{\sin\zeta} + \frac{\rho(\varphi - \varphi')\sin 1''\sin A'}{\sin\zeta}\,d\pi$$

$$dA_1' = dA'$$

We neglect errors in c and b which are practically eliminated by comparing the moon with a star of nearly the same declination, and combining observations in the reverse positions of the axis.

The total differential of A is, by (51), after reducing dt to arc,

$$dA = \frac{\cos\delta \cos q}{\sin\zeta} \cdot 15\,dt + \frac{\sin q}{\sin\zeta}\,d\delta - \cot\zeta \sin A\,d\varphi$$

consequently, also,

$$dA' = \frac{\cos\delta' \cos q'}{\sin\zeta'} \cdot 15dt' + \frac{\sin q'}{\sin\zeta'}\,d\delta' - \cot\zeta' \sin A' d\varphi$$

Since $t = T + \Delta T - \alpha$, we have

$$dt = dT + d\Delta T - d\alpha$$

where dT and $d\Delta T$ may be at once exchanged for δT and $\delta\Delta T$; but $d\alpha$ is composed of two parts: 1st, the correction of the Ephemeris, and 2d, $\lambda(\Delta L + \delta T + \delta\Delta T)$, which results from our having taken α for the uncorrected time. Hence we have, in arc,

$$15\,dt = 15\,\delta T + 15\ \delta\Delta T - 15\,\delta\alpha - \lambda(\Delta L + \delta T + \delta\Delta T)$$

The correction $d\delta$ is likewise composed of two parts, namely,

$$d\delta = \delta\delta + \beta(\Delta L + \delta T + \delta\Delta T)$$

Further, we have simply $d\delta' = \delta\delta'$ and

$$dt' = \delta T' + \delta\Delta T' - \delta\alpha'$$

but, as we may neglect the error in the rate of the chronometer for the brief interval between the observation of the moon and the star, we can take $\delta\Delta T' = \delta\Delta T$, and, consequently,

$$dt' = \delta T' + \delta\Delta T - \delta\alpha'$$

When the substitutions here indicated are made in (432), we obtain the expression

$$\begin{aligned} x = a\Delta L + 15f\,.\,\delta\alpha - \frac{\sin q}{\sin\zeta} \cdot \delta\delta\ - (15f - a)\,\delta T \\ - 15f'.\,\delta\alpha' + \frac{\sin q'}{\sin\zeta'} \cdot \delta\delta' + 15f'.\,\delta T' \\ \mp \frac{\delta S}{\sin\zeta} - \frac{\rho(\varphi - \varphi')\sin 1'' \sin A'}{\sin\zeta}\,\delta\pi \\ - [15(f - f') - a]\ \delta\Delta T + \frac{\sin(\zeta' - \zeta)\sin A'}{\sin\zeta \sin\zeta'}\,\delta\varphi \end{aligned} \qquad (433)$$

in which the following abbreviations are used:

$$f = \frac{\cos\delta\cos q}{\sin\zeta} \qquad f' = \frac{\cos\delta'\cos q'}{\sin\zeta'}$$

$$a = \lambda f - \beta\,\frac{\sin q}{\sin\zeta}$$

and in the coefficient of $\delta\varphi$ we have put $A = A'$.

By the aid of this equation we can now trace the effect of each source of error.

1st. The coefficients of $\delta\delta$, $\delta\delta'$, $\delta\pi$, $\delta\varphi$ have different signs for observations on different sides of the meridian, and therefore the errors of declination, parallax, and latitude will be eliminated by taking the mean of a pair of observations equidistant from the meridian.

2d. The star's declination being nearly equal to that of the moon, we shall have very nearly $f = f'$, and the coefficient of $\delta\Delta T$ will be $= a$; and since to find ΔL we have yet to divide the equation by a, it follows that an error in the assumed clock correction produces an equal error (but with a different sign) in the longitude, as in the case of meridian observations.

3d. An error δT in the observed time of the moon's transit produces in the longitude the error

$$\left(\frac{15f}{a} - 1\right)\delta T$$

The mean of the values of a for two observations equidistant from the meridian is λf. The mean effect of the error δT is therefore

$$\left(\frac{15}{\lambda} - 1\right)\delta T$$

which is the same as in the case of a meridian observation.

The effect of an error $\delta T'$ in the observed time of the star's transit is

$$\frac{15f'}{a}\,\delta T'$$

and for two observations equidistant from the meridian, the star being in the same parallel as the moon, the mean effect is

$$\frac{15}{\lambda}\,\delta T'$$

also the same as for a meridian observation.

4th. An error δS in the tabular semidiameter is always eliminated in the case of meridian observations when they are compared with observations at another meridian, since the same semidiameter is employed in reducing the observations at both meridians. But in the case of an extra-meridian observation the effect upon the longitude is

$$\frac{\delta S}{a \sin \zeta} = \frac{\delta S}{\lambda \cos \delta \cos q - \beta \sin q}$$

and in the mean of two observations equidistant from the meridian, the values of q being small, it is

$$\frac{\delta S}{\lambda \cos \delta \cos q} = \frac{\delta S}{\lambda \cos \delta}(1 + 2 \sin^2 \tfrac{1}{2} q) \text{ nearly.}$$

For a meridian observation the error will be

$$\frac{\delta S}{\lambda \cos \delta}$$

The error in the case of extra-meridian observations, therefore, remains somewhat greater than in the case of meridian ones, the excess being nearly

$$\frac{2\,\delta S \,.\, \sin^2 \tfrac{1}{2} q}{.\,\lambda \cos \delta}$$

which, however, is practically insignificant; for we have not to fear that δS can be as great as $1''$, and therefore, taking $q = 15°$, $\delta = 30°$, and $\lambda = 0.4$, which are extreme values, the difference cannot amount to $0^s.1$ in the longitude.

5th. The error $\delta\alpha$ of the tabular right ascension of the moon produces in the longitude the error

$$-\frac{15 f}{a}\,\delta\alpha$$

and from the mean of two observations equidistant from the meridian, the error is

$$-\frac{15\,\delta\alpha}{\lambda}$$

as in the case of the meridian observation.

The error $\delta\alpha'$ in the star's right ascension produces the error $\frac{15\,\delta\alpha'}{\lambda}$ when the star is in the same parallel as the moon.

From this discussion it follows that, by arranging the observations *symmetrically* with respect to the meridian, the mean result will be liable to no sensible errors which do not equally affect meridian observations. But for the large culmination error in the case of the moon (Art. 236), which equally affects extra-meridian observations, the latter would have a great advantage by diminishing the effect of accidental errors. But the probable error of the mean of two observations equidistant from the meridian, seven threads being employed, will be, by (422),

$$\epsilon_1 = \sqrt{\left[(0.091)^2 + \frac{(0.104)^2}{14} + \frac{(0.06)^2}{2}\right]} = 0^s.10$$

and that of a single meridian observation, *even where only one star is compared with the moon*, is, by the same formula, $= 0^s.11$. When we take into account the extreme simplicity of the computation, the method of moon culminations must evidently be preferred; and that of extra-meridian observations will be resorted to only in the case already referred to (Art. 239), where the traveller may wish to determine his position in the shortest possible time and without waiting to adjust his instrument accurately in the meridian.

EXAMPLE.—At the U. S. Naval Academy, 1857 May 9, I observed the following transits of the moon's second limb and of *σ Scorpii*, at an approximate azimuth of 10° East, with an ERTEL universal instrument of 15 inches focal length:

	Chronometer.	Level.	Collim.	
☽ II Limb.	$T = 16^h\ 11^m\ 30^s.17$	$b = +\ 2''.2$	$c = 0.0$	Vertical circle
σ Scorpii	$T' = 16\ \ 27\ \ 49.83$	$b' = +\ 2\ .2$		*left.*

These times are the means of three threads. The chronometer correction, found by transits of stars in the meridian, was $-\ 55^m\ 9^s.16$ at 13^h sidereal time, and its hourly rate $-\ 0^s.32$. The assumed latitude and longitude were

$$\varphi = 38°\ 58'\ 53''.5 \qquad L' = 5^h\ 5^m\ 55^s$$

The star's place was

$$\alpha' = 16^h\ 12^m\ 31^s.90 \qquad \delta' = -\ 25°\ 14'\ 58''.5$$

We first find the sidereal times of the observations of the moon and star respectively, and the Greenwich mean time of the observation of the moon: we have

$$
\begin{array}{rlrrr}
\Delta T & = & - & 55^m & 9^s.89 \\
T + \Delta T & = & 15^h & 16^m & 20^s.28 \\
L' & = & 5 & 5 & 55. \\
\hline
\text{Gr. sidereal time} & = & 20 & 22 & 15.28 \\
\text{Sid. time Gr. moon} & = & 3 & 8 & 58.91 \\
\hline
\text{Sidereal interval} & = & 17 & 13 & 16.37 \\
\text{Red. to mean time} & = & - & 2 & 49.28 \\
\hline
\text{Gr. mean time} & = & \text{May } 9, 17^h & 10^m & 27^s.09
\end{array}
\qquad
\begin{array}{rl}
\Delta T' & = - \; 55^m \;\; 9^s.97 \\
T' + \Delta T' & = 15^h \; 32^m \; 39^s.89
\end{array}
$$

Hence from the Ephemeris we find

$$
\begin{array}{rl}
\alpha & = 15^h \; 54^m \; 45^s.32 \\
\Delta\alpha & = 2^s.1135 \\
S & = 14' \; 47''.2
\end{array}
\qquad
\begin{array}{rl}
\delta & = - \; 24^\circ \; 42' \; 54''.4 \\
\Delta\delta & = - \; 7''.619 \\
\pi & = \;\; 54' \;\; 9''.2
\end{array}
$$

By (426) we find

$$
\begin{array}{rl}
A & = - \; 9^\circ \; 40' \; 51''.0 \\
\log \sin q & = n9.1581 \\
\zeta & = 64^\circ \; 19'.5 \\
\pi \sin \zeta & = + \;\; 48.8 \\
\text{Refraction} & = - \;\;\; 2.1 \\
\zeta_1 & = 65 \;\;\; 6.2
\end{array}
\qquad
\begin{array}{rl}
A' & = - \; 9^\circ \; 57' \; 14''.8 \\
\log \sin q' & = n9.1719 \\
\zeta' & = 64^\circ \; 54'.1 \\
 & \\
\text{Refraction} & = - \;\;\; 2.1 \\
\zeta_1' & = 64 \;\; 52.0
\end{array}
$$

For the latitude φ we find, from Table III.,

$$
\log \rho = 9.9994 \qquad\qquad \varphi - \varphi' = 11' \; 15''
$$

and then, by (427), we find

$$
\begin{array}{rlr|rlr}
A & = - \; 9^\circ \; 40' \; 51''.0 & & A' & = - \; 9^\circ \; 57' \; 14''.8 & \\
-\dfrac{S}{\sin \zeta} & = - & 16 \;\; 24.4 & & & \\
\dfrac{\rho\pi(\varphi - \varphi')\sin 1'' \sin A'}{\sin \zeta} & = - & 2.0 & & & \\
-\dfrac{c}{\sin \zeta_1} & = & 0.0 & -\dfrac{c}{\sin \zeta_1'} & = & 0.0 \\
-\dfrac{b}{\tan \zeta_1} & = - & 1.0 & -\dfrac{b'}{\tan \zeta_1'} & = - & 1.0 \\
\hline
A_1 & = - \; 9 \;\; 57 & 18.4 & A_1' & = - \; 9 \;\; 57 & 15.8
\end{array}
$$

whence

$$x = -2''.6$$

By (429), (430), and (431), we find

$$\log \lambda = 9.72175 \qquad \log \beta = n9.10256 \qquad a = 0.5054$$

$$\Delta L = \frac{-2.6}{0.5054} = -5^s.14$$

If we wish to see the effect of all the sources of error in this example, we find, by (433),

$$\begin{aligned} 0.5054\,\Delta L = & -2''.6 - 14.96\,\delta\alpha - 0.16\,\delta\delta + 14.45\,\delta T - 14.82\,\delta T' - 0.36\,\delta\Delta T \\ & + 14.82\,\delta\alpha' + 0.16\,\delta\delta' + 1.11\,\delta S - 0.001\,\delta\pi + 0.002\,\delta\varphi \end{aligned}$$

The proper combination of observations is supposed to eliminate, or at least reduce to a minimum, all the errors except that of the moon's right ascension as given in the Ephemeris. In practice, therefore, it will be necessary to retain the term involving $\delta\alpha$. Thus, in the present case we take only

$$0.5054\,\Delta L = -2''.6 - 14.96\,\delta\alpha$$

A second observation on the same day at an azimuth 10° west gave

$$0.5458\,\Delta L = -5''.7 - 14.92\,\delta\alpha$$

The elimination of the errors of declination requires that we take the arithmetical mean of these equations; whence we have, finally,

$$\Delta L = -7^s.89 - 28.43\,\delta\alpha$$

SIXTH METHOD.—BY ALTITUDES OF THE MOON.

242. The hour angle (t) of the moon may be computed from an observed altitude, the latitude and declination being known, and hence with the local sidereal time of the observation ($= \Theta$) the moon's right ascension by the equation $\alpha = \Theta - t$, with which the Greenwich time can be found, as in Art. 234, and, consequently, also the longitude.

The hour angle is most accurately found from an altitude when the observed body is on the prime vertical, and more accurately in low latitudes than in high ones (Art. 149). This method, therefore, is especially suited to low latitudes.

The method may be considered under two forms:—(A) that in which the moon's absolute altitude is directly observed and

employed in the computation of the hour angle; and (B) that in which the moon's altitude is compared differentially with that of a neighboring star,—*i.e.* when the moon and a star are observed either at the *same* altitude, or at altitudes which differ only by a quantity which can be measured with a micrometer.

243. (A.) *By the moon's absolute altitude.*—This method being practised only with portable instruments, it would be quite superfluous to employ the rigorous processes of correcting for the parallax, which require the azimuth of the moon to be given. The process of Art. 97 will, therefore, be employed in this case with advantage, by which the observed zenith distance is reduced not to the centre of the earth, but to the point of the earth's axis which lies in the vertical line of the observer, and which we briefly designate as *the point O.* Let

ζ'' = the observed zenith distance, or complement of the observed altitude, of the moon's limb,
Θ = the local sidereal time,
L' = the assumed longitude,
ΔL = the required correction of L',
L = the true longitude $= L' + \Delta L$.

Find the Greenwich sidereal time $\Theta + L'$, and convert it into mean time, for which take from the Ephemeris the quantities

δ = the moon's declination,
π = " eq. hor. parallax,
S = " semidiameter.

Let S' be the apparent semidiameter obtained by adding to S the augmentation computed by (251) or taken from Table XII. Let r be the refraction for the apparent zenith distance ζ''; and put

$$\zeta' = \zeta'' + r \pm S' \tag{434}$$

Let π_1 be the corrected parallax for the point O, found by (127), or by adding to π the correction of Table XIII. (which in the present application will never be in error 0″.1); and put

$$\left.\begin{aligned}\delta_1 &= \delta + e^2\pi_1 \sin\varphi \cos\delta \\ \zeta_1 &= \zeta' - \pi_1 \sin\zeta'\end{aligned}\right\} \tag{435}$$

in which $\log e^2 = 7.8244$.

The hour angle (which is the same for the point O as for the centre of the earth) is then found by (267), *i.e.*

$$\sin \tfrac{1}{2} t = \sqrt{\left(\frac{\sin \frac{1}{2} [\zeta_1 + (\varphi - \delta_1)] \sin \frac{1}{2} [\zeta_1 - (\varphi - \delta_1)]}{\cos \varphi \cos \delta_1} \right)} \tag{436}$$

after which the moon's right ascension is found by the formula

$$\alpha = \Theta - t \tag{437}$$

and hence the Greenwich time and the longitude as above stated. But since we have taken δ for an approximate Greenwich time depending on the assumed longitude, the first computation of t will not be quite correct; a second one with a corrected value of δ will give a nearer approximation; and thus by successive approximations the true value of t and of the longitude will at last be found.

But instead of these successive approximations we may obtain at once the correction of the assumed longitude, as follows. We have taken δ for the Greenwich time $\Theta + L'$, when we should have taken it for the time $\Theta + L' + \Delta L$. Hence, putting

$\beta =$ the increase of δ in a unit of time,

it follows that δ requires the correction $\beta \Delta L$; and therefore, by (51), the correction of the computed hour angle will be

$$\frac{\beta \Delta L}{\cos \delta \tan q}$$

in which q is the parallactic angle. Since $\alpha = \Theta - t$, the computed right ascension requires the correction (in seconds of time)

$$-\frac{\beta \Delta L}{15 \cos \delta \tan q}$$

Therefore, if we put

$\lambda =$ the increase of α in a unit of time,

the computed Greenwich time and, consequently, also the longitude derived from it requires the correction

$$-\frac{\beta \Delta L}{15 \lambda \cos \delta \tan q}$$

Hence, denoting the longitude computed from the right ascension $\alpha = \Theta - t$ by L'', we have

$$\text{True longitude} = L' + \Delta L = L'' - \frac{\beta \Delta L}{15\,\lambda \cos\delta \tan q}$$

whence

$$\Delta L = \frac{L'' - L'}{1 + \dfrac{\beta}{15\,\lambda}\sec\delta \cot q}$$

If we denote the denominator of this expression by $1 + a$, we shall have, by (18),

$$a = \frac{\beta}{15\,\lambda}\left(\frac{\tan\varphi}{\sin t} - \frac{\tan\delta}{\tan t}\right) \tag{438}$$

and then

$$\Delta L = \frac{L'' - L'}{1 + a} \qquad L = L' + \Delta L \tag{439}$$

EXAMPLE.—At the U. S. Naval Academy, in latitude $\varphi = 38°\ 58'\ 53''$ and assumed longitude $L' = 5^h\ 6^m\ 0^s$, I observed the double altitude of the moon's upper limb with a sextant and artificial horizon as below:

1849 May 2.—Moon east of the meridian.

Chronometer		10^h	14^m	$21^s.6$
Fast		4	41	0 .0
Local mean time	=	5	33	21 .6
Assumed L'	=	5	6	0 .
Approx. Gr. time	=	10	39	21 .6

(For which time we take π, S, and δ from the Nautical Almanac)

δ	= +	3°	47′	47″ 6
$e^2 \pi_1 \sin\phi \cos\delta$	=		+	14 .1
δ_1	= +	3	48	1 .7

Mean of 6 obs. 2 $\overline{☽}$	=	64°	40′	0″	
Index corr. of sextant	=	—	14	57	
	2)	64	25	3	
App. alt. $\overline{☽}$	=	32	12	31	.5
ζ''	=	57	47	28	.5
Barom. $30^{in}.45$; Att. Therm. 63° F.; Ext. " 65° F. — r	=	+	1	30	.9
$S = 15'\ 16''.4$; ΔS (Tab. XII.) $= +\ 8\ .1$ — S'	=	+	15	24	.5
ζ'	=	58	4	23	.9
$\pi = 56'\ 3''.1$; $\Delta\pi$ (Tab. XIII.) $= +\ 4\ .4$; $\pi_1 = 56\ 7\ .5$ — $\pi_1 \sin\zeta'$	=		47	38	.1
ζ_1	=	57	16	45	.8

With these values of δ_1, ζ_1, and $\varphi = 38°\ 58'\ 53''$, we find, by (436),

$$t = -\ 3^h\ 19^m\ 53^s.64$$

The sidereal time at Greenwich mean noon, 1849 May 2, was $2^h\ 41^m\ 7^s.98$; whence

$$\Theta = \ 8^h\ 16^m\ 14^s.61$$
$$\alpha = 11\ \ 36\ \ \ 8\ .25$$

Corresponding to this right ascension we find by the hourly Ephemeris the Greenwich mean time, and hence the longitude L'', as follows:

Greenwich mean time	=	10^h	39^m	$48^s.7$
Local " "	=	5	33	21 .6
L''	=	5	6	27 .1
$L'' - L'$	=		+	$27^s.1$

By the hourly Ephemeris we also have for the Greenwich time $10^h\ 39^m\ 48^s.7$,

$$\text{Increase of } \alpha \text{ in } 1^m = \lambda = +\ 2^s.014$$
$$\text{"} \quad \delta \text{ in } 1^m = \beta = +\ 10''.01$$

and hence, by (438) and (439),

$$a = -\ 0.3317 \qquad \Delta L = +\ 40^s.6$$
$$L = L' + \Delta L = 5^h\ 6^m\ 40^s.6$$

244. The result thus obtained involves the errors of the tabular right ascension and declination and the instrumental error. The tabular errors are removed by means of observations of the same data made at some of the principal observatories, as in the case of moon culminations. The instrumental error will be nearly eliminated by determining the local time from a star at the same altitude and as nearly as possible the same declination; for the instrumental error will then produce the same error in both Θ and t, and, therefore, will be eliminated from their difference $\Theta - t = \alpha$. The error in the longitude will then be no greater than the error in Θ. But to give complete effect to this mode of eliminating the error, an instrument, such as the zenith telescope, should be employed, which is capable of indicating the same altitude with great certainty and does not involve the errors of graduation of divided circles. A very different method of observation and computation must then be resorted to, which I proceed to consider.

245. (B.) *By equal altitudes of the moon and a star, observed with the zenith telescope.*—The reticule of this instrument should for these observations be provided with a system of fixed horizontal threads: nevertheless, we may dispense with them, and employ only the single movable micrometer thread, by setting it successively at convenient intervals.

Having selected a well determined star as nearly as possible in the moon's path and differing but little in right ascension, a preliminary computation of the approximate time when each body will arrive at some assumed altitude (not less than 10°) must be made, as well as of their approximate azimuths, in order to point the instrument properly. The instrument being pointed for the first object, the level is clamped so that the bubble plays near the middle of the tube, and is then not to be moved between the observation of the moon and the star. After the object enters the field, and before it reaches the first thread, it may be necessary to move the instrument in azimuth in order that the transits over the horizontal threads may all be observed without moving the instrument *during* these transits. The times by chronometer of the several transits are then noted, and the level is read off. The instrument is then set upon the azimuth of the second object, the observation of which is made in the same manner, and then the level is again read off. This completes one observation. The instrument may then be set for another assumed altitude, and a second observation may be taken in the same manner.* Each observation is then to be separately reduced as follows: Let

i, i', i'', &c. = the distances in arc of the several threads from their mean,

m, m' = the mean of the values of i for the observed threads, in the case of the moon and star respectively,

l, l' = the level readings, in arc, for the moon and star,

Θ, Θ' = the mean of the sidereal times of the observed transits of the moon and star;

then the excess of the observed zenith distance of the moon's limb at the time Θ above that of the star at the time Θ' is†

$$m - m' + l - l'$$

the quantities m and l being supposed to increase with increasing zenith distance.

* The same method of observation may be followed with the ordinary universal instrument, but, as the level is generally much smaller than that of the zenith telescope, the same degree of accuracy will not be possible.

† When the micrometer is set successively upon assumed readings, m and m' will be the means of these readings, converted into arc, with the known value of the screw.

Also, let

$$\begin{aligned}
\alpha, \delta, t, \zeta, A, q &= \text{the R. A., decl., hour angle, geocentric zenith distance, azimuth, and parallactic angle of the moon's centre at the time } \Theta;\\
\alpha', \delta', t', \zeta', A', q' &= \text{the same for the star at the time } \Theta';\\
\pi, S &= \text{the moon's equatorial hor. parallax and semidiameter;}\\
\lambda &= \text{the increase of } \alpha \text{ in } 1^s \text{ of sid. time;}\\
\beta &= \quad \text{"} \quad \delta \quad \text{"} \quad \text{"} \quad \text{"}\\
\varphi &= \text{the latitude;}\\
L' &= \text{the assumed longitude;}\\
\Delta L &= \text{the required correction of } L';
\end{aligned}$$

The quantities α, δ, π, and S are to be taken from the Ephemeris for the Greenwich sidereal time $\Theta + L'$ (converted into mean time); α and δ being interpolated with second differences by the hourly Ephemeris. Then the required correction of the longitude will be found by comparing the computed value of ζ with the observed value. For this purpose we first compute ζ and ζ' with the greatest precision. and also A and q approximately. If the differential formula of the next article is also to be computed, A' and q' will also be required. The most convenient formulæ will be—

For the moon. For the star.

$$t = \Theta - \alpha \qquad\qquad t' = \Theta' - \alpha'$$

$$\left.\begin{aligned}\tan M &= \tan\delta \sec t\\ \cos\zeta &= \frac{\sin\delta\cos(\varphi - M)}{\sin M}\end{aligned}\right\} \text{with six decimals;} \left\{\begin{aligned}\tan M' &= \tan\delta' \sec t'\\ \cos\zeta' &= \frac{\sin\delta'\cos(\varphi - M')}{\sin M'}\end{aligned}\right. \tag{440}$$

$$\left.\begin{aligned}\cos A &= \tan(\varphi - M)\cot\zeta\\ \tan N &= \cot\varphi\cos t\\ \tan q &= \frac{\tan t \sin N}{\cos(\delta + N)}\end{aligned}\right\} \text{with four decimals;} \left\{\begin{aligned}\cos A' &= \tan(\varphi - M')\cot\zeta'\\ \tan N' &= \cot\varphi\cos t'\\ \tan q' &= \frac{\tan t' \sin N'}{\cos(\delta' + N')}\end{aligned}\right.$$

The zenith distance ζ thus computed will not strictly correspond to the time Θ unless the assumed longitude is correct. Let its true value be $\zeta + d\zeta$. Also put

$$\begin{aligned}
\zeta_1 &= \text{the observed zenith distance of the moon's limb,}\\
\zeta_1' &= \text{the observed zenith distance of the star,}\\
r, r' &= \text{the refraction for } \zeta_1 \text{ and } \zeta_1',
\end{aligned}$$

then

$$\zeta_1' = \zeta' - r'$$
$$\zeta_1 = \zeta_1' + m - m' + l - l'$$

Putting then

$$\left.\begin{array}{l} \zeta'' = \zeta_1 + r = \zeta' + m - m' + l - l' + (r - r') \\ \text{and, by Art. (136),} \\ \gamma = (\varphi - \varphi')\cos A \qquad \sin p = \rho \sin\pi \sin(\zeta'' - \gamma) \\ k = p \mp S \mp \tfrac{1}{2}(p \mp S)\sin p \sin S \end{array}\right\} \quad (441)$$

the $\left\{\begin{matrix}\text{upper}\\\text{lower}\end{matrix}\right\}$ sign being used for the moon's $\left\{\begin{matrix}\text{upper}\\\text{lower}\end{matrix}\right\}$ limb, we have

$$\zeta'' - (\zeta + d\zeta) = k$$

This equation determines $d\zeta$. We have, therefore, only to determine the relation between $d\zeta$ and ΔL. Now, we have taken α and δ for the Greenwich sidereal time $\Theta + L'$, when we should have taken them for the time $\Theta + L' + \Delta L'$: hence

α	requires the correction	$\lambda \Delta L$
δ	" "	$\beta \Delta L$
t	" "	$-\lambda \Delta L$

and then, by (51),

$$d\zeta = -\cos q \,.\, \beta \Delta L - \sin q \cos\delta \,.\, 15\,\lambda \Delta L$$

Hence, putting $x = -d\zeta$, or

$$\left.\begin{array}{lll} & x = \zeta - \zeta'' + k & \\ \text{and} & a = 15\,\lambda \sin q \cos\delta + \beta \cos q & \\ \text{we have} & \Delta L = \dfrac{x}{a} & L = L' + \Delta L \end{array}\right\} \quad (442)$$

The solution of the problem, upon the supposition that all the data are correct, is completely expressed by the equations (440), (441), and (442).

246. The quantity x is in fact produced not only by the error in the assumed longitude, but also by the errors of observation and of the Ephemeris. In order to obtain a general expression

in which the effect of every source of error may be represented, let

$T, T' =$ the chronometer times of observation of the moon and star,

$\Delta T =$ the assumed chronometer correction,

$\delta T, \delta T' =$ the corrections of T and T' for errors of observation,

$\delta \Delta T =$ the correction of ΔT,

$\delta \alpha, \delta \delta, \delta \pi, \delta S =$ the corrections of the elements taken from the Ephemeris,

$\delta \varphi =$ the correction of the assumed latitude.

If, when the corrected values of all the elements are substituted, ζ, ζ', k become $\zeta + d\zeta, \zeta' + d\zeta', k + dk$, instead of the equation $\zeta'' - (\zeta + d\zeta) = k$ we shall have

$$\zeta'' + d\zeta' - (\zeta + d\zeta) = k + dk$$

and hence

$$x = - d\zeta + d\zeta' - dk \tag{443}$$

and we have now to find expressions for $d\zeta$, $d\zeta'$, and dk in terms of the above corrections of the elements.

Taking all the quantities as variables, we have

$$\begin{aligned} d\zeta &= 15 \sin q \cos \delta \; dt - \cos q \; d\delta + \cos A \; d\varphi \\ d\zeta' &= 15 \sin q' \cos \delta' \, dt' - \cos q' \, d\delta' + \cos A' \, d\varphi \end{aligned}$$

Since $t = T + \Delta T - \alpha$, we have

$$dt = dT + d\Delta T - d\alpha$$

where dT and $d\Delta T$ may be exchanged for δT and $\delta \Delta T$, but $d\alpha$ is composed of two parts: 1st, of the actual correction of the Ephemeris; and 2d, of $\lambda(\Delta L + \delta T + \delta \Delta T)$ resulting from our having taken α for the uncorrected time: hence we have

$$dt = \delta T + \delta \Delta T - \delta \alpha - \lambda(\Delta L + \delta T + \delta \Delta T)$$

The correction $d\delta$ is also composed of two parts, so that

$$d\delta = \delta\delta + \beta(\Delta L + \delta T + \delta \Delta T)$$

Further, we have simply $d\delta' = \delta\delta'$, and

$$dt' = \delta T' + \delta \Delta T - \delta \alpha'$$

in which $\delta \Delta T$ at the time T' is assumed to be the same as at the

time T, an error in the rate of chronometer being insensible in the brief interval between the observations of the moon and the star.

Again, we have, from (441),

$$\cos p\, dp = \rho \cos \pi \sin(\zeta'' - \gamma)\, d\pi + \rho \sin \pi \cos(\zeta'' - \gamma)\, d\zeta''$$
$$dk = dp \mp dS$$

or, with sufficient accuracy,

$$dk = \sin \zeta'\, \delta\pi \mp \delta S + \sin \pi \cos \zeta'\, d\zeta'$$

Now, substituting in $d\zeta$ and $d\zeta'$ the values of dt, $d\delta$, &c., and then substituting the values of $d\zeta$ and $d\zeta'$ thus found, in (443), together with the value of dk, we obtain the final equation desired, which may be written as follows:*

$$\left.\begin{aligned} x = a\Delta L &+ f\,.\,\delta\alpha + \cos q\,.\,\delta\delta - (f - a)\,\delta T \\ &- mf'\,.\,\delta\alpha' - m\cos q'\,\delta\delta' + mf'\,.\,\delta T' \\ &\pm \delta S - \sin\zeta'\,\delta\pi - (f - mf' - a)\,\delta\Delta T \\ &- (\cos A - m\cos A')\,\delta\varphi \end{aligned}\right\} \quad (444)$$

where the following abbreviations are employed:

$$f = 15 \sin q \cos \delta \qquad f' = 15 \sin q' \cos \delta'$$
$$a = \lambda f + \beta \cos q \qquad m = 1 - \sin \pi \cos \zeta'$$

Having computed the equation in this form, every term is to be divided by a, and then ΔL will be obtained in terms of x and all the corrections of the elements.

A discussion of this equation, quite similar to that of (433), will readily show that the observations will give the best result when taken near the prime vertical and in low latitudes, and, farther, that the combination of observations equidistant from the meridian, east and west, eliminates almost wholly errors of declination and parallax and of the chronometer correction.

EXAMPLE.†—At Batavia, on the 11th of October, 1853, Mr. DE LANGE, among other observations of the same kind, noted the following times by a sidereal chronometer, when the moon's

---

* The formula (444) is essentially the same as that given by OUDEMANS, *Astronom. Journal*, Vol. IV. p. 164. The method itself is the suggestion of Professor KAISER of the Netherlands.

† Astronomical Journal, Vol. IV. p. 165.

lower limb and 36 *Capricorni* passed the same fixed horizontal threads:

$$T = 0^h\ 38^m\ 8^s.62 \qquad T' = 0^h\ 49^m\ 53^s.77$$

The difference of the zenith distances indicated by the level was

$$l - l' = +\ 2''.0$$

The chronometer correction was $\Delta T = +\ 1^m\ 3^s.32$, and the rate in the interval $T' - T$ was insensible.

The assumed latitude was $\varphi = -\ 6°\ 9'\ 57''.0$
" longitude " $L' = -\ 7^h\ 7^m\ 37^s.0$

We have

$$\Theta = 0^h\ 39^m\ 11^s.94 \qquad \Theta' = 0^h\ 50^m\ 57^s.09$$

For the Greenwich sid. time $\dot{\Theta} + L' = 17^h\ 31^m\ 34^s.94$, or mean time $4^h\ 10^m\ 57^s.00$, we find, from the Nautical Almanac,

$$\begin{aligned} \alpha &= 21^h\ 12^m\ 5^s.45 & \lambda &= +\ 0^s.0387 \\ \delta &= -\ 20°\ 55'\ 8''.9 & \beta &= +\ 0''.1440 \\ \pi &= 57'\ 51''.4 & \alpha' &= 21^h\ 20^m\ 22^s.45 \\ S &= 15'\ 47''.8 & \delta' &= -\ 22°\ 26'\ 30''.5 \end{aligned}$$

The computation by (440) gives

$$\begin{aligned} \zeta &= 52°\ 11'\ 49''.44 & \zeta' &= 53°\ 13'\ 57''.30 \\ A &= 68°\ 14'.4 & A' &= 66°\ 30'.6 \\ q &= 81°\ 18'.9 & q' &= 80°\ 35'.2 \end{aligned}$$

From Table III. we find

$$\varphi - \varphi' = -\ 2'\ 27'' \qquad \log \rho = 9.999983$$

Since the same fixed threads were used for both moon and star, we have $m = m'$, and hence also sensibly $r = r'$; therefore, by (441), we find

$$\begin{aligned} \zeta'' &= 53°\ 13'\ 59''.30 & \gamma &= -\ 54''.5 & p &= 46'\ 21''.25 \\ \zeta - \zeta'' &= -\ 62'\ 9''.86 & k &= +\ 62'\ 9''.17 \end{aligned}$$

Hence, by (442),

$$x = -\ 0''.69 \qquad a = +\ 0.5575 \qquad \Delta L = -\ 1^s.24$$

The longitude by this observation, if the Ephemeris is correct, is therefore

$$L = L' + \Delta L = -\ 7^h\ 7^m\ 38^s.24$$

If we compute all the terms of (444), we shall find

$$\begin{aligned}\Delta L = -1^s.24 - 24.84\,\delta\alpha - 0.27\,\delta\delta + 23.84\,\delta T - 24.24\,\delta T' - 0.44\,\delta\Delta T \\ + 24.28\,\delta\alpha' + 0.29\,\delta\delta' + 1.79\,\delta S + 1.44\,\delta\pi - 0.04\,\delta\varphi\end{aligned}$$

This shows clearly the effect of each source of error; but in practice it will usually be sufficient to compute only the coefficients of $\delta\alpha$ and $\delta\delta$. In the present example, therefore, we should take

$$\Delta L = -1^s.24 - 24.84\,\delta\alpha - 0.27\,\delta\delta$$

which will finally be fully determined when $\delta\alpha$ and $\delta\delta$ have been found from nearly corresponding observations at Greenwich or elsewhere.

SEVENTH METHOD.—BY LUNAR DISTANCES.

247. The distance of the moon from a star may be employed in the same manner as the right ascension was employed in Arts. 229, &c., to determine the Greenwich time, and hence the longitude. If the star lies directly in the moon's path, the change of distance will be even more rapid than the change of right ascension; and therefore if the distance could be measured with the same degree of accuracy as the right ascension, it would give a more accurate determination of the Greenwich time The distance, however, is observed with a sextant, or other reflecting instrument (see Vol. II.), which being usually held in the hand is necessarily of small dimensions and relatively inferior accuracy. Nevertheless, this method is of the greatest importance to the travelling astronomer, and especially to the navigator, as the observation is not only extremely simple and requires no preparation, but may be practised at almost any time when the moon is visible.

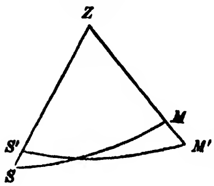
The Ephemerides, therefore, give the true distance of the centre of the moon from the sun, from the brightest planets, and from nine bright fixed stars, selected in the path of the moon, for every third hour of mean Greenwich time. The planets employed are Saturn, Jupiter, Mars, and Venus. The nine stars, known as lunar-distance stars, are α Arietis, α Tauri (*Aldebaran*), β Geminorum (*Pollux*), α Leonis (*Regulus*), α Virginis (*Spica*), α Scorpii (*Antares*), α Aquilæ (*Altair*), α Piscis Australis (*Fomalhaut*), and α Pegasi (*Markab*).

The distance *observed* is that of the moon's bright limb from a

star, from the estimated centre of a planet, or from the limb of the sun. The *apparent* distance of the moon's centre from a star or planet is found by adding or subtracting the moon's apparent (augmented) semidiameter, according as the bright limb is nearer to or farther from the star or planet than the centre. The observed distance of the sun and moon is always that of the nearest limbs, and therefore the apparent distance of the centres is found by adding both semidiameters.*

The apparent distance thus found differs from the *true* (geocentric) distance, in consequence of the parallax and refraction which affect the altitudes of the objects, and consequently also the distance. The true distance is therefore to be obtained by computation, the general principle of which may be exhibited in a simple manner as follows. Let *Z*, Fig. 29, be the zenith of the observer, *M′* and *S′* the observed places of the moon and star, *MM′* the parallax and refraction of the moon, *SS′* the refraction of the star, so that *M* and *S* are the geocentric places. The apparent altitudes of the objects may either be measured at the same time as the distance, or, the local time being known, they may be computed (Art. 14). The apparent zenith distances, and, consequently, also the true zenith distances, are therefore known. In the triangle *ZM′S′* there are known the three sides, *M′S′* the apparent distance of the objects, *ZM′* the apparent zenith distance of the moon, and *ZS′* the apparent zenith distance of the star; from which the angle *Z* is computed. Then, in the triangle *ZMS* there are known the sides, *ZM* the moon's true zenith distance, and *ZS* the star's true zenith distance, and the angle *Z*; from which the required true distance *MS* is computed.

Fig. 29.



In this elementary explanation the parallax and refraction of the moon are supposed to act in the same vertical circle *ZM*, whereas parallax acts in a circle drawn through the moon and the geocentric zenith (Art. 81), while refraction acts in the vertical circle drawn through the astronomical zenith. Again, when the moon, or the sun, is observed at an altitude less than 50°, it is necessary to take into account the distortion of the disc produced

* We may also observe the distance from the limb of a planet, provided the sextant telescope is of sufficient power to give the planet a well-defined disc; and the planet's semidiameter is then also to be added or subtracted.

by refraction if we wish to compute the true distance to the nearest second of arc (Art. 133). These features, which add very materially to the labor of computation, cannot be overlooked in any complete discussion of the problem.

Simple as the problem appears when stated generally, the strict computation of it is by no means brief; and its importance and the frequency of its application at sea, where long computations are not in favor, have led to numerous attempts to abridge it. In most instances the abbreviations have been made at the expense of precision; but in the methods given below the error in the computation will always be much less than the probable error of the best observation with reflecting instruments: so that these methods are entitled to be considered as practically perfect.

With the single exception of that proposed by BESSEL,* all the solutions depend upon the two triangles of Fig. 29, and may be divided into two classes, *rigorous* and *approximative.* In the rigorous methods the true distance is directly deduced by the rigorous formulæ of Spherical Trigonometry; but in the approximative methods the difference between the apparent and the true distance is deduced either by successive approximations or from a development in series of which the smaller terms are neglected. Practically, the latter may be quite as correct as the former, and, indeed, with the same amount of labor, more correct, since they require the use of less extended tables of logarithms. I propose to give two methods, one from each of these classes.

A.—*The Rigorous Method.*

248. For brevity, I shall call the body from which the moon's distance is observed *the sun*, for our formulæ will be the same for a planet, and for a fixed star they will require no other change than making the parallax and semidiameter of the star zero.

---

* *Astron. Nach.* Vol. X. No. 218, and *Astron. Untersuchungen*, Vol. II. BESSEL'S method requires a different form of lunar Ephemeris from that adopted in our Nautical Almanacs. But even with the Ephemeris arranged as he proposes, the computation is not so brief as the approximative method here given, and its superiority in respect of precision is so slight as to give it no important practical advantage. It is, however, the only theoretically *exact* solution that has been given, and might still come into use if the *measurement* of the distance could be rendered much more precise than is now possible with instruments of reflection.

Let us suppose that at the given local mean time T the observation (or, in the case of the altitudes, computation) has given

d'' = the apparent distance of the limbs of the moon and sun,
h' = the apparent altitude of the moon's centre,
H' = the apparent altitude of the sun's centre,

and that in order to compute the refraction accurately the barometer and thermometer have also been observed. For the Greenwich time corresponding to T, which will be found with sufficient accuracy for the purpose by employing the supposed longitude, take from the Ephemeris

s = the moon's semidiameter,
S = the sun's "

then, putting

d' = the apparent distance of the centres,
s' = the moon's augmented semidiameter,
= s + correction of Table XII.

we have

$$d' = d'' \pm s' \pm S$$

upper signs for nearest (inner) limbs, lower signs for farthest (outer) limbs.

But if the altitude of either body is less than 50°, we must take into account the elliptical figure of the disc produced by refraction. For this purpose we must employ, instead of s' and S, those semidiameters which lie in the direction of the lunar distance. Putting

$$q = ZM'S', \qquad Q = ZS'M' \quad \text{(Fig. 29)}$$

Δs, ΔS = the contraction of the *vertical* semidiameters of the moon and sun for the altitudes h' and H',

the required inclined semidiameters will be (Art. 133)

$$s' - \Delta s \cos^2 q \qquad \text{and} \qquad S - \Delta S \cos^2 Q$$

The angles q and Q will be found from the three sides of the triangle $ZM'S'$, taking for d' its approximate value $d'' \pm s' \pm S$ (which is sufficiently exact for this purpose, as great precision in q and Q is not required), and for the other sides $90° - h'$ and $90° - H'$. If we put

$$m = \tfrac{1}{2}(h' + H' + d')$$

we shall have

$$\sin^2 \tfrac{1}{2} q = \frac{\cos m \sin (m - H')}{\sin d' \cos h'} \qquad \sin^2 \tfrac{1}{2} Q = \frac{\cos m \sin (m - h')}{\sin d' \cos H'} \tag{445}$$

and then the apparent distance by the formula

$$d' = d'' \pm (s' - \Delta s \cos^2 q) \pm (S - \Delta S \cos^2 Q) \tag{446}$$

We are now to reduce the distance to the centre of the earth. We shall first reduce it to that point of the earth's axis which lies in the vertical line of the observer. Designating this point as *the point* O, Art. 97, let

$d_1, h_1, H_1 =$ the distance and altitudes reduced to the point O,
$r, R =$ the refraction for the altitudes h' and H',
$\pi, P =$ the equatorial hor. parallax of the moon and sun.

The moon's parallax for the point O will be found rigorously by (127), but with even more than sufficient precision for the present problem by adding to π the correction given by Table XIII. Denoting this correction by $\Delta\pi$, we have

$$\pi_1 = \pi + \Delta\pi$$

$$h_1 = h' - r + \pi_1 \cos (h' - r) \qquad H_1 = H' - R + P \cos (H' - R) \tag{447}$$

The parallax P is in all cases so small that its reduction to the point O is insignificant.

If, then, in Fig. 29, M and S represent the moon's and sun's places reduced to the point O, and we put

$$Z = \text{the angle at the zenith, } MZS,$$

we shall have given in the triangle $M'ZS'$ the three sides d', $90° - h'$, $90° - H'$, whence

$$\cos^2 \tfrac{1}{2} Z = \frac{\cos \tfrac{1}{2} (h' + H' + d') \cos \tfrac{1}{2} (h' + H' - d')}{\cos h' \cos H'}$$

and, then, in the triangle MZS we shall have given the angle Z with the sides $90° - h_1$ and $90° - H_1$, whence the side $MS = d_1$ will be found by the formula [Sph. Trig. (17)],

$$\sin^2 \tfrac{1}{2} d_1 = \cos^2 \tfrac{1}{2} (h_1 + H_1) - \cos h_1 \cos H_1 \cos^2 \tfrac{1}{2} Z$$

To simplify the computation, put

$$m = \tfrac{1}{2}(h' + H' + d')$$

then the last formula, after substituting the value of Z, becomes,

$$\sin^2 \tfrac{1}{2} d_1 = \cos^2 \tfrac{1}{2}(h_1 + H_1) - \frac{\cos h_1 \cos H_1}{\cos h' \cos H'} \cos m \cos (m - d')$$

Let the auxiliary angle M be determined by the equation

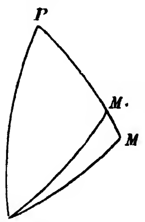
$$\sin^2 M = \frac{\cos h_1 \cos H_1}{\cos h' \cos H'} \cdot \frac{\cos m \cos (m - d')}{\cos^2 \tfrac{1}{2}(h_1 + H_1)} \tag{448}$$

then we have*

$$\sin \tfrac{1}{2} d_1 = \cos \tfrac{1}{2}(h_1 + H_1) \cos M \tag{449}$$

Finally, to reduce the distance from the point O to the centre of the earth, let P (Fig. 30) be the north pole of the heavens, M_1 the moon's place as seen from the point O, M the moon's geocentric place, S the sun's place (which is sensibly the same for either point). The point O being in the axis of the celestial sphere, the points M_1 and M evidently lie in the same declination circle PM_1M. Hence, putting

Fig. 30.



d = the geocentric distance of the moon and sun = SM,
$d_1 = SM_1$,
δ = the moon's geocentric declination = $90° - PM$,
δ_1 = the declination reduced to the point $O = 90° - PM_1$,
Δ = the sun's declination = $90° - PS$,

we have, in the triangles PMS and M_1MS,

$$\cos PMS = \frac{\cos d_1 - \cos(\delta_1 - \delta) \cos d}{\sin(\delta_1 - \delta) \sin d} = \frac{\sin \Delta - \sin \delta \cos d}{\cos \delta \sin d}$$

We may put $\cos(\delta_1 - \delta) = 1$, and, therefore,

$$\cos d_1 - \cos d = \frac{\sin(\delta_1 - \delta)}{\cos \delta}(\sin \Delta - \sin \delta \cos d)$$

---

* This transformation of the formulæ is due to BORDA, *Description et usage du cercle de réflexion.*

and since $d - d_1$ is very small, we may put $\cos d_1 - \cos d = \sin (d - d_1) \sin d_1$, and hence, very nearly,

$$d - d_1 = \frac{\delta_1 - \delta}{\cos \delta}\left(\frac{\sin \varDelta}{\sin d_1} - \frac{\sin \delta}{\tan d_1}\right)$$

Substituting the value of $\delta_1 - \delta$ from (122),

$$d - d_1 = A \pi \sin \varphi \left(\frac{\sin \varDelta}{\sin d_1} - \frac{\sin \delta}{\tan d_1}\right) \tag{450}$$

in which φ is the latitude of the observer, and log A may be taken from the small table given on p. 116. The correction given by this equation being added to d_1, we have the geocentric distance d according to the observation.

To find the longitude, we have now only to find the Greenwich mean time T_0 corresponding to d, by Art. 66, and then

$$L = T_0 - T \tag{451}$$

EXAMPLE.—In latitude 35° N. and assumed longitude 150° W., 1856 March 9, at the local mean time $T = 5^h\, 14^m\, 6^s$, the observed altitudes of the lower limbs and the observed distance of the nearest limbs of the moon and sun were as follows, corrected for error of the sextant:

$$h'' = 52° \, 34' \, 0'' \qquad H'' = 8° \, 56' \, 23'' \qquad d'' = 44° \, 36' \, 58''.6$$

The height of the barometer was 29.5 inches, Attached therm. 60° F., External therm. 58° F.

I shall put down nearly all the figures of the computation, in order to compare it with that of the approximative method to be given in the next article.

1st. The approximate Greenwich mean time is $5^h\, 14^m\, 6^s + 10^h = 15^h\, 14^m\, 6^s$, with which we take from the American Ephemeris

$$s = 16' \, 23''.1 \qquad \pi = 60' \, 1''.9 \qquad \delta = +\, 14° \, 19'$$
$$S = 16' \;\; 8''.0 \qquad P = \;\; 8''.6 \qquad \varDelta = -\;\; 4° \;\; 3'$$

2d. To find the apparent semidiameters, we first take the augmentation of the moon's semidiameter from Table XII., $= 14''.0$, and hence find

$$s' = 16' \, 37''.1$$

Then to compute the contraction produced by refraction we find from the refraction table, for the given observed altitudes, the contractions of the *vertical* semidiameters (Art. 132),

$$\Delta s = 0''.4 \qquad \Delta S = 9''.6$$

With the approximate altitudes and distance of the centres we then proceed by (445), as follows:

$d' = 45° \ 10'$	log cosec d' 0.1493	log cosec d'	0.1493
$h' = 52 \ 51$	log sec h' 0.2190		
$H' = 9 \ 12$		log sec H'	0.0056
$m = 53 \ 37$	log cos m 9.7782	log cos m	9.7732
$m - H' = 44 \ 25$	log sin $(m - H')$ 9.8450		
$m - h' = 0 \ 46$		log sin $(m - h')$	8.1265
	9.9865		8.0546
	log sin $\frac{1}{2} q$ 9.9933	log sin $\frac{1}{2} Q$	9.0273
	$q =$ 159° 56′	$Q =$	12° 14′
	log cos² q 9.9456	log cos² Q	9.9800
	log Δs 9.6021	log ΔS	0.9823
	9.5477		0.9623
	$\Delta s \cos^2 q =$ 0.″4	$\Delta S \cos^2 Q =$	9″.2

Hence we have, by (446),

$$\begin{aligned} d'' &= 44° \ 36' \ 58''.6 \\ s' - \Delta s \cos^2 q &= \ 16 \ 36 \ .7 \\ S - \Delta S \cos^2 Q &= \ 15 \ 58 \ .8 \\ d' &= 45 \quad 9 \ 34 \ .1 \end{aligned}$$

3d. To find the apparent and true altitudes of the centres.—The apparent altitudes of the centres will be found by adding the contracted vertical semidiameters to the observed altitudes of the limbs. The apparent altitudes, however, need not be computed with extreme precision, provided that the differences between them and the true altitudes are correct; for it is mainly upon these differences that the difference between the apparent and true distance depends.

The reduction of the moon's horizontal parallax to the point O for the latitude 35° is, by Table XIII., $\Delta\pi = 3''.9$; and hence we have

$$\pi_1 = \pi + \Delta\pi = 60' \ 5''.8$$

and the computation of the altitudes by (447) is as follows:

$h'' =$	$52° 34' 0''$	$H'' =$	$8° 56' 23''$
Vert. semid. $=$	$16\ 37$	Vert. semid. $=$	$15\ 58$
$h' =$	$52\ 50\ 37$	$H' =$	$9\ 12\ 21$
Table II. $r =$	$42\ .7$	$R =$	$5\ 33\ .6$
$h' - r =$	$52\ 49\ 54\ .3$	$H' - R =$	$9\ 6\ 47\ .4$

$\log \pi_1$	3.55700	$\log P$	0.9345
$\log \cos (h' - r)$	9.78115	$\log \cos (H' - R)$	9.9945
	3.33815		0.9290
$\pi_1 \cos (h' - r) =$	$36' 18''.5$	$P \cos (H' - R) =$	$8''.5$
$h_1 =$	$53° 26' 12''.8$	$H_1 =$	$9° 6' 55''.9$

4th. We now find the distance d_1 by (448) and (449), as follows:

$d' =$	$45° 9' 34''.1$				
$h' =$	$52\ 50\ 37$	log sec	0.2189683		
$H' =$	$9\ 12\ 21$	log sec	0.0056300		
$m =$	$53\ 36\ 16\ .1$	log cos	9.7733154		
$m - d' =$	$8\ 26\ 42\ .$	log cos	9.9952654		
$h_1 =$	$53\ 26\ 12\ .8$	log cos	9.7750333		
$H_1 =$	$9\ 6\ 55\ .9$	log cos	9.9944803		
		2)	9.7626927		
			9.8813464		
$\frac{1}{2}(h_1 + H_1) =$	$31\ 16\ 34\ .4$	log cos	9.9318007	....	9.9318007
		log sin M	9.9495457	log cos M	9.6583330
$\frac{1}{2} d_1 =$	$22\ 54\ 9\ .2$			log sin $\frac{1}{2} d_1$	9.5901337
$d_1 =$	$45\ 48\ 18\ .4$				

5th. To find the geocentric distance, we have, by (450), for $\varphi = 35°$,

$\log A$	7.8249	$\delta = + 14° 19'$	
$\log \pi$	3.5565	$\Delta = - 4\ \ 3$	
$\log \sin \varphi$	9.7586		
	1.1400	....	1.1400
$\log \sin \Delta$	$n8.8490$	$\log \sin \delta$	9.3932
$\log \text{cosec}\, d_1$	0.1445	$\log \cot d_1$	9.9878
	$n0.1335$		$n0.5210$
	$-1''.4$		$-3''.3$

$$d - d_1 = -4''.7$$
$$d = 45° 48' 13''.7$$

6th. To find the Greenwich mean time corresponding to d.

and hence the longitude, according to Art. 66, we find an approximate time $(T) + t$ by simple interpolation, and then the required time $T_0 = (T) + t + \Delta t$, taking Δt from Table XX., with the arguments t and ΔQ (= increase of the logarithms in the Ephemeris in 3^h), as follows:

By the American Ephemeris of 1856 for March 9, we have

$(T) = 15^h\ 0^m\ 0^s$	$(d) = 45°\ 40'\ 54''$	$Q = 0.2510$	$\Delta Q = +17$
	$d\ = 45\ \ 48\ \ 13.7$		
$t\ = \ \ 0\ \ 13\ \ \ 4$	$7\ \ 19.7$	$\log\ = 2.6432$	
$\Delta t = \ \ \ \ -\ \ \ 1$		$\log t = 2.8942$	
$T_0 = 15\ \ 13\ \ \ 3$			
$T\ = \ \ 5\ \ 14\ \ \ 6$			
$L\ = \ \ 9\ \ 58\ \ 57$			

B.—*The Approximative Method.*

249. I shall here give my own method (first published in the *Astronomical Journal*, Vol. II.), as it yet appears to me to be the shortest and most simple of the approximative methods *when these are rendered sufficiently accurate by the introduction of all the necessary corrections.* Its value must be decided by the importance attached to a precise result. There are briefer methods to be found in every work on Navigation, which will (and should) be preferred in cases where only a rude approximation to the longitude is required.

As before, let

h', H' = the apparent altitudes of the centres of the moon and sun,
d'' = the observed distance of the limbs,
s, S = their geocentric semidiameters,
π, P = their equatorial horizontal parallaxes,
s' = the moon's semidiameter, augmented by Table XII.,
π_1 = the moon's parallax, augmented by Table XIII.

We shall here also first reduce the distance to the point O of Art. 97. The contractions of the semidiameters produced by refraction will be at first disregarded, and a correction on that account will be subsequently investigated. If then in Fig. 29, p. 394, M' and S' denote the apparent places, M and S the places reduced to the point O, we shall here have

$$d' = d'' \pm s' \pm S = M'S', \qquad d_1 = MS,$$

$$h' = 90° - ZM', \qquad H' = 90° - ZS',$$
$$h_1 = 90° - ZM, \qquad H_1 = 90° - ZS,$$

and the two triangles give

$$\cos Z = \frac{\cos d_1 - \sin h_1 \sin H_1}{\cos h_1 \cos H_1} = \frac{\cos d' - \sin h' \sin H'}{\cos h' \cos H'}$$

from which, if we put

$$m = \frac{\sin h_1 \sin H_1}{\sin h' \sin H'} \qquad n = \frac{\cos h_1 \cos H_1}{\cos h' \cos H'}$$

we derive

$$\cos d' - \cos d_1 = (1 - n) \cos d' + (n - m) \sin h' \sin H' \qquad (a)$$

Put

$$\Delta d = d_1 - d' \qquad \Delta h = h_1 - h' \qquad \Delta H = H' - H_1 \qquad (b)$$

then we have

$$\cos d' - \cos d_1 = 2 \sin \tfrac{1}{2} \Delta d \sin (d' + \tfrac{1}{2} \Delta d) \qquad (c)$$

and

$$n = \frac{\cos (h' + \Delta h)}{\cos h'} \cdot \frac{\cos (H' - \Delta H)}{\cos H'}$$

$$= \left(1 - \frac{2 \sin \tfrac{1}{2} \Delta h \sin (h' + \tfrac{1}{2} \Delta h)}{\cos h'}\right) \times \left(1 + \frac{2 \sin \tfrac{1}{2} \Delta H \sin (H' - \tfrac{1}{2} \Delta H)}{\cos H'}\right)$$

$$1 - n = \frac{2 \sin \tfrac{1}{2} \Delta h \sin (h' + \tfrac{1}{2} \Delta h)}{\cos h'} - \frac{2 \sin \tfrac{1}{2} \Delta H \sin (H' - \tfrac{1}{2} \Delta H)}{\cos H'}$$

$$+ \frac{4 \sin \tfrac{1}{2} \Delta h \sin \tfrac{1}{2} \Delta H \sin (h' + \tfrac{1}{2} \Delta h) \sin (H' - \tfrac{1}{2} \Delta H)}{\cos h' \cos H'} \qquad (d)$$

Also

$$n - m = \frac{\sin h' \cos h_1 \sin H' \cos H_1 - \cos h' \sin h_1 \cos H' \sin H_1}{\sin h' \cos h' \sin H' \cos H'}$$

substituting in which the values

$$\begin{aligned}
2 \sin h' \cos h_1 &= \sin (2 h' + \Delta h) - \sin \Delta h \\
2 \cos h' \sin h_1 &= \sin (2 h' + \Delta h) + \sin \Delta h \\
2 \sin H' \cos H_1 &= \sin (2 H' - \Delta H) + \sin \Delta H \\
2 \cos H' \sin H_1 &= \sin (2 H' - \Delta H) - \sin \Delta H
\end{aligned}$$

we find

$$n - m = \frac{\sin \Delta H \sin (2 h' + \Delta h) - \sin \Delta h \sin (2 H' - \Delta H)}{2 \sin h' \cos h' \sin H' \cos H'} \qquad (e)$$

Substituting (c), (d), and (e) in (a), and at the same time, for brevity, putting

$$A_1 = \frac{2 \sin \frac{1}{2} \Delta h \sin (h' + \frac{1}{2} \Delta h)}{\cos h'}$$

$$B_1 = -\frac{\sin \Delta h \sin (2 H' - \Delta H)}{2 \cos h' \cos H'}$$

$$C_1 = -\frac{2 \sin \frac{1}{2} \Delta H \sin (H' - \frac{1}{2} \Delta H)}{\cos H'}$$

$$D_1 = \frac{\sin \Delta H \sin (2 h' + \Delta h)}{2 \cos h' \cos H'}$$

we have

$$2 \sin \tfrac{1}{2} \Delta d \sin (d' + \tfrac{1}{2} \Delta d) = A_1 \cos d' + B_1 + C_1 \cos d' + D_1 - A_1 C_1 \cos d' \quad (f)$$

This formula is rigorously exact; but, since Δd is always less than 1°, it will not produce an error of $0''.1$ to substitute the arcs $\frac{1}{2} \Delta d$, $\frac{1}{2} \Delta h$, &c. for their sines, or $\frac{1}{2} \Delta d \sin 1''$, $\frac{1}{2} \Delta h \sin 1''$, &c. for $\sin \frac{1}{2} \Delta d$, $\sin \frac{1}{2} \Delta h$, &c.; and therefore we may write

$$\Delta d \sin (d' + \tfrac{1}{2} \Delta d) = A_1 \cos d' + B_1 + C_1 \cos d' + D_1 - A_1 C_1 \sin 1'' \cos d' \quad (g)$$

in which A_1, B_1, C_1, D_1, now have the following signification:

$$A_1 = \frac{\Delta h}{\cos h'} \cdot \sin (h' + \tfrac{1}{2} \Delta h)$$

$$B_1 = -\frac{\Delta h}{\cos h'} \cdot \frac{\sin (2 H' - \Delta H)}{2 \cos H'}$$

$$C_1 = -\frac{\Delta H}{\cos H'} \cdot \sin (H' - \tfrac{1}{2} \Delta H)$$

$$D_1 = \frac{\Delta H}{\cos H'} \cdot \frac{\sin (2 h' + \Delta h)}{2 \cos h'}$$

The next step in our transformation consists in finding convenient and at the same time sufficiently accurate expressions of Δh and ΔH. Let

r, R = the true refractions for the apparent altitudes h' and H';

then we have, within less than $0''.1$,

$$\Delta h = \pi_1 \cos (h' - r) - r$$

If we neglect r in the term $\pi_1 \cos (h' - r)$, the error in this term will never exceed $1''$; but even this error will be avoided by taking the approximate expression

$$\cos(h' - r) = \cos h' + \sin r \sin h'$$

and we shall then have

$$\Delta h = \pi_1 \cos h' - r + \pi_1 \sin r \sin h'$$
$$= (\pi_1 \cos h' - r)\left(1 + \frac{\pi_1 \sin r \sin h'}{\pi_1 \cos h' - r}\right)$$

Since the second term of the second factor produces but $1''$ in Δh, we may employ for it an approximate value, which will still give Δh with great precision. Denoting this term by k, we have

$$k = \frac{\pi_1 \sin r \sin h'}{\pi_1 \cos h' - r} = \frac{\sin r \tan h'}{1 - \dfrac{r}{\pi_1 \cos h'}}$$

or, very nearly,

$$k = \sin r \tan h'\left(1 + \frac{r}{\pi_1 \cos h'}\right)$$

If we put

$$r = \alpha \cot h',$$

in which α has the value given in Table II., we have

$$k = \alpha \sin 1''\left(1 + \frac{\alpha}{\pi_1 \sin h'}\right)$$

Now, α increases with h', but in such a ratio that k remains very nearly constant for a constant value of π_1. We may without sensible error take $\pi_1 = 57'\ 30'' = 3450''$, which is about the mean value of π_1, and we shall find for a mean state of the air, by the values of α given in Table II.,

$$\begin{array}{ll} \text{for } h' = \ \ 5^\circ & k = 0.000291 \\ \phantom{\text{for }} h' = 45 & k = 0.000286 \\ \phantom{\text{for }} h' = 90 & k = 0.000285 \end{array}$$

Hence, if we take

$$k = 0.00029$$

the formula

$$\Delta h = (\pi_1 \cos h' - r)\ (1 + k) \tag{452}$$

will give Δh within $\frac{1}{200000}$ of its whole amount, that is, within less than $0''.02$ in a mean state of the air. For extreme variations

of the density of the air, it is possible that the refraction may be increased by its one-sixth part, and k will also be increased by its one-sixth part. But, as the term depending on k is not more than $1''$, the error in Δh, even in the improbable case supposed, will not be greater than $0''.16$. The formula (452) may therefore be regarded as practically exact with the value $k = 0.00029$.

A strict computation of the sun's or a planet's altitude requires the formula

$$\Delta H = R - P \cos (H' - R)$$

but P is in all cases so small that the formula

$$\Delta H = R - P \cos H' \tag{453}$$

will always be correct within a very small fraction of a second.

Now, let

$$r' = \frac{r}{\cos h'} \qquad R' = \frac{R}{\cos H'} \tag{454}$$

The quantities r' and R′ computed from the mean values of the refraction are given in Table XIV. under the name "Mean Reduced Refraction for Lunars." The numbers of the table are corrected for the height of the barometer and thermometer by means of Table XIV.A and B. These tables are computed from Bessel's refraction table, assuming the attached thermometer of the barometer, and the external thermometer, to indicate the same temperature, which is allowable in our present problem.* By the introduction of r' and R', we obtain

$$\frac{\Delta h}{\cos h'} = (\pi_1 - r')(1 + k) \qquad \frac{\Delta H}{\cos H'} = R' - P$$

and the coefficients of formula (g) become

---

* If it is desired to compute r' and R' with the utmost rigor, it can be done by Table II., by taking (Art. 107)

$$r' = \frac{\alpha\beta^A\gamma^\lambda}{\sin h'} \qquad R' = \frac{\alpha\beta^A\gamma^\lambda}{\sin H'}$$

The tables XIV. and XIV.A and B give the correct values to the nearest second in all practical cases.

$$\begin{aligned} A_1 &= \quad (\pi_1 - r')\,(1+k)\sin(h' + \tfrac{1}{2}\Delta h) \\ B_1 &= -(\pi_1 - r')\,(1+k)\,\frac{\sin(2H' - \Delta H)}{2\cos H'} \\ C_1 &= -(R' - P)\sin(H' - \tfrac{1}{2}\Delta H) \\ D_1 &= \quad (R' - P)\,\frac{\sin(2h' + \Delta h)}{2\cos h'} \end{aligned}$$

The term $A_1\,C_1 \sin 1'' \cos d'$ is very small, its maximum being only $1''$. It is easy to obtain an approximate expression for it and to combine it with the term $A_1 \cos d'$. In so small a term we may take

$$C_1 \sin 1'' \doteq -R' \sin 1'' \sin H' = -\sin R \tan H' = -k$$

and hence

$$A_1 - A_1 C_1 \sin 1'' = A_1\,(1+k) = (\pi_1 - r')\,(1+k)^2 \sin(h' + \tfrac{1}{2}\Delta h)$$

If now we put

$$\left.\begin{aligned} A &= (1+k)^2 \cdot \frac{\sin(h' + \frac{1}{2}\Delta h)}{\sin h'} \\ B &= (1+k)\,\frac{\sin(2H' - \Delta H)}{\sin 2H'} \\ C &= \frac{\sin(H' - \frac{1}{2}\Delta H)}{\sin H'} \\ D &= \frac{\sin(2h' + \Delta h)}{\sin 2h'} \end{aligned}\right\} \quad (455)$$

and

$$\left.\begin{aligned} A' &= \quad (\pi_1 - r')\,A \sin h' \cot d' \\ B' &= -(\pi_1 - r')\,B \sin H' \operatorname{cosec} d' \\ C' &= -(R' - P)\,C \sin H' \cot d' \\ D' &= \quad (R' - P)\,D \sin h' \operatorname{cosec} d' \end{aligned}\right\} \quad (456)$$

the formula (g) becomes, when divided by $\sin d'$,

$$\Delta d \cdot \frac{\sin(d' + \frac{1}{2}\Delta d)}{\sin d'} = A' + B' + C' + D'$$

the first member of which may be put under the form

$$\Delta d\left(1 + \frac{2\sin\frac{1}{4}\Delta d\,\cos(d' + \frac{1}{4}\Delta d)}{\sin d'}\right)$$

so that if we put

$$x = -\frac{\Delta d^2 \sin 1'' \cos (d' + \frac{1}{2}\Delta d)}{2 \sin d'}$$

or, within 0''.15,

$$x = -\tfrac{1}{2}\Delta d^2 \sin 1'' \cot d' \tag{457}$$

we have

$$\Delta d = A' + B' + C' + D' + x \tag{458}$$

The terms A', B', C', and D' are computed directly from the apparent distance and altitudes by (456), and with sufficient accuracy with four-figure logarithms. The logarithms of A, B, C, D, are given in Table XV., log A and log D with the arguments $\pi_1 - r'$ and h'; log B and log C with the arguments $R' - P$ and H'. In the construction of this table Δh and ΔH are computed by (452) and (453), and then the logarithms of A, B, C, D, by (455).

The sum $A' + B' + C' + D'$ is called the "first correction of the distance," and, being very nearly equal to Δd, is used as the argument of Table XVI., which gives x, or the "second correction of the distance," computed by (457). When x is greater than 30'' and the distance small, it will be necessary to enter this table a second time with the more correct value of Δd found by employing the first value of x.

The correction Δd being thus found and added to d', we have d_1, or the distance reduced to the point O. The reduction to the centre of the earth is then made by (450). This reduction is also facilitated by a table. If we put

$$N = A\pi\left(\frac{\sin \Delta}{\sin d_1} - \frac{\sin \delta}{\tan d_1}\right)$$

and then

$$a = -A\pi\frac{\sin \delta}{\tan d_1} \qquad b = A\pi\frac{\sin \Delta}{\sin d_1}$$

we shall have

$$N = a + b \tag{459}$$

and a and b can be taken from Table XIX. where a is called "the first part of N," and b "the second part of N." We then have

$$d - d_1 = N \sin \varphi \tag{460}$$

which is the correction to be added to d_1 to obtain the geocentric distance d. Table XIX. is computed with the mean value of

$\pi = 57' \ 30''$, which will not produce more than $1''$ error in $d - d_1$ in any case. But, if we wish to compute the correction for the actual parallax, we shall have, after finding N by the table,

$$d - d_1 = N \sin \varphi \times \frac{\pi}{3450''} \qquad (460^*)$$

π being in seconds.

The trouble of finding the declinations of the bodies and the use of Table XIX. would be saved if the Almanac contained the logarithm of N in connection with the lunar Ephemeris. The value of $\log N$ in the Almanac would, of course, be computed with the actual parallax, and (460) would be perfectly exact.

We have yet to introduce corrections for the elliptical figure of the discs of the moon and sun produced by refraction. These corrections are obtained by Tables XVII. and XVIII., which are constructed upon the following principles. Let

$\Delta s_1, \Delta S_1 =$ the contractions of the vertical semidiameters,
$\Delta s, \Delta S =$ the contractions of the inclined semidiameters;

then we have (Art. 133)

$$\Delta s = \Delta s_1 \cos^2 q \qquad \Delta S = \Delta S_1 \cos^2 Q$$

where $q =$ the angle $ZM'S'$ (Fig. 29) and $Q = ZS'M'$. We have

$$\cos q = \frac{\sin H' - \sin h' \cos d'}{\cos h' \sin d'}$$

But, by (456),

$$\frac{\sin H'}{\cos h' \sin d'} = - \frac{B'}{B(\pi_1 - r') \cos h'} \qquad \frac{\sin h' \cos d'}{\cos h' \sin d'} = \frac{A'}{A(\pi_1 - r') \cos h'}$$

so that

$$\cos q = -\left(\frac{A'}{A} + \frac{B'}{B}\right) \frac{1}{(\pi_1 - r') \cos h'}$$

If we put $A = 1$ and $B = 1$, which are approximate values, we shall have

$$\cos q = - \frac{A' + B'}{(\pi_1 - r') \cos h'}$$

$$\Delta s = \Delta s_1 \left[\frac{A' + B'}{(\pi_1 - r') \cos h'}\right]^2 \qquad (461)$$

In order to ascertain the degree of accuracy of this formula, we observe that the errors in $\cos q$ produced by the assumption $A = 1$, $B = 1$, are

$$e = (A - 1)\frac{\tan h'}{\tan d'} \qquad e' = (1 - B)\frac{\sin H'}{\cos h' \sin d'}$$

the errors in $\cos^2 q$ are

$$2e\cos q \qquad 2e'\cos q$$

and the errors in Δs are, therefore,

$$e_1 = \frac{2\Delta s_1 (A - 1)\tan h' \cos q}{\tan d'} \qquad e_1' = \frac{2\Delta s_1 (1 - B)\sin H' \cos q}{\cos h' \sin d'}$$

In order to represent extreme cases, let us suppose $q = 0$ and $H' = 90°$, which will give e_1 and e_1' their greatest values; then we shall find for the different values of h' the following errors:

h'	$e_1 \tan d'$	$e_1' \sin d'$
5°	0″.45	0″.02
10	.16	.00
15	.08	.00
30	.02	.00
50	.00	.00

It can only be for very small values of d' that the error e_1 can be important, even for $h' = 5°$; and, as these small values of the distance are always avoided in practice, our formula (461) may be considered quite perfect.

In the same manner, we shall find

$$\Delta S = \Delta S_1 \left[\frac{C' + D'}{(R' - P)\cos H'}\right]^2 \tag{462}$$

which is even more accurate than (461).

These formulæ are put into tables as follows. For the moon, Table XVII.A, with the arguments h' and $\pi_1 - r'$, gives the value of

$$g = \frac{\Delta s_1}{(\pi_1 - r')^2 \cos^2 h'} \times f$$

where f is an arbitrary factor (= 18000000) employed to give g convenient integral values. Then Table XVII.B, with the arguments g and $A' + B'$, gives

$$\Delta s = (A' + B')^2 \times \frac{g}{f}$$

For the sun, Table XVIII.A, with the arguments H' and $R' - P$, gives the value of

$$G = \frac{(R' - P)^2 \cos^2 H'}{\Delta S_1} \times F$$

in which $F = \frac{1}{200}$; and Table XVIII.B gives

$$\Delta S = (C' + D')^2 \times \frac{F}{G}$$

In these tables $A' + B'$ is called the "whole correction of the moon," and $C' + D'$ the "whole correction of the sun." As these quantities are furnished by the previous computation of the true distance, the required corrections are taken from the tables without any additional computation.

The values of Δs and ΔS are applied to the distance as follows: when the limb of the moon nearest to the star or planet is observed, Δs is to be subtracted, and when the farthest limb is observed, Δs is to be added; when the sun is observed, both Δs and ΔS are to be subtracted from d.

In strictness, these corrections should be applied to the distance d', and the distance thus corrected should be employed in computing the values of A', B', C', and D'. This would require a repetition of the computation after Δs and ΔS had been found by a first computation; but this repetition will rarely change the result by $0''.5$. In the extreme and improbable case when the distance is only 20° and one body is at the altitude 5° and the other directly above it in the same vertical circle (so that the entire contraction of the vertical semidiameter comes into account), such a repetition would change the result only $1''.8$; and even this error is much less than the probable error of sextant observations at this small altitude, where the sun and moon already cease to present perfectly defined discs.

250. I shall now recapitulate the steps of this method.

1st. The local mean time of the observation being T, and the assumed longitude L, take from the Ephemeris, for the approxi-

mate Greenwich time $T + L$, the quantities s, S, π, P, δ, and Δ. (For the sun we may always take $P = 8''.5$; for a star, $S = 0$, $P = 0$.)

2d. If h'', H'', d'' denote the observed altitudes and distance of the limbs, find

$$s' = s + \text{correction of Table XII.,}$$
$$\pi_1 = \pi + \text{correction of Table XIII.,}$$

and the apparent altitudes and distance of the centres,

$$h' = h'' \mp s', \qquad H' = H'' \mp S, \qquad d' = d'' \pm s' \pm S$$

upper signs for upper and nearest limbs, lower signs for lower and farthest limbs.

For the altitudes h' and H', take the "reduced refractions" r' and R' from Table XIV., correcting them by Table XIV.A and B for the barometer and thermometer. Then compute the quantities

$$A' = \quad (\pi_1 - r')\, A \sin h' \cot d' \qquad C' = -(R' - P)\, C \sin H' \cot d'$$
$$B' = -(\pi_1 - r')\, B \sin H' \operatorname{cosec} d' \qquad D' = \quad (R' - P)\, D \sin h' \operatorname{cosec} d'$$

for which the logarithms of A, B, C, and D are taken from Table XV. In this table the argument $\pi_1 - r'$ is called the "reduced parallax and refraction of the moon," and $R' - P$ the "reduced refraction and parallax of the sun (or planet) or star." For a star this argument is simply R'.

When $d' > 90°$, the signs of A' and C' will be reversed. It may be convenient for the computer to determine the signs by referring to the following table:

	A'	B'	C'	D'
$d' < 90°$	+	—	—	+
$d' > 90°$	—	—	+	+

3d. The terms A' and B', which depend upon the moon's parallax and refraction, may be called the first and second parts of the moon's correction, and the sum $A' + B'$ the "whole correction of the moon." In like manner, C' and D' may be called the first and second parts of the sun's, planet's, or star's correc-

tion, and the sum $C' + D'$ the "whole correction of the sun, planet, or star."

The sum of these corrections $= A' + B' + C' + D'$ may be called the "first correction of the distance." Taking it as the upper argument in Table XVI., find the second correction $= x$, the sign of which is indicated in the table.

4th. Take from Table XVII.A and B the contraction of its inclined semidiameter $= \Delta s$. If the sun is the other body, take also the contraction from Table XVIII.A and B, $= \Delta S$. The sign of either of these corrections will be positive when the farthest limb is observed, and negative when the nearest limb is observed.

5th. The correction for the compression of the earth is $= N \sin \varphi$, φ being the latitude; and N may be accurately computed by the formula

$$N = A\pi\left(\frac{\sin \varDelta}{\sin d_1} - \frac{\sin \delta}{\tan d_1}\right)$$

or it may be found within 1″ by Table XIX., the mode of consulting which is evident. The sign of $N \sin \varphi$ will be determined by the signs of N and $\sin \varphi$, remembering that for south latitudes $\sin \varphi$ is negative.

All the corrections being applied to d', we have the geocentric distance d; and hence the corresponding Greenwich time and the longitude.

EXAMPLE.—Let us take the example of the preceding article (p. 399), in which the observation gives

	1856, March 9th, $\phi = 35°$		
T	$= 5^h\ 14^m\ 6^s$	☽ $h'' = 52°\ 34'\ 0''$	Barom. 29.5 in
Assumed L	$= 10\ \ 0\ \ 0$	☉ $H'' = 8\ \ 56\ \ 23$	Therm. 58° F.
Approx. Gr. T.	$= 15\ \ 14\ \ 6$	☽ ☉ $d'' = 44\ \ 36\ \ 58.6$	

By the Ephemeris, we have

	$s = 16'\ 23''.1$		$\pi = 60'\ 1''.9$	$S = 16'\ 8''.0$	$P = 8''.6$
Table XII.	$+\ 14\ .0$	Tab. XIII.	$+\ 3\ .9$	$\delta = +\ 14°$	$\varDelta = -\ 4°$
	$s' = 16\ 37\ .1$		$\pi_1 = 60\ 5\ .8$		

The computation may be arranged as follows:

☽ $h''=$	52° 34'.0	☉ $H''=$	8° 56'.4	$d''=$	44° 36' 58".6
$s'=$	+ 16.6	$S=$	16.1	$s'=$	16 37 .1
$h'=$	52 50.6	$H'=$	9 12.5	$S=$	16 8 .0
				$d'=$	45 9 43 .7

Table XIV.	1'13".1		5'49".6
" " A.	— 1 .		— 6 .
" " B.	— 1 .		— 6 .
$r'=$	1 11 .1	$R'=$	5 37 .6
$\pi_1=$	60 5 .8	$P=$	8 .6
$\pi_1 - r'=$	58 54 .7	$R' - P=$	5 29 .0

(Table XV.) log A	0.0019	(Table XV.) log C	9.9978
log $(\pi_1 - r')$	3 5484	log $(R' - P)$	2.5172
log sin h'	9.9015	log sin H'	9.2042
log cot d'	9 9975	log cot d'	9.9975
log A'	3.4493	log C'	n1.7167
$A' = + 46'53".9$		$C' = - 52".1$	

(Table XV.) log B	9.9981	(Table XV.) log D	9.9987
log $(\pi_1 - r')$	3.5484	log $(R' - P)$	2.5172
log sin H'	9.2042	log sin h'	9.9015
log cosec d'	0.1493	log cosec d'	0.1493
log B'	n2.9000	log D'	2.5667
$B' = -13'14".3$		$D' = + 6'\ 8".7$	
$A' + B' = + 33\ 39\ .6$		$C' + D' = + 5\ 16\ .6$	

	1st corr. =	+ 38' 56".2
(Table XVI.)	2d corr. =	— 18 .5
(Table XVII.)	$\Delta\epsilon$ =	0 .
(Table XVIII.)	ΔS =	— 9 .
	$N \sin \phi$ =	— 4 .6
	d =	45 48 12 .8

Table XIX. 1st Part of $N = - 6"$
2d " " $= - 2$
$N = - 8.$ $\quad \phi = 35°.$

This result agrees with that found by the rigorous method on p. 401, within 1".

To find the longitude, we now have, by the American Ephemeris for March 9,

$(T) = 15^h\ 0^m\ 0^s \quad (d) = 45°\ 40'\ 54" \quad Q = 0.2510 \quad \Delta Q = + 17$

$d = 45\ 48\ 13$

$7\ 19 \quad \log = 2.6425$

$\log t = 2.8935$

t =	0 13 3
Table XX.	— 1
T_0 =	15 13 2
T =	5 14 6
L =	9 58 56

251. In consequence of the neglect of the fractions of a second in several parts of the above method, it is possible that the computed distance may be in error several seconds, but it is easily seen that the error from this cause will be most sensible in cases where the distance is small; and, since the lunar distances are given in the Ephemeris for a number of objects, the observer can rarely be obliged to employ a small distance. If he confines himself to distances greater than 45° (as he may readily do), the method will rarely be in error so much as 2″, especially if he also avoids altitudes less than 10°. When we remember that the least count of the sextant reading is 10″, and that to the probable error of observation we must add the errors of graduation, of eccentricity, and of the index correction, it must be conceded that we cannot hope to reduce the probable error of an observed distance below 5″, if indeed we can reduce it below 10″. Our approximate method is, therefore, for all practical purposes, a perfect method, in relation to our present means of observation.

252. If the altitudes have not been observed, they may be computed from the hour angles and declinations of the bodies, the hour angles being found from the local time and the right ascensions. But the declination and right ascension of the moon will be taken from the Ephemeris for the approximate Greenwich time found with the assumed longitude. If, then, the assumed longitude is greatly in error, a repetition of the computation may be necessary, starting from the Greenwich time furnished by the first. As a practical rule, we may be satisfied with the first computation when the error in the assumed longitude is not more than 30ᵐ. In the determination of the longitude of a fixed point on land, it will be advisable to omit the observation of the altitudes, as thereby the observer gains time to multiply the observations of the distance. But at sea, where an immediate result is required with the least expenditure of figures, the altitudes should be observed.

253. At sea, the observation is noted by a chronometer regulated to Greenwich time, and the most direct employment of the resulting Greenwich time will then be to determine the true correction of the chronometer. This proceeding has the advan-

tage of not requiring an exact determination of the local time at the instant of the observation.

For example, suppose the observation in the example above computed had been noted by a Greenwich mean time chronometer which gave $15^h\ 10^m\ 0^s$, and was supposed to be *slow* $4^m\ 6^s$. The true Greenwich time according to the lunar observation was $15^h\ 13^m\ 0^s$, and hence the true correction was $+\ 3^m\ 0^s$. With this correction we may at any convenient time afterwards determine the longitude by the chronometer (Art. 214).

In this way the navigator may from time to time during a voyage determine the correction of the chronometer, and, by taking the mean of all his results, obtain a very reliable correction to be used when approaching the land. He may even determine the *rate* of the chronometer with considerable accuracy by comparing the mean of a number of observations in the first part of the voyage with a similar mean in the latter part of it.

254. *To correct the longitude found by a lunar distance for errors of the Ephemeris.*—In relation to the degree of accuracy of the observation, we may in the present state of the Ephemeris regard all its errors as insensible except those which affect the moon's place. If, therefore, the longitude of a fixed point has been found by a lunar distance on a certain date, the corrections of the moon's right ascension and declination are first to be found for that date from the observations at one or more of the principal observatories, and then the correction of the longitude will be found as follows. Let

$\alpha,\ \delta =$ the right ascension and declination of the moon given in the Ephemeris for the date of the observation,

$A,\ \Delta =$ those of the sun, planet, or star,

$\delta\alpha, \delta\delta =$ the corrections of the moon's right ascension and declination,

$\delta d =$ the corresponding correction of the lunar distance,

$\delta L =$ the corresponding correction of the computed longitude;

In Fig. 30, M and S being the geocentric places of the two bodies, as given in the Ephemeris, and d denoting the distance MS, we have

$$\cos d = \sin \delta \sin \Delta + \cos \delta \cos \Delta \cos (\alpha - A) \qquad (463)$$

by differentiating which we find

$$\delta d = \frac{\cos\delta\cos\varDelta\sin(\alpha - A)}{\sin d}\cdot\delta\alpha$$

$$- \frac{\cos\delta\sin\varDelta - \sin\delta\cos\varDelta\cos(\alpha - A)}{\sin d}\cdot\delta\delta \qquad (464)$$

If then

$$v = \text{the change of distance in } 3^h,$$

we shall have

$$\delta L = -\delta d \times \frac{3^h}{v} \qquad (465)$$

in computing which we employ the proportional logarithm of the Ephemeris, $Q = \log\frac{3^h}{v}$, reduced to the time of the observation.

Example.—At the time of the observation computed in Art 250, we have

Moon, $\alpha =$	$2^h\ 11^m\ 14^s$	$\delta =$	$+\ 14°\ 18'.4$
Sun, $A =$	$23\ \ 22\ \ 25$	$\varDelta =$	$-\ \ 4\ \ \ 3\ .1$
$\alpha - A =$	$2\ \ 49\ \ 19$	$d =$	$45\ \ 48\ .2$
$=$	$42°\ 19'.8$		

with which we find, by (464),

$$\delta d = 0.908\ \delta\alpha + 0.350\ \delta\delta$$

and hence, by (465), with $\log Q = 0.2511$,

$$\delta L = -1.62\ \delta\alpha - 0.62\ \delta\delta$$

Suppose then we find from the Greenwich observations $\delta\alpha = -0^s.38 = -5''.7$ and $\delta\delta = -4''.0$, the correction of the longitude above found will be

$$\delta L = +11^s.7$$

255. *To find the longitude by a lunar distance not given in the Ephemeris.*—The regular lunar-distance stars mentioned in Art. 247 are selected nearly in the moon's path, and are therefore in general most favorable for the accurate determination of the Greenwich time. Nevertheless, it may occasionally be found expedient to employ other stars, not too far from the ecliptic. Sometimes, too, a different star may have been observed by mistake, and it may be important to make use of the observation

The true distance d is to be found from the observed distance by the preceding methods, as in any other case. Let the local time of the observation be T, and the assumed longitude L. Take from the Ephemeris the moon's right ascension α and declination δ for the Greenwich time $T + L$, and also the star's right ascension A and declination $\varDelta$; with which the corresponding true distance d_0 is found by the formula

$$\cos d_0 = \sin \delta \sin \varDelta + \cos \delta \cos \varDelta \cos (\alpha - A)$$

Then, if $d = d_0$, the assumed longitude is correct; if otherwise, put

λ = the increase of α in one minute of mean time,
β = the increase of δ " " " "
γ = the increase of d " " " "

then we have, by (464),

$$\gamma = \frac{\cos \delta \cos \varDelta \sin (\alpha - A)}{\sin d_0} \cdot \lambda - \frac{\cos \delta \sin \varDelta - \sin \delta \cos \varDelta \cos (\alpha - A)}{\sin d_0} \cdot \beta$$

and hence the correction of the assumed longitude in seconds of time,

$$\delta L = \frac{60}{\gamma} (d - d_0)$$

For computation by logarithms, these formulæ may be arranged as follows:

$$\left.\begin{aligned}
\tan M &= \frac{\tan \varDelta}{\cos (\alpha - A)} \\
\cos d_0 &= \frac{\sin \varDelta \cos (\delta - M)}{\sin M} \\
\gamma &= \lambda \cdot \frac{\cos \delta \cos \varDelta \sin (\alpha - A)}{\sin d_0} + \beta . \cot d_0 \tan (\delta - M) \\
\delta L &= \frac{60 (d - d_0)}{\gamma}
\end{aligned}\right\} \quad (466)$$

Example.—Suppose an observer has measured the distance of the moon from *Arcturus*, at the local mean time 1856 March 16, $T = 10^h\ 30^m\ 0^s$, in the assumed longitude $L = 6^h\ 0^m\ 0^s$, and, reducing his observation, finds the true distance

$$d = 73° \ 55' \ 10''$$

what is the true longitude?

For the Greenwich time $T + L = 16^h\ 30^m$ we find

$$\begin{array}{llll} \alpha = & 8^h\ 47^m\ 6^s.54 & \delta = +\ 23°\ 12'\ \ 7''.1 & \lambda = +\ 31''.40 \\ A = & 14\ \ \ 9\ \ \ 7\ .04 & \Delta = +\ 19\ \ \ 55\ \ 44\ .8 & \beta = -\ \ \ 8\ .62 \\ \alpha - A = - & 5^h\ 22^m\ 0^s.50 = -\ 80°\ 30'\ \ 7''.5 & & \end{array}$$

with which we find by (466),

$$\begin{array}{llll} d_0 = & 73°\ 55'\ 35''. & \gamma = & -\ 25''.59 \\ d - d_0 = & -\ 25'' & \delta L = & +\ 58^s.6 \end{array}$$

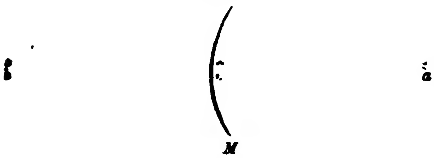
and therefore the longitude is $6^h\ 0^m\ 58^s.6$.

256. In order to eliminate as far as possible any constant errors of the instrument used in measuring the distance, we should observe distances from stars both east and west of the moon. If the index correction of the sextant is in error, the errors produced in the computed Greenwich time, and consequently in the longitude, will have different signs for the two observations, and will be very nearly equal numerically: they will therefore be nearly eliminated in the mean. If, moreover, the distances are nearly equal, the eccentricity of the sextant will have nearly the same effect upon each distance, and will therefore be eliminated at the same time with the index error. Since even the best sextants are liable to an error of eccentricity of as much as 20'', according to the confession of the most skilful makers, and this error is not readily determined, it is important to eliminate it in this manner whenever practicable. If a circle of reflexion is employed which is read off by two opposite verniers, the eccentricity is eliminated from each observation; but even with such an instrument the same method of observation should be followed, in order to eliminate other constant errors.

It has been stated by some writers that by observing distances *of stars* on opposite sides of the moon we also eliminate a constant error of *observation*, such, for example, as arises from a faulty habit of the observer in making the contact of the moon's limb with the star. This, however, is a mistake; for if the habit of the observer is to make the contact *too close*, that is, to bring the reflected image of the moon's limb somewhat over the star, the effect will be to increase a distance on one side of the moon while it diminishes that on the opposite side, and the effect upon the deduced Greenwich time will be the same in

both cases. This will be evident from the following diagram, (Fig. 31). Suppose a and b are the two stars, M the moon's limb. If the observer judges a contact to exist when the star appears within the moon's disc as at c, the distance ac is too small and the distance bc too great. But, supposing the moon to be moving in the direction from a to b, each distance will give too early a Greenwich time, for each will give the time when the moon's limb was actually at c.

Fig. 31.



If, however, we observe the *sun* in both positions, this kind of error, if really constant, will be eliminated; for, the moon's bright limb being always turned towards the sun, the error will increase both distances, and will produce errors of opposite sign in the Greenwich time. Hence, if a series of lunar distances from the sun has been observed, it will be advisable to form two distinct means,—one, of all the results obtained from increasing distances, the other, of all those obtained from decreasing distances: the mean of these means will be nearly or quite free from a constant error of observation, and also from constant instrumental errors.

FINDING THE LONGITUDE AT SEA.

257. *By chronometers.*—This method is now in almost universal use. The form under which it is applied at sea differs very slightly from that given in Art. 214. The correction of the chronometer on the time of the first meridian (that of Greenwich among American and English navigators) is found at any place whose longitude is known, and at the same time also its daily rate is to be established with all possible care. The rate being duly allowed for from day to day during the voyage, the Greenwich time is constantly known, and therefore at any instant when the local time is obtained by observation, the longitude of the ship is determined.

The local time on shipboard is always found from an altitude of some celestial object, observed with the sextant from the sea horizon. (Art. 156.) The computation of the hour angle is then made by (268), and the resulting local time is compared directly with the Greenwich time given by the chronometer at

the instant of the observation. The data from the Ephemeris required in computing the local time are taken for the Greenwich time given by the chronometer.

Example.—A ship being about to sail from New York, the master determined the correction on Greenwich time and the rate of his chronometer by observations on two dates, as follows:

1860 April 22, at Greenwich noon, chron. correction	=	+ 3^m 10^s.0
" " 30, " " " "	=	+ 3 43 .6
Rate in 8 days	=	+ 33 .6
Daily rate	=	+ 4 .2

On May 18 following, about 7^h 30^m A.M., the ship being in latitude 41° 33′ N., three altitudes of the sun's lower limb were observed from the sea horizon as below. The correction of the chronometer on that day is found from the correction on April 30 by adding the rate for 18 days. (It will not usually be worth while to regard the fraction of a day in computing the total rate at sea.) The record of the observation and the whole computation may be arranged as follows:

1860 May 18. $\phi = 41° 33'$

Chronometer	9^h 37^m 21^s.	☉̲	29° 40′ 10″	Barom. 30.32 in.	
	" 37 53.		" 46 0	Therm. 59° F.	
	" 38 20.		" 50 50		
Mean	= 9 37 51.3	Mean	= 29 45 40		
Correction	= + 4 59.2	Index corr.	= − 1 10		
Gr. date = May 17,	21 42 50.5	Dip	= − 4 2		
for which time we take from the			29 40 28		
Ephemeris the quantities		Semid.	= + 15 50		
☉'s δ	= 19° 38′ 39″	Refraction	= − 1 42		
Semidiameter	= 15′ 50″	Parallax	= + 8		
Equation of time	= − 3^m 49^s.8	h	= 29 54 44		
		ϕ	= 41 33 0	sec	0.12588
		P	= 70 21 21	cosec	0.02604
		s	= 70 54 33	cos	9.51464
		$s - h$	= 40 59 49	sin	9.81692
					9.48348
		Apparent time	= 7^h 32^m 6^s.8	sin	9.74174
		Eq. of time	= − 3 49.8		
		Local mean time	= 19 28 16.5		
		Gr. " "	= 21 42 50.5		
		Longitude	= 2 14 34	= 33° 38′.5 W.	

In this observation, the sun was near the prime vertical, a position most favorable to accuracy (Art. 149).

The method by equal altitudes may also be used for finding the time at sea in low latitudes, as in Arts. 158, 159.

258. In order that the longitude thus found shall be worthy of confidence, the greatest care must be bestowed upon the determination of the rate. As a single chronometer might deviate very greatly without being distrusted by the navigator, it is well to have at least three chronometers, and to take the mean of the longitudes which they severally give in every case.

But, whatever care may have been taken in determining the rate on shore, the sea rate will generally be found to differ from it more or less, as the instrument is affected by the motion of the ship; and, since a cause which accelerates or retards one chronometer may produce the same effect upon the others, the agreement of even three chronometers is not an absolutely certain proof of their correctness. The sea rate may be found by determining the chronometer correction at two ports whose *difference* of longitude is well known, although the absolute longitudes of both ports may be somewhat uncertain. For this purpose, a "Table of Chronometric Differences of Longitude" is given in RAPER'S *Practice of Navigation*, the use of which is illustrated in the following example.

EXAMPLE.—At St. Helena, May 2, the correction of a chronometer on the local time was $-\ 0^h\ 23^m\ 10^s.3$. At the Cape of Good Hope, May 17, the correction on the local time was $+\ 1^h\ 14^m\ 28^s.6$; what was the sea rate?

We have

Corr. at St. Helena, May 2d	$= -$	0^h	23^m	$10^s.3$
Chron. diff. of long. from Raper	$= +$	1	36	45 .
Corr. for Cape of G. H., May 2d	$= +$	1	13	34 .7
" " " " 17th	$= +$	1	14	28 .6
Rate in 15 days	$=$		$+$	53 .9
Daily sea rate	$=$		$+$	3 .59

259. *By lunar distances.*—Chronometers, however perfectly made, are liable to derangement, and cannot be implicitly relied upon in a long voyage. The method of lunar distances (Arts. 247–256) is, therefore, employed as an occasional check upon the chronometers even where the latter are used for finding the longitude from day to day. When there is no chronometer on

board, the method of lunar distances is the only regularly available method for finding the longitude at sea, at once sufficiently accurate and sufficiently simple.

As a check upon the chronometer, the result of a lunar distance is used as in Art. 253.

In long voyages an assiduous observer may determine the sea rates of his chronometers with considerable precision. For this purpose, it is expedient to combine observations taken at various times during a lunation in such a manner as to eliminate as far as possible constant errors of the sextant and of the observer (Art. 256). Suppose distances of the sun are employed exclusively. Let two chronometer corrections be found from two nearly equal distances measured on opposite sides of the sun on two different dates, in the first and second half of the lunation respectively. The mean of these corrections will be the correction for the mean date, very nearly free from constant instrumental and personal errors. In like manner, any number of *pairs* of equal, or nearly equal, distances may be combined, and a mean chronometer correction determined for a mean date from all the observations of the lunation. The sea rate will be found by comparing two corrections thus determined in two different lunations. This method has been successfully applied in voyages between England and India.

260. *By the eclipses of Jupiter's satellites.*—An observed eclipse of one of Jupiter's satellites furnishes immediately the Greenwich time without any computation (Art. 225.) But the eclipse is not sufficiently instantaneous to give great accuracy; for, with the ordinary spy-glass with which the eclipse may be observed on board ship, the time of the disappearance of the satellite may precede the true time of total eclipse by even a whole minute. The time of disappearance will also vary with the clearness of the atmosphere. Since, however, the same causes which accelerate the disappearance will retard the reappearance, if both phenomena are observed on the same evening under nearly the same atmospheric conditions, the mean of the two resulting longitudes will be nearly correct. Still, the method has not the advantage possessed by lunar distances of being almost always available at times suited to the convenience of the navigator.

261. *By the moon's altitude.*—This method, as given in Art. 243,

may be used at sea in low latitudes; but, on account of the unavoidable inaccuracy of an altitude observed from the sea horizon, it is even less accurate than the method of the preceding article, and always far inferior to the method of lunar distances, although on shore it is one which admits of a high degree of precision when carried out as in Art. 245.

262. *By occultations of stars by the moon.*—This method, which will be treated of in the chapter on eclipses, may be successfully used at sea, as the disappearance of a star behind the moon's limb may be observed with a common spy-glass at sea with nearly as great a degree of precision as on shore; but, on account of the length of the preliminary computations as well as of the subsequent reduction of the observation, it is seldom that a navigator would think of resorting to it as a substitute for the convenient method of lunar distances.

CHAPTER VIII.

FINDING A SHIP'S PLACE AT SEA BY CIRCLES OF POSITION.

263. In the preceding two chapters we have treated of methods of finding the position of a point on the earth's surface by the two co-ordinates *latitude* and *longitude*; and therefore in all these methods the required position is determined by the intersection of two circles, one a parallel of latitude and the other a meridian. In the following method it is determined by circles *oblique* to the parallels of latitude and the meridians. The principle which underlies the method has often been applied; but its value as a practical nautical method was first clearly shown by Capt. Thomas H. Sumner.*

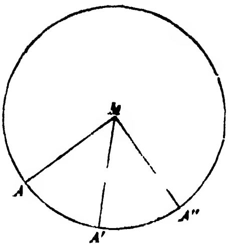
Let an altitude of the sun (or any other object) be observed *at any time*, the time being noted by a chronometer regulated to Greenwich time. Suppose that at this Greenwich time the sun

---

* *A new and accurate method of finding a ship's position at sea by projection on Mercator's chart* by Capt. Thomas H. Sumner. Boston, 1843.

is vertical to an observer at the point M of the globe (Fig 32). Let a small circle $AA'A''$ be described on the globe from M as a pole, with a polar distance MA equal to the zenith distance, or complement of the observed altitude, of the sun. It is evident that at all places *within* this circle an observer would at the given time observe a smaller zenith distance, and at all places without this circle a greater zenith distance; and therefore the observation fully determines the observer to be *on* the circumference of the small circle $AA'A''$. If, then, the navigator can project this small circle upon an artificial globe or a chart, *the knowledge that he is upon this circle will be just as valuable to him in enabling him to avoid dangers as the knowledge of either his latitude alone or his longitude alone;* since one of the latter elements only determines a point to be in a certain circle, without fixing upon any particular point of that circle.

Fig. 32.



The small circle of the globe described from the projection of the celestial object as a pole we shall call a *circle of position.*

264. *To find the place on the globe at which the sun is vertical (or the sun's projection on the globe) at a given Greenwich time.*—The sun's hour angle from the Greenwich meridian is the Greenwich apparent time. The diurnal motion of the earth brings the sun into the zenith of all the places whose latitude is just equal to the sun's declination. Hence the required projection of the sun is a place whose longitude (reckoned westward from Greenwich from 0^h to 24^h) is equal to the Greenwich apparent time, and whose latitude is equal to the sun's declination at that time.

265. *From an altitude of the sun taken at a given Greenwich time, to find the circle of position of the observer, by projection on an artificial globe.*—Find the Greenwich apparent time and the sun's declination, and put down on the globe the sun's projection by the preceding article. From this point as a pole, describe a small circle with a circular radius equal to the true zenith distance deduced from the observation. This will be the required circle of position.

266. The preceding problem may be extended to any celestial

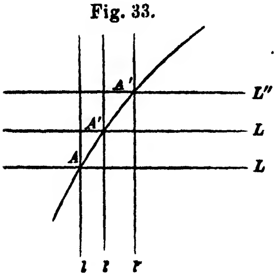
object. The pole of the circle of position will always be the place whose west longitude is the Greenwich hour angle of the object (reckoned from 0^h to 24^h) and whose latitude is the declination of the object. The hour angle is found by Art. 54.

267. *To find both the latitude and the longitude of a ship by circles of position projected on an artificial globe.—First.* Take the altitudes of two different objects at the same time by the Greenwich chronometer. Put down on the globe, by the preceding problem, their two circles of position. The observer, being in the circumference of each of these circles, must be at one of their two points of intersection; which of the two, he can generally determine from an approximate knowledge of his position.

Second. Let the same object be observed at two different times, and project a circle of position for each. Their intersection gives the position of the ship as before. If between the observations the ship has moved, the first altitude must be reduced to the second place of observation by applying the correction of Art. 209, formula (380). The projection then gives the ship's position at the second observation.

268. *From an altitude of a celestial body taken at a given Greenwich time, to find the circle of position of the observer, by projection on a Mercator chart.*—The scale upon which the largest artificial globes are constructed is much smaller than that of the working charts used by navigators. But on the Mercator chart a circle of position will be distorted, and can only be laid down by points. Let L, L', L'' (Fig. 33) be any parallels of latitude crossed by the required circle. For each of these latitudes, with the true altitude found from the observation and the polar distance of the celestial body taken for the Greenwich time, compute the local time, and hence the longitude, "by chronometer" (Art. 257). Let l, l', l'' be the longitudes thus found. Let A, A', A'' be the points whose latitudes and longitudes are, respectively, L, l; L', l'; L'', l''; these are evidently points of the required circle. The ship is consequently in the curve $AA'A''$, traced through these points.

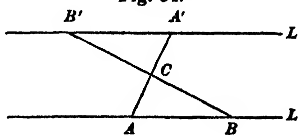
Fig. 33.



In practice it is generally sufficient to lay down only two points; for, the approximate position of the ship being known, if L and L' are two latitudes between which the ship may be assumed to be, her position is known to be on the curve AA' somewhere between A and A'. When the difference between L and L' is small, the arc AA' will appear on the chart as a straight line.

269. *To find the latitude and longitude of a ship by circles of position projected on a Mercator chart.—First.* Let the altitudes of two objects be taken at the same time. Assume two latitudes embracing between them the ship's probable position, and find two points of each of their two circles of position by the preceding problem, and project these points on the chart. Each pair of points being joined by a straight line, the intersection of the two lines is very nearly the ship's position. Thus, if one object gives the points A, A' (Fig. 34) corresponding to the latitudes L, L', and the other object the points B, B' corresponding to the same latitudes, the ship's position is the point C, the intersection of AA' and BB'.

Fig. 34.



It is, of course, not essential that the same latitudes should be used in computing the points of the two circles; but it is more convenient, and saves some logarithms.

If greater accuracy is desired, the circles may be more fully laid down by three or more points of each.

Second.—The altitude of the same object may be taken at two different times, and the circles laid down as before; the usual reduction of the first altitude being applied when the ship changes her position between the observations.

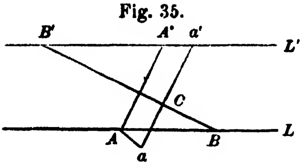
It is evident from the nature of the above projection that the most favorable case for the accurate determination of the intersection C is that in which the circles of position intersect at right angles. Hence the two objects observed, or the two positions of the same object, should, if possible, differ about 90° in azimuth. This agrees with the results of the analytical discussion of the method of finding the latitude by two altitudes, Art. 183.

If the chronometer does not give the true Greenwich time, the only effect of the error will be to shift the point C towards the east or the west, without changing its latitude, unless the error is

so great as to affect sensibly the declination which is taken from the Ephemeris for the time given by the chronometer. This method is, therefore, a convenient substitute for the usual method of finding the latitude at sea by two altitudes, a projection on the sailing chart being always sufficient for the purposes of the navigator.

Instead of reducing the first altitude for the change of the ship s position between the observations, we may put down the circle of position for each observation and afterwards shift one of them by a quantity due to the ship's run. Thus, let the first observation give the position line AA' (Fig. 35), and let Aa represent, in direction and length, the ship's course and distance sailed between the observations. Draw aa' parallel to AA'. Then, BB' being the position line by the second observation, its intersection C with aa' is the required position of the ship at the second observation.

Fig. 35.



270. If the latitude is desired by computation, independently of the projection, it is readily found as follows. Let

l_1, l_2 = the longitudes (of A and B) found from the first and second altitudes respectively with the latitude L,

l_1', l_2' = the longitudes (of A' and B') found from the same altitudes with the latitude L',

L_0 = the latitude of C.

From Fig. 34 we have, by the similarity of the triangles ABC and $A'B'C$,

$$l_2' - l_1' : l_1 - l_2 = B'C : BC$$

whence

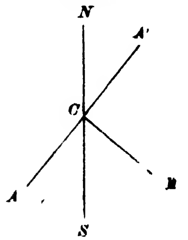
$$(l_2' - l_1') + (l_1 - l_2) : l_1 - l_2 = BB' : BC = L' - L : L_0 - L$$

$$L_0 = L + \frac{(L' - L)(l_1 - l_2)}{(l_2' - l_1') + (l_1 - l_2)} \tag{467}$$

This formula reduces SUMNER's method of "double altitudes" to that given long ago by LALANDE (*Astronomie*, Art. 3992, and *Abrégé de Navigation*, p. 68). The distinctive feature of SUMNER's process, however, is that a single altitude taken at any time is made available for determining a line of the globe on which the ship is situated.

271. *To find the azimuth of the sun by a position line projected on the chart.*—Let AA' (Fig. 36) be a position line on the chart, derived from an observed altitude by Art. 268. At any point C of this line draw CM perpendicular to AA', and let NCS be the meridian passing through C; then SCM is evidently the sun's azimuth. The line CM is, of course, drawn on that side of the meridian NS upon which the sun was known to be at the time of the observation.

Fig. 36.



The solution is but approximate, since AA' should be a curve line, and the azimuth of the normal CM would be different for different points of AA'. It is, however, quite accurate enough for the purpose of determining the variation of the compass at sea, which is the only practical application of this problem.

CHAPTER IX.

THE MERIDIAN LINE AND VARIATION OF THE COMPASS.

272. The *meridian line* is the intersection of the plane of the meridian with the plane of the horizon. Some of the most useful methods of finding the direction of this line will here be briefly treated of; but the full discussion of the subject belongs to geodesy.

273. *By the meridian passage of a star.*—If the precise instant when a star arrives at its greatest altitude could be accurately distinguished, the direction of the star at that instant, referred to the horizon, would give the direction of the meridian line; but the altitude varies so slowly near the meridian that this method only serves to give a first approximation.

274. *By shadows.*—A good approximation may be made as follows. Plant a stake upon a level piece of ground, and give it a vertical position by means of a plumb line. Describe one or

more concentric circles on the ground from the foot of the stake as a centre. At the two instants before and after noon when the shadow of the stake extends to the same circle, the azimuths of the shadow east and west are equal. The points of the circle at which the shadow terminates at these instants being marked, let the included arc be bisected; the point of bisection and the centre of the stake then determine the meridian line. Theoretically, a small correction should be made for the sun's change of declination, but it would be quite superfluous in this method.

275. *By single altitudes.*—With an altitude and azimuth instrument, observe the altitude of a star at the instant of its passage over the middle vertical thread (at any time), and read the horizontal circle. Correct the observed altitude for refraction. Then, if

h = the true altitude,
φ = the latitude of the place of observation,
p = the star's polar distance,
A = the star's azimuth,
A' = the reading of the horizontal circle,

we have, from the triangle formed by the zenith, the pole, and the star,

$$\tan^2 \tfrac{1}{2} A = \frac{\sin(s - \varphi)\sin(s - h)}{\cos s \cos(s - p)} \tag{468}$$

in which

$$s = \tfrac{1}{2}(\varphi + h + p)$$

In this formula the latitude may be taken with the positive sign, whether north or south, and p is then to be reckoned from the elevated pole; consequently, also, A will be the azimuth reckoned from the elevated pole.

It is evident that in order to bring the telescope into the plane of the meridian we have only to revolve the instrument through the angle A, and therefore either $A' + A$ or $A' - A$, according to the direction of the graduations of the circle, will be the reading of the horizontal circle when the telescope is in the meridian.

The same method can be followed when the azimuth is observed with a compass and the altitude is measured with a sextant; and then $A' - A$ is the *variation of the compass.*

276. From the first equation of (50), φ and δ being constant, we have

$$dA = -\frac{dh}{\cos h \tan q}$$

and therefore an error in the observed altitude will have the least effect upon the computed azimuth when $\tan q$ is a maximum; that is, when the star is on the prime vertical. Therefore, in the practice of the preceding method the star should be as far from the meridian as possible.

277. *By equal altitudes of a star.*—Observe the azimuth of a star with an altitude and azimuth instrument, or a compass, when at the same altitude east and west of the meridian. The mean of the two readings of the instrument is the reading when its sight line is in the direction of the meridian. This is the method of Article 274, rendered accurate by the introduction of proper instruments for observing both the altitude and the azimuth.

278. If equal altitudes of the sun are employed, a correction for the change of the sun's declination is necessary, since equal azimuths will no longer correspond to equal altitudes. Let

$A' =$ the east azimuth at the first observation,
$A =$ " west " " second "
$\delta =$ the declination at noon,
$\Delta\delta =$ the increase of declination from the first to the second observation,

then, by (1), we have, h being the altitude in each case,

$$\sin(\delta - \tfrac{1}{2}\Delta\delta) = \sin\varphi \sin h - \cos\varphi \cos h \cos A'$$
$$\sin(\delta + \tfrac{1}{2}\Delta\delta) = \sin\varphi \sin h - \cos\varphi \cos h \cos A$$

the difference of which gives

$$2\cos\delta \sin\tfrac{1}{2}\Delta\delta = 2\cos\varphi \cos h \sin\tfrac{1}{2}(A + A') \sin\tfrac{1}{2}(A - A')$$

whence, since $\Delta\delta$ is but a few minutes, we have, with sufficient accuracy,

$$A - A' = \frac{\Delta\delta \cos\delta}{\cos\varphi \cos h \sin A} \qquad (469)$$

It will be necessary to note the times of the two observations in order to find $\Delta\delta$. If we take half the elapsed time as the hour angle t of the western observation, we shall have, instead of (469), the more convenient formula

$$A - A' = \frac{\Delta\delta}{\cos\varphi \sin t} \tag{470}$$

It will not be necessary to know the exact value of h, if only the same *instrumental* altitude is employed at both observations.

Now let A_1' and A_1 be the readings of the horizontal circle at the two observations, then the readings corresponding to equal azimuths are

$$A_1' \text{ and } A_1 - (A - A')$$

and, consequently, the reading for the meridian is the mean of these, or

$$\tfrac{1}{2}(A_1' + A_1) - \tfrac{1}{2}(A - A')$$

That is, the reading for the meridian is the mean of the observed readings diminished by one-half the correction (470). We here suppose the graduations to proceed from 0° to 360°, and from left to right.

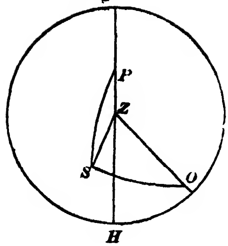
279. *By the angular distance of the sun from any terrestrial object.*— If the true azimuth of any object in view is known, the direction of the meridian is, of course, known also. The following method can be carried out with the sextant alone. Measure the angular distance of the sun's limb from any well-defined point of a distant terrestrial object, and note the time by a chronometer. Measure also the angular height of the terrestrial point above the horizontal plane. The correction of the chronometer being known, deduce the local apparent time, or the sun's hour angle t (Art. 54), and then with the sun's declination δ and the latitude φ compute the true altitude h and azimuth A of the sun by the formulæ (16), or

$$\tan M = \frac{\tan\delta}{\cos t}, \quad \tan A = \frac{\tan t \cos M}{\sin(\varphi - M)}, \quad \tan h = \cot(\varphi - M)\cos A \tag{471}$$

Now, let O, Fig. 37, be the apparent position of the terrestrial point, projected upon the celestial sphere; S the apparent place of the sun, Z the zenith, P the pole; and put

D = the apparent angular distance of the sun's centre from the terrestrial point
= the observed distance increased by the sun's semidiameter,
H = the apparent altitude of the point,
h' = the sun's apparent altitude,
a = the difference of the azimuth of the sun and the point,
A' = the azimuth of the point.

Fig. 37.



The apparent altitude h' will be deduced from the true altitude by adding the refraction and subtracting the parallax. Then in the triangle SZO we have given the three sides $ZS = 90° - h'$, $ZO = 90° - H$, $SO = D$, and hence the angle $SZO = a$ can be found by the formula

$$\tan^2 \tfrac{1}{2} a = \frac{\sin (s - H) \sin (s - h')}{\cos s \cos (s - D)} \qquad (472)$$

in which

$$s = \tfrac{1}{2} (H + h' + D)$$

Then we have

$$A' = A \pm a \qquad (473)$$

and the proper sign of a to be used in this equation must be determined by the position of the sun with respect to the object at the time of the observation.

If the altitude of the sun is observed, we can dispense with the computation of (471), and compute A by the formula (468). The chronometer will not then be required, but an approximate knowledge of the local time and the longitude is necessary in order to find δ from the Ephemeris.

If the terrestrial object is very remote, it will often suffice to regard its altitude as zero, and then we shall find that (472) reduces to

$$\tan \tfrac{1}{2} a = \sqrt{[\tan \tfrac{1}{2} (D + h') \tan \tfrac{1}{2} (D - h')]} \qquad (474)$$

This method is frequently used in hydrographic surveying to determine the meridian line of the chart.

Example.—From a certain point B in a survey the azimuth of a point C is required from the following observation:

Chronometer time	=	$4^h\ 12^m\ 12^s$		
Chronom. correction	=	$-\ 2\ \ 0$		
Local mean time	=	$4\ 10\ 12$		
Equation of time	=	$-\ 4\ 10.9$		
Local app. time, t	=	$4\ 6\ 1.1$		

Altitude of $C = H$	=	$0°\ 30'\ 20''$
Distance of the nearest limb of the sun from the point C	=	$48°\ 17'\ 10''$
Semidiameter	=	$16\ \ 1$
D	=	$48\ 33\ 11$

The sun's declination was $\delta = + 4° 16' 55''$, the latitude was $\varphi = + 38° 58' 50''$; and hence, by (471), we find

$$A = 74° 36' 36'' \qquad \begin{array}{rr} h = & 24° 37' 58'' \\ \text{Refraction and parallax} = & 1\ \ 54 \\ \hline h' = & 24\ \ 39\ \ 52 \end{array}$$

and, by (472),

$$a = 43° 35' 6''$$

Now, the sun was on the right of the object, and hence

$$A' = A - a = 31° 1' 30''$$

Therefore, a line drawn on the chart from B on the left of the line BC, making with it the angle $31° 1' 30''$, will represent the meridian.

280. *By two measures of the distance of the sun from a terrestrial object.*—In the practice of the preceding method with the sextant, it is not always practicable to measure the apparent altitude of the terrestrial object. We may then measure the distance of the sun from the object at two different times, and, first computing the altitude and azimuth of the sun at each observation, we may from these data compute the altitude of the object and the difference between its azimuth and that of the sun at either observation, by formulæ entirely analogous to those employed in computing the latitude and time from two altitudes, Art. 178, (304), (305), (306), and (307).

281. *By the azimuth of a star at a given time.*—When the time is known, the azimuth of the star is found by (471): hence we have only to direct the telescope of an altitude and azimuth instrument to the star at any time, and then compare the reading of its horizontal circle with the computed azimuth.

This method will be very accurate if a star near the pole is employed, since in that case an error in the time will produce a comparatively small error in the azimuth. It will be most accurate if the star is observed at its greatest elongation, as in the following article.

282. *By the greatest elongation of a circumpolar star.*—At the instant of the greatest elongation we have, by Art. 18,

$$\sin A = \frac{\cos \delta}{\cos \varphi}$$

in which A is the azimuth reckoned from the elevated pole. At this instant the star's azimuth reaches its maximum, and for a certain small interval of time appears to be stationary, so that the observer has time to set his instrument accurately upon the star.

In order to be prepared for the observation, the time of the elongation must be (at least approximately) known. The hour angle of the star is found by the formula

$$\cos t = \frac{\tan \varphi}{\tan \delta}$$

and from t and the star's right ascension the local time is found, Art. 55.

The pole star is preferred, on account of its extremely slow motion.

If the latitude is unknown, the direction of the meridian may nevertheless be obtained by observing the star at both its eastern and its western greatest elongations. The mean of the readings of the horizontal circle at the two observations is the reading for the meridian.

283. One of the most refined methods of determining the direction of the meridian is that by which the transit instrument is adjusted, or by which its deviation from the plane of the meridian is measured; for which see Vol. II.

284. At sea, the direction of the meridian, or the variation of the compass, is found with sufficient accuracy by the graphic process of Art. 271.

CHAPTER X.

ECLIPSES.

285. THE term *eclipse*, in astronomy, may be applied to any obscuration, total or partial, of the light of one celestial body by another. But the term *solar eclipse* is usually confined to an eclipse of the sun by the moon; while an eclipse of the sun by one of the inferior planets is called a *transit* of the planet. An eclipse of a star or a planet by the moon is called an *occultation* of the star or planet. A *lunar eclipse* is an eclipse of the moon by the earth.

All these phenomena may be computed upon the same general principles; and the investigation of solar eclipses, with which we shall set out, will involve nearly every thing required in the other cases.

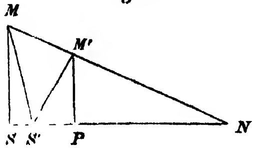
SOLAR ECLIPSES.

PREDICTION OF SOLAR ECLIPSES FOR THE EARTH GENERALLY.

286. For the purposes of general prediction, and before entering upon any precise computation, it is convenient to know the limits which determine the possibility of the occurrence of an eclipse for any part of the earth. These limits are determined in the following problem.

287. *To find whether near a given conjunction of the sun and moon, an eclipse of the sun will occur.*—In order that an eclipse may occur, the moon must be near the ecliptic, and, therefore, near one of the nodes of her orbit. Let NS (Fig. 38) be the ecliptic, N the moon's node, NM the moon's orbit, S and M the centres of the sun and moon at the time of conjunction in longitude, so that MS is a part of a circle of latitude and is perpendicular to

Fig. 38.



NS. Let S', M', be the centres of the sun and moon when at their least true distance, and put

β = the moon's latitude at conjunction = SM,
I = the inclination of the moon's orbit to the ecliptic,
λ = the quotient of the moon's motion in longitude divided by the sun's,
Σ = the least true distance = $S'M'$,
γ = the angle SMS'

We may regard NMS as a plane triangle; and, drawing $M'P$ perpendicular to NS, we find

$$SS' = \beta \tan \gamma \qquad SP = \lambda \beta \tan \gamma$$

and hence

$$S'P = \beta (\lambda - 1) \tan \gamma \qquad M'P = \beta - \lambda \beta \tan \gamma \tan I$$
$$\Sigma^2 = \beta^2 [(\lambda - 1)^2 \tan^2 \gamma + (1 - \lambda \tan I \tan \gamma)^2]$$

To find the value of γ for which this expression becomes a minimum, we put its derivative taken relatively to γ equal to zero, whence

$$\tan \gamma = \frac{\lambda \tan I}{(\lambda - 1)^2 + \lambda^2 \tan^2 I}$$

which substituted in the value of Σ^2 reduces it to

$$\Sigma^2 = \frac{\beta^2 (\lambda - 1)^2}{(\lambda - 1)^2 + \lambda^2 \tan^2 I}$$

If then we assume I' such that

$$\tan I' = \frac{\lambda}{\lambda - 1} \tan I \tag{475}$$

we have for the least true distance

$$\Sigma = \beta \cos I' \tag{476}$$

The apparent distance of the centres of the sun and moon as seen from the surface of the earth may be less than Σ by the difference of the horizontal parallaxes of the two bodies: so that if we put

π = the moon's horizontal parallax,
π' = the sun's " "

we have

$$\text{minimum apparent distance} = \Sigma - (\pi - \pi')$$

An eclipse will occur when this least apparent distance of the centres is less than the sum of the semidiameters of the bodies; and therefore, putting

s = the moon's semidiameter,
s' = the sun's "

we shall have, in case of eclipse,

$$\Sigma - (\pi - \pi') < s + s'$$

or

$$\beta \cos I' < \pi - \pi' + s + s' \qquad (477)$$

This formula gives the required limit with great precision; but, since I' is small, its cosine does not vary much for different eclipses, and we may in most cases employ its mean value. We have, by observation,

	Greatest values.	Least values.	Mean values.
I	5° 20′ 6″	4° 57′ 22″	5° 8′ 44″
π	61′ 32″	52′ 50″	57′ 11″
π'	9	8	8.5
s	16 46	14 24	15 35
s'	16 18	15 45	16 1
λ	16.19	10.89	13.5

From the mean values of I and λ we find the mean value of $\sec I' = 1.00472$, and the condition (477) becomes

$$\beta < (\pi - \pi' + s + s') \times 1.00472$$

or

$$\beta < \pi - \pi' + s + s' + (\pi - \pi' + s + s') \times .00472$$

where the small fractional term varies between 20″ and 30″. Taking its mean value, we have, with sufficient precision for all but very unusual cases,

$$\beta < \pi - \pi' + s + s' + 25'' \qquad (478)$$

If in this formula we substitute the greatest values of π, s, and s', and the least value of π', the limit

$$\beta < 1^\circ\ 34'\ 53''$$

is the greatest limit of the moon's latitude at the time of conjunction, for which an eclipse can occur.

If in (478) we substitute the least values of π, s, and s', and the greatest value of π', the limit

$$\beta < 1^\circ\ 23'\ 15''$$

is the least limit of the moon's latitude at the time of conjunction for which an eclipse can fail to occur.

Hence a solar eclipse is *certain* if at new moon $\beta < 1^\circ\ 23'\ 15''$, *impossible* if $\beta > 1^\circ\ 34'\ 53''$, and *doubtful* between these limits. For the doubtful cases we must apply (478), or for greater precision (477), using the actual values of π, π', s, s', λ, and I for the date.

EXAMPLE.—On July 18, 1860, the conjunction of the moon and sun in longitude occurs at $2^h\ 19^m.2$ Greenwich mean time: will an eclipse occur? We find at this time, from the Ephemeris,

$$\beta = 0^\circ\ 33'\ 18''\ 6$$

which, being within the limit $1^\circ\ 23'\ 15''$, renders an eclipse certain at this time.

Having thus found that an eclipse will be visible in some part of the earth, we can proceed to the exact computation of the phenomenon. The method here adopted is a modified form of BESSEL'S,* which is at once rigorous in theory and simple in practice. For the sake of clearness, I shall develop it in a series of problems.

Fundamental Equations of the Theory of Eclipses.

288. *To investigate the condition of the beginning or ending of a solar eclipse at a given place on the earth's surface.*—The observer sees the limbs of the sun and moon in apparent contact when he is situated in the surface of a cone which envelops and is in contact with the two bodies. We may have two such cones:

* See *Astronomische Nachrichten*, Nos. 151, 152, and, for the full development of the method with the utmost rigor, BESSEL'S *Astronomische Untersuchungen*, Vol. II. HANSEN'S development, based upon the same fundamental equations, but theoretically less accurate, may also be consulted with advantage: it is given in *Astronom. Nach.*, Nos. 339-342.

First. The cone whose vertex falls between the sun and the moon, as at V, Fig. 39, and which is called the *penumbral cone.* An observer at C, in one of the elements CBV of the cone, sees the points A and B of the limbs of the sun and moon in apparent *exterior* contact, which is either the first or the last contact; that is, either the beginning or the ending of the whole eclipse.

Fig. 39.

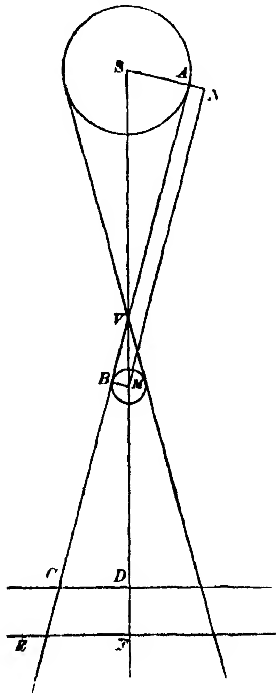
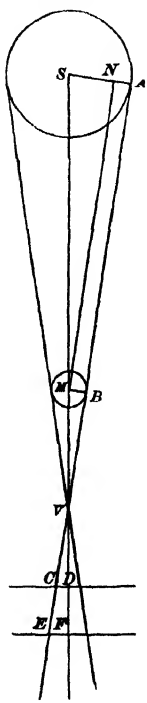


Fig. 40.



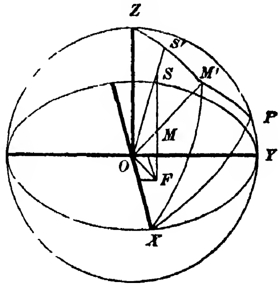
Second. The cone whose vertex is beyond the moon (in the direction of the earth), as at V, Fig. 40, and which is called the *umbral* cone, or cone of total shadow. An observer at C, in the element $CVBA$, sees the points A and B of the limbs of the sun and moon in apparent *interior* contact, which is the beginning or the ending of *annular* eclipse in case the observer is farther from the moon than the vertex of the cone (as in the figure), and which is either the beginning or the ending of *total* eclipse in case the observer is between the vertex of the cone and the moon.

If now a plane is passed through the point C, at right angles to the axis SVD of the cone, its intersection with the cone will

be a circle (the sun and moon being regarded as spherical) whose radius, CD, we shall call the *radius of the shadow* (penumbral or umbral) for that point. The condition of the occurrence of one of the above phases to an observer is, then, that *the distance of the point of observation from the axis of the shadow is equal to the radius of the shadow for that point.* The problems which follow will enable us to translate this condition into analytical language

289. *To find for any given time the position of the axis of the shadow.*—The axis of the cone of shadow produced to the celestial sphere meets it in that point in which the sun would be projected upon the sphere by an observer at the centre of the moon. Let O, Fig. 41, be the centre of the earth; S, that of the sun; M, that of the moon. The line MS produced to the infinite celestial sphere meets it in the common vanishing point of all lines parallel to MS; that is, in *the point* Z, in which the line OZ, drawn through the centre of the earth parallel to MS, meets the sphere. The position of the axis of the cone will be determined by the right ascension and declination of the point Z.

Fig. 41.



In order to determine the point Z, let the positions of the sun and moon be expressed by rectangular co-ordinates (Art. 32), of which the axis of x is the straight line drawn through the centre of the earth and the equinoctial points, the axis of y the intersection of the planes of the equator and solstitial colure, and the axis of z the axis of the equator. Let x be taken as positive towards the vernal equinox; y as positive towards the point of the equator whose right ascension is 90°; z as positive towards the north.

Let

α, δ, r = the right ascension, declination, and distance from the centre of the earth, respectively, of the moon's centre,

α', δ', r' = the right ascension, declination, and distance from the centre of the earth, respectively, of the sun's centre;

The co-ordinates x, y, z will be, by (41),

Of the sun.	*Of the moon.*
$r' \cos \delta' \cos \alpha'$	$r \cos \delta \cos \alpha$
$r' \cos \delta' \sin \alpha'$	$r \cos \delta \sin \alpha$
$r' \sin \delta'$	$r \sin \delta$

Now let another system of co-ordinates be taken parallel to the first, the centre of the moon being the origin. The position of the sun in this system will be determined by the right ascension and declination of the sun as seen from the moon; that is, by the right ascension and declination of the point Z.

If we put

$a, d =$ the right ascension and declination of the point Z,
$G =$ the distance of the centres of the sun and moon,

the co-ordinates of the sun in the new system are

$$\begin{array}{l} G \cos d \cos a \\ G \cos d \sin a \\ G \sin d \end{array}$$

But these co-ordinates are evidently equal respectively to the difference of the corresponding co-ordinates of the sun and moon in the first system; so that we have

$$\begin{aligned} G \cos d \cos a &= r' \cos \delta' \cos \alpha' - r \cos \delta \cos \alpha \\ G \cos d \sin a &= r' \cos \delta' \sin \alpha' - r \cos \delta \sin \alpha \\ G \sin d &= r' \sin \delta' - r \sin \delta \end{aligned}$$

which fully determine a, d, and G in terms of quantities which may be derived from the Ephemeris for a given time.

But, as a and d differ but little from α' and δ', it is expedient to put these equations under the following form. (See the similar transformation, Art. 92.)

$$\begin{aligned} G \cos d \sin (a - \alpha') &= - r \cos \delta \sin (\alpha - \alpha') \\ G \cos d \cos (a - \alpha') &= r' \cos \delta' - r \cos \delta \cos (\alpha - \alpha') \\ G \sin d &= r' \sin \delta' - r \sin \delta \end{aligned}$$

If these are divided by r', and we put

$$\frac{G}{r'} = g \qquad \frac{r}{r'} = b$$

they become

$$\left.\begin{aligned} g \cos d \sin (a - \alpha') &= - b \cos \delta \sin (\alpha - \alpha') \\ g \cos d \cos (a - \alpha') &= \cos \delta' - b \cos \delta \cos (\alpha - \alpha') \\ g \sin d &= \sin \delta' - b \sin \delta \end{aligned}\right\} \quad (479)$$

where the second members, besides the right ascensions and declinations, involve only the quantity b, which may be expressed in terms of the parallaxes as follows:

Let

$\pi =$ the moon's equatorial horizontal parallax,
$\pi' =$ the sun's " " "

then we have (Art. 89)

$$b = \frac{r}{r'} = \frac{\sin \pi'}{\sin \pi}$$

If, further,

$\pi_0 =$ the sun's mean horizontal parallax,

and r' is expressed in terms of the sun's mean distance from the earth, we have, as in (146),

$$\sin \pi' = \frac{\sin \pi_0}{r'}$$

and hence

$$b = \frac{\sin \pi_0}{r' \sin \pi} \tag{480}$$

which is the most convenient form for computing b, because r' and π are given in the Ephemeris, and π_0 is a constant.

290. The equations (479) are rigorously exact, but as b is only about $\frac{1}{400}$, and $\alpha - \alpha'$ at the time of an eclipse cannot exceed $1° \, 43'$, $a - \alpha'$ is a small arc never exceeding $17''$, which may be found by a brief approximative process with great precision The quotient of the first equation divided by the second gives

$$\tan (a - \alpha') = - \frac{b \cos \delta \sec \delta' \sin (\alpha - \alpha')}{1 - b \cos \delta \sec \delta' \cos (\alpha - \alpha')}$$

where the denominator differs from unity by the small quantity $b \cos \delta \sec \delta' \cos (\alpha - \alpha')$; and, since δ and δ' are nearly equal, this small difference may be put equal to b, and we may then write the formula thus:*

$$a - \alpha' = - \frac{b}{1 - b} \cos \delta \sec \delta' (\alpha - \alpha')$$

* Developing the formula for tan $(a - \alpha')$ in series, we have

$$a - \alpha' = - \frac{b \cos \delta \sec \delta' \sin (\alpha - \alpha')}{\sin 1''} - \frac{b^2 \cos^2 \delta \sec^2 \delta' \sin 2 (\alpha - \alpha')}{2 \sin 1''} - \&c.$$

where the second term cannot exceed $0''.04$, and the third term is altogether inap

If we take $\cos(a - \alpha') = 1$ and $\cos(\alpha - \alpha') = 1$, we have, from the second and third of (479),

$$g \cos d = \cos \delta' - b \cos \delta$$
$$g \sin d = \sin \delta' - b \sin \delta$$

whence

$$g \sin (d - \delta') = - b \sin (\delta - \delta')$$
$$g \cos (d - \delta') = 1 - b \cos (\delta - \delta')$$

from which follows

$$\tan (d - \delta') = - \frac{b \sin (\delta - \delta')}{1 - b \cos (\delta - \delta')}$$

or, nearly,*

$$d - \delta' = - \frac{b}{1 - b} (\delta - \delta')$$

From the above we also have, with sufficient precision for the subsequent application of g, the formula

$$g = 1 - b$$

The formulæ which determine the point Z, together with the quantity G, will, therefore, be

$$\left.\begin{aligned} a &= \alpha' - \frac{b}{1 - b} \cos \delta \sec \delta' (\alpha - \alpha') \\ d &= \delta' - \frac{b}{1 - b} (\delta - \delta') \\ g &= 1 - b, \qquad G = r'g \end{aligned}\right\} \quad (481)$$

and in many cases it will suffice to take the extremely simple forms

$$a = \alpha' - b (\alpha - \alpha') \qquad d = \delta' - b (\delta - \delta')$$

291. *To find the distance of a given place of observation from the axis of the shadow at a given time.*—Let the positions of the sun,

---

preciable. The formula adopted in the text is the same as

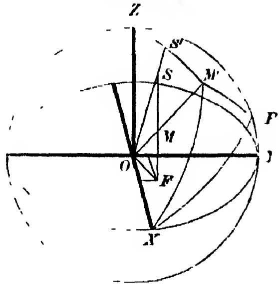
$$a - \alpha' = - b \cos \delta \sec \delta' (\alpha - \alpha') (1 - b)^{-1}$$
$$= - b \cos \delta \sec \delta' (\alpha - \alpha') - b^2 \cos \delta \sec \delta' (\alpha - \alpha') - \&c$$

which, since $\cos \delta \sec \delta'$ may in the second term be put equal to unity, differs from the complete series only by terms of the third order. The error of the approximate formula is, therefore, something less than 0''.01.

* The error of this formula, as can be easily shown, will never exceed 0''.088.

the moon, and the observer be referred by rectangular co-ordinates to three planes passing through the centre of the earth, of which the plane of xy shall always be at right angles to the axis of the shadow, and will here be called the *principal plane of reference.* Let the plane of yz be the plane of the declination circle passing through the point Z. The plane of xz will, of course, be at right angles to the other two.

Fig. 41 (bis).



The axis of z will then be the line OZ, Fig. 41, drawn through the centre of the earth parallel to the axis of the shadow, and will be reckoned as positive towards Z. The axis of y will be the intersection, OY, of the plane of the declination circle through Z with the principal plane, and will be taken as positive towards the north. The axis of x will be the intersection, OX, of the plane of the equator with the principal plane, and will be taken as positive towards that point, X, whose right ascension is $90° + a$.

Let M' and S' be the true places of the moon and sun upon the celestial sphere, P the north pole; then, if we put

x, y, z = the co-ordinates of the moon,

we have, by (Art. 31),

$$x = r \cos M'X$$
$$y = r \cos M'Y$$
$$z = r \cos M'Z$$

which, by the formulæ of Spherical Trigonometry applied to the triangles $M'PX$, $M'PY$, $M'PZ$, become

$$\left.\begin{aligned} x &= r \cos \delta \sin (\alpha - a) \\ y &= r [\sin \delta \cos d - \cos \delta \sin d \cos (\alpha - a)] \\ z &= r [\sin \delta \sin d + \cos \delta \cos d \cos (\alpha - a)] \end{aligned}\right\} \quad (482)$$

or

$$\left.\begin{aligned} x &= r \cos \delta \sin (\alpha - a) \\ y &= r [\sin (\delta - d) \cos^2 \tfrac{1}{2} (\alpha - a) + \sin (\delta + d) \sin^2 \tfrac{1}{2} (\alpha - a)] \\ z &= r [\cos (\delta - d) \cos^2 \tfrac{1}{2} (\alpha - a) - \cos (\delta + d) \sin^2 \tfrac{1}{2} (\alpha - a)] \end{aligned}\right\} \quad (482^*)$$

and if the equatorial radius of the earth is taken as the unit of

r, x, y, z, we shall have the value of r, required in these equations, by the formula

$$r = \frac{1}{\sin \pi}$$

The co-ordinates x and y of the sun in this system are the same as those of the moon, and the third co-ordinate is $z + G$; but the method of investigation which we are here following does not require their use.

Now let

ξ, η, ζ = the co-ordinates of the place of observation,
φ = the latitude of the place,
φ' = the reduced latitude (Art. 81),
ρ = the radius of the terrestrial spheroid for the latitude φ,
μ = the given sidereal time;

then, if in Fig. 41 we had taken M for the place of observation, M' would have been the geocentric zenith with the right ascension μ and declination φ', and, the distance of the place from the origin being ρ, we should have found

$$\left.\begin{aligned}\xi &= \rho \cos \varphi' \sin (\mu - a)\\ \eta &= \rho\,[\sin \varphi' \cos d - \cos \varphi' \sin d \cos (\mu - a)]\\ \zeta &= \rho\,[\sin \varphi' \sin d + \cos \varphi' \cos d \cos (\mu - a)]\end{aligned}\right\}(483)$$

These equations, if we determine A and B by the conditions

$$\begin{aligned}A \sin B &= \rho \sin \varphi'\\ A \cos B &= \rho \cos \varphi' \cos (\mu - a)\end{aligned}$$

may be computed under the form

$$\left.\begin{aligned}\xi &= \rho \cos \varphi' \sin (\mu - a)\\ \eta &= A \sin (B - d)\\ \zeta &= A \cos (B - d)\end{aligned}\right\}(483^*).$$

The equations (482) might be similarly treated; but the most accurate form for their computation is (482*).

The quantity $\mu - a$ is the hour angle of *the point Z* for the meridian of the given place. To facilitate its computation, it is convenient to find first its value for the Greenwich meridian. Thus, if we put for any given Greenwich mean time T

μ_1 = the hour angle of the point Z at the Greenwich meridian,
ω = the longitude of the given place,

we have

$$\mu - a = \mu_1 - \omega$$

To find μ_1 we have only to convert the Greenwich mean time T into sidereal time and to subtract a.

By means of the formulæ (482) and (483) the co-ordinates of the moon and of the place of observation can be accurately computed for any given time. Now, the co-ordinates x and y of the moon are also those of every point of the axis of the shadow: so that if we put

Δ = the distance of the place of observation from the axis of the shadow,

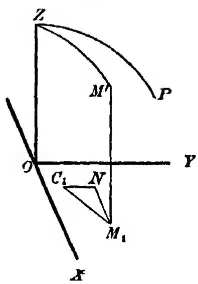
we have, evidently,

$$\Delta^2 = (x - \xi)^2 + (y - \eta)^2 \tag{484}$$

[The co-ordinates z and ζ have also been found, as they will be required hereafter.]

292. The distance Δ may be determined under another form, which we shall hereafter find useful. Let M', Fig. 42, be the apparent position of the moon's centre in the celestial sphere as seen from the place of observation; P the north pole; Z the point where the axis of the cone of shadow meets the sphere, as in Fig. 41; M_1, C_1, the projections of the moon's centre and of the place of observation on the principal plane. The distance C_1M_1 is equal to Δ, and is the projection of the line joining the place of observation and the moon's centre. The plane by which this line is projected contains the axis of the cone of shadow, and its intersection with the celestial sphere is, therefore, a great circle which passes through Z, and of which ZM' is a portion. Hence it follows that C_1M_1 makes the same angle with the axis of y that $M'Z$ makes with PZ: so that if we draw C_1N and M_1N parallel to the axes of y and x respectively, and put

Fig. 42.



$$Q = PZM' = NC_1M_1$$

we have, from the right triangle C_1M_1N,

$$\left.\begin{aligned} \Delta \sin Q &= x - \xi \\ \Delta \cos Q &= y - \eta \end{aligned}\right\} \tag{485}$$

the sum of the squares of which gives again the formula (484).

293. *To find the radius of the shadow on the principal plane, or on any given plane parallel to the principal plane.*—This radius is evidently equal to the distance of the vertex of the cone of shadow from the given plane, multiplied by the tangent of the angle of the cone. In Figs. 39 and 40, p. 440, let EF be the radius of the shadow on the principal plane, CD the radius on a parallel plane drawn through C. Let

H the apparent semidiameter of the sun at its mean distance,
$k =$ the ratio of the moon's radius to the earth's equatorial radius,
$f =$ the angle of the cone $= EVF$,
$c =$ the distance of the vertex of the cone above the principal plane $= VF$,
$\zeta =$ the distance of the given parallel plane above the principal plane $= DF$,
$l =$ the radius of the shadow on the principal plane $= EF$,
$L =$ the radius of the shadow on the parallel plane $= CD$.

If the mean distance of the sun from the earth is taken as unity, we have

$$\begin{aligned} &\text{the earth's radius} = \sin \pi_0, \\ &\text{the moon's radius} = k \sin \pi_0 = MB, \\ &\text{the sun's radius} \quad = \sin H \quad = SA, \end{aligned}$$

and, remembering that $G = r'g$ found by (481) is the distance MS, we easily deduce from the figures

$$\sin f = \frac{\sin H \pm k \sin \pi_0}{r'g} \tag{486}$$

in which the upper sign corresponds to the penumbral and the lower to the umbral cone.

The numerator of this expression involves only constant quantities. According to Bessel, $H = 959''.788$; Encke found $\pi_0 = 8''.57116$; and the value of k, found by Burckhardt from eclipses and occultations, is $k = 0.27227$;* whence we have

$$\log [\sin H + k \sin \pi_0] = 7.6688033 \text{ for exterior contacts,}$$
$$\log [\sin H - k \sin \pi_0] = 7.6666913 \text{ for interior contacts.}$$

* The value of k here adopted is precisely that which the more recent investigation of Oudemans (*Astron. Nach.*, Vol. LI. p. 30) gives for eclipses of the sun. For occultations, a slightly increased value seems to be required.

Now, taking the earth's equatorial radius as unity, we have

$$VM = \frac{k}{\sin f}$$

$$MF = z \quad \text{(Art. 291)}$$

and hence

$$c = z \pm \frac{k}{\sin f} \tag{487}$$

the upper sign being used for the penumbra and the lower for the umbra.

We have, then,

$$\left.\begin{aligned} l &= c \tan f = z \tan f \pm k \sec f \\ L &= (c - \zeta) \tan f = l - \zeta \tan f \end{aligned}\right\} \tag{488}$$

For the penumbral cone, $c - \zeta$ is always positive, and therefore L is positive also.

For the umbral cone, $c - \zeta$ is negative when the vertex of the cone falls below the plane of the observer, and in this case we have total eclipse: therefore for the case of total eclipse we shall have $L = (c - \zeta) \tan f$ a negative quantity. It is usual to regard the radius of the shadow as a positive quantity, and therefore to change its sign for this case; but the analytical discussion of our equations will be more general if we preserve the negative sign of L as the characteristic of total eclipse.

When the vertex of the umbral cone falls above the plane of the observer, L is positive, and we have the case of annular eclipse.

For brevity we shall put

$$\left.\begin{aligned} i &= \tan f \\ l &= ic \\ L &= l - i\zeta \end{aligned}\right\} \tag{489}$$

294. *The analytical expression of the condition of beginning or ending of eclipse* is

$$\varDelta = L$$

or, by (484) and (489),

$$(x - \xi)^2 + (y - \eta)^2 = (l - i\zeta)^2 \tag{490}$$

It is convenient, however, to substitute the two equations (485) for this single one, after putting L for $\varDelta$, so that

$$\left.\begin{aligned} (l - i\zeta) \sin Q &= x - \xi \\ (l - i\zeta) \cos Q &= y - \eta \end{aligned}\right\} \tag{491}$$

may be taken as the conditions which determine the beginning or ending of an eclipse at a given place.

The equation (490), which is only expressed in a different form by (491), is to be regarded as the fundamental equation of the theory of eclipses.

295. By Art. 292, so long as Δ is regarded as a positive quantity, Q is the position angle of the moon's centre at the point Z; and since the arc joining the point Z and the centre of the moon also passes through the centre of the sun, Q is the common position angle of both bodies.

Again, since in the case of a contact of the limbs the arc joining the centres passes through the point of contact, Q will also be the position angle of this point when all three points—sun's centre, moon's centre, and point of contact—lie on the same side of Z. In the case of total eclipse, however, the point of contact and the moon's centre evidently lie on opposite sides of the point Z; and if $l - i\zeta$ in (490) were a positive quantity, the angle Q which would satisfy these equations would still be the position angle of the moon's centre, but would differ 180° from the position angle of the point of contact. But, since we shall preserve the negative sign of $l - i\zeta$ for total eclipse (Art. 293), (and thereby give Q values which differ 180° from those which follow from a positive value), *the angle Q will in all cases be the position angle of the point of contact.*

296. The quantities a, d, x, y, l, and i may be computed by the formulæ (480), (481), (482), (486), (487), (488), for any given time at the first meridian, since they are all independent of the place of observation. In order to facilitate the application of the equations (490) and (491), it is therefore expedient to compute these general quantities for several equidistant instants preceding and following the time of conjunction of the sun and moon, and to arrange them in tables from which their values for any time may be readily found by interpolation.

The quantities x and y do not vary uniformly; and in order to obtain their values with accuracy from the tables for any time, we should employ the second and even the third differences in the interpolation. This is effected in the most simple manner by the following process. Let the times for which x and y have been computed be denoted by $T_0 - 2^h$, $T_0 - 1^h$, T_0, $T_0 + 1^h$,

$T_0 + 2^h$, the interval being one hour of mean time; and let the values of x and y for these times be denoted by x_{-2}, x_{-1}, &c., y_{-2}, y_{-1}, &c. Let the *mean* hourly changes of x and y from the epoch T_0 to any time $T = T_0 + \tau$ be denoted by x' and y'. Then the values of x' and y' for the instants $T_0 - 2^h$, $T_0 - 1^h$, &c. will be formed as in the following scheme, where c denotes the third difference of the values of x as found from the series x_{-2}, x_{-1}, &c. according to the form in Art. 69, and the difference for the instant T_0 is found by the first formula of (77). The form for computing y' is the same.

Time.	x	x'
$T_0 - 2^h$	x_{-2}	$\frac{1}{2}(x_0 - x_{-2})$
$T_0 - 1^h$	x_{-1}	$x_0 - x_{-1}$
T_0	x_0	$\frac{1}{2}(x_1 - x_{-1}) - \frac{1}{6}c$
$T_0 + 1^h$	x_1	$x_1 - x_0$
$T_0 + 2^h$	x_2	$\frac{1}{2}(x_2 - x_0)$

If then we require x and y for a time $T = T_0 + \tau$, we take x' and y' from the table for this time, and we have

$$x = x_0 + x'\tau$$
$$y = y_0 + y'\tau$$

297. EXAMPLE.—Compute the elements of the solar eclipse of July 18, 1860.

The mean Greenwich time of conjunction of the sun and moon in right ascension is July 18, $2^h\ 8^m\ 56^s$. The computation of the elements will therefore be made for the Greenwich hours 0, 1, 2, 3, 4, and 5. For these hours we take the following quantities from the American Ephemeris:

For the Moon.

Greenwich mean time.	α	δ	π
July 18, 0^h	116° 44′ 24″.30	21° 52′ 20″.3	59′ 45″.80
1	117 21 59 .10	42 32 8	47 .13
2	117 59 30 .45	32 36 .4	48 .44
3	118 36 58 .35	22 31 .2	49 .72
4	119 14 22 .65	12 17 .2	50 .98
5	119 51 43 .35	1 54 .6	52 .22

For the Sun.

Greenwich mean time.	α'	δ'	$\log r'$
July 18, 0^h	117° 59′ 41″.85	20° 57′ 56″.20	0.0069675
1	118 2 12 .50	57 29 .42	61
2	118 4 43 .14	57 2 .60	47
3	118 7 13 .77	56 35 .75	33
4	118 9 44 .39	56 8 .86	19
5	118 12 15 .09	55 41 .94	05

The formulæ to be employed will be here recapitulated, for convenient reference.

I. For the elements of *the point Z:*

$$b = \frac{\sin \pi_0}{r' \sin \pi} \qquad \log \sin \pi_0 = 5.61894$$

$$a = \alpha' - \frac{b}{1-b} \cos \delta \sec \delta' (\alpha - \alpha') \quad \text{or, nearly,} \quad a = \alpha' - b(\alpha - \alpha')$$

$$d = \delta' - \frac{b}{1-b} (\delta - \delta') \qquad \text{"} \qquad d = \delta' - b(\delta - \delta')$$

$$g = 1 - b$$

II. The moon's co-ordinates:

$$r = \frac{1}{\sin \pi}$$

$$x = r \cos \delta \sin (\alpha - a)$$

$$y = r \sin (\delta - d) \cos^2 \tfrac{1}{2}(\alpha - a) + r \sin (\delta + d) \sin^2 \tfrac{1}{2}(\alpha - a)$$

$$z = r \cos (\delta - d) \cos^2 \tfrac{1}{2}(\alpha - a) - r \cos (\delta + d) \sin^2 \tfrac{1}{2}(\alpha - a)$$

III. The angle of the cone of shadow and the radius of the shadow:

For penumbra: or *exterior contacts.*	*For umbra:* or *interior contacts*
$\sin f = \frac{[7.668803]}{r'g}$	$\sin f = \frac{[7.666691]}{r'g}$
$c = z + \frac{k}{\sin f}$, $\log k = 9.435000$,	$c = z - \frac{k}{\sin f}$
$i = \tan f$	$i = \tan f$
$l = ic$	$l = ic$

IV. The values of a, d, x, y, $\log i$, and l, will then be tabulated and the differences x' and y' formed according to Art. 296.

I give the computation for the three hours 1^h, 2^h, and 3^h, *in extenso.*

I. Elements of *the point Z.*

		1^h	2^h	3^h
	$\alpha - \alpha'$	$-0°\ 40'\ 13''.40$	$-0°\ 5'\ 12''.69$	$+0°\ 29'\ 44''.58$
	$\delta - \delta'$	$+\ 45\ \ 3\ .38$	$+\ 35\ 33\ .80$	$+\ 25\ 55\ .45$
	$\log \operatorname{cosec} \pi = \log r$	1.7596999	1.7595414	1.7593865
	ar. co. $\log r'$	9.9930339	9.9930353	9.9930367
	Constant $\log \sin \pi_0$	5.61894		
(1)	$\log b$	7.37167	7.37152	7.37136
(2)	ar. co. $\log (1 - b)$	0.001023	0.001023	0.001022
	$\log \cos \delta$	9.96805	9.96855	9.96905
	$\log \sec \delta'$	0.02973	0.02970	0.02968
	$\log (\alpha - \alpha')$	$n3.38263$	$n2.49511$	3.25154
	$\log (a - \alpha')$	0.75310	9.86590	$n0.62265$
	$a - \alpha'$	$+\ 5''.66$	$+\ 0''.73$	$-\ 4''.19$
(1) + (2)		7.37269	7.37254	7.37238
	$\log (\delta - \delta')$	3.43191	3.32915	3.19185
	$\log (d - \delta')$	$n0.80460$	$n0.70169$	$n0.56423$
	$d - \delta'$	$-\ 6''.38$	$-\ 5''.03$	$-\ 3''.67$
	a	$118°\ 2'\ 18''.16$	$118°\ 4'\ 43''.87$	$118°\ 7'\ 9''.58$
	d	$20\ 57\ 23\ .04$	$20\ 56\ 57\ .57$	$20\ 56\ 32\ .08$
	$\log (1 - b) = \log g$	9.998977	9.998977	9.998978

II. Co-ordinates x, y, and z.

		1^h	2^h	3^h
	$\alpha - a$	$-0°\ 40'\ 19''.06$	$-0°\ 5'\ 13''.42$	$+0°\ 29'\ 48''.77$
	$\delta - d$	$+\ 45\ \ 9\ .76$	$+\ 35\ 38\ .83$	$+\ 25\ 59\ .12$
	$\delta + d$	$42\ 39\ 55\ .84$	$42\ 29\ 33\ .97$	$42\ 19\ \ 3\ .28$
	$\log \sin (\alpha - a)$	$n8.0692116$	$n7.1817014$	7.9381239
	$\log \cos \delta$	9.9680502	9.9685481	9.9690490
	$\log r \cos \delta \sin (\alpha - a) = \log x$	$n9.7969617$	$n8.9097909$	9.6665594
	x	-0.626559	-0.081244	0.464044
	$\log \cos^2 \frac{1}{2} (\alpha - a)$	9.9999850	9.9999998	9.9999920
	$\log \sin (\delta - d)$	8.1184932	8.0157434	7.8784502
	$\log (3) = \log r \sin (\delta - d) \cos^2 \frac{1}{2} (\alpha - a)$	9.8781781	9.7752846	9.6378287
	$\log \sin^2 \frac{1}{2} (\alpha - a)$	5.5363780	3.7613394	5.2741910
	$\log \sin (\delta + d)$	9.8310485	9.8296235	9.8281695
	$\log (4) = \log r \sin (\delta + d) \sin^2 \frac{1}{2} (\alpha - a)$	7.1271264	5.3405043	6.8617470
	(3)	$+0.755402$	$+0.596053$	$+0.434329$
	(4)	$+0.001340$	$+0.000022$	$+0.000727$
(3) + (4)	y	$+0.756742$	$+0.596075$	$+0.435056$
	$\log \cos (\delta - d)$	9.9999625	9.9999766	9.9999876
	$\log (5) = \log r \cos (\delta - d) \cos^2 \frac{1}{2} (\alpha - a)$	1.7596474	1.7595178	1.7593661
	$\log \cos (\delta + d)$	9.8664780	9.8676822	9.8688939
	$\log (6) = \log r \cos (\delta + d) \sin^2 \frac{1}{2} (\alpha - a)$	7.1625559	5.3885630	6.9024714
	$\log [(5) - (6)] = \log z$	1.7596364	1.7595176	1.7593601

III. Log i and l, for exterior contacts. [Constant log = 7.668803]

	1^h	2^h	3^h
$\log r'g$	0.005943	0.005942	0.005941
Const. — $\log r'g = \log \sin f$	7.662860	7.662861	7.662862
$\log \sec f$	0.000005		
$\log k \operatorname{cosec} f$	1.772140	1.772139	1.772138
$\log [z + k \operatorname{cosec} f] = \log c$	2.066963	2.066904	2.066826
$\log \tan f = \log i$	7.662865	7.662866	7.662867
$\log ic = \log l$	9.729828	9.729770	9.729693
l	0.536819	0.536747	0.536652

Log i and l for interior contacts. [Constant log = 7.666691]

Const. — $\log r'g = \log \sin f$	7.660748	7.660749	7.660750
$\log \sec f$	0.000005		
$\log k \operatorname{cosec} f$	1.774252	1.774251	1.774250
$\log [z - k \operatorname{cosec} f] = \log c$	n0.293985	n0.297413	n0.301919
$\log \tan f = \log i$	7.660753	7.660754	7.660755
$\log ic = \log l$	n7.954738	n7.958167	n7.962674
l	—0.009010	—0.009082	—0.009176

IV. The computation being made for the other hours in the same manner, the results are collected in the following tables.

	a	d	Exterior Contacts.		Interior Contacts.	
			l	$\log i$	l	$\log i$
0^h	117° 59′ 52″.44	20° 57′ 48″.50	0.536867	7.662864	— 0.008960	7.660752
1	118 2 18 .16	57 23 .04	0.536819	65	0.009010	53
2	4 43 .87	56 57 .57	0.536747	66	0.009082	54
3	7 9 .58	56 32 .08	0.536652	67	0.009176	55
4	9 35 .27	56 6 .58	0.536533	68	0.009293	56
5	12 0 .95	55 41 .06	0.536391	69	0.009434	57

	x	Δ_1	Δ_2	Δ_3	y	Δ_1	Δ_2	Δ_3
0^h	— 1.171856				+ 0.917040			
		+ 0.545297				— 0.160298		
1	— 0.626559		+ 18		+ 0.756742		—369	
		0.545315		—45		— 0.160667		+17
2	— 0.081244		— 27		+ 0.596075		—352	
		0.545288		—60		— 0.161019		+19
3	+ 0.464044		— 87		+ 0.435056		—333	
		0.545201		—75		— 0.161352		+20
4	+ 1.009245		—162		+ 0.273704		—313	
		0.545039				— 0.161665		
5	+ 1.554284				+ 0.112039			

For the values of the hourly differences of x and y, we find from the above, by Art. 296,

	x'	$\log x'$	y'	$\log y'$
0^h	0.545306	9.736640	-0.160483	$n9.205429$
1	0.545315	648	-0.160667	5927
$T_0 = 2$	0.545310	644	-0.160846	6410
3	0.545288	626	-0.161019	6877
4	0.545245	592	-0.161186	7327
5	0.545176	537	-0.161345	7756

and for any given time $T = T_0 + \tau$, we have

$$\left.\begin{aligned} x &= -0.081244 + x'\tau \\ y &= +0.596075 + y'\tau \end{aligned}\right\} \quad (492)$$

Finally, to facilitate the computation of the hour angle $\mu - a = \mu_1 - \omega$ (Art. 291), we prepare the values of μ_1 for each of the Greenwich hours. Thus, for $T = 1^h$, we have

From the Ephemeris, July 18, 1860,

Sid. time at mean noon	$=$	$7^h\ 46^m\ 4^s.03$
Sid. equivalent of 1^h mean t.	$=$	1 0 9 .86
Greenwich sid. time	$=$	8 46 13 .89
" " " in arc,	$=$	$131°\ 33'\ 28''.35$
a	$=$	118 2 18 .16
μ_1	$=$	13 31 10 .19

Thus we form the following table, to which is also added for future use the value of the logarithm of

μ' = the hourly difference of μ_1 in parts of the radius;

	μ_1	Hourly diff.
0^h	$358°\ 31'\ 8''.0$	
1	13 31 10 .2	
2	28 31 12 .3	$54002''.15$
3	43 31 14 .4	
4	58 31 16 .6	
5	73 31 18 .7	

$$\log \mu' = \log 54002''.15 \sin 1'' = 9.417986$$

I proceed to consider the principal problems relating to the general prediction of eclipses, in which the preceding results will be applied.

Outline of the Shadow on the Surface of the Earth.

298. *To find the outline of the moon's shadow upon the earth at a given time.*—This outline is the intersection of the cone of shadow with the earth's surface; or, it is the curve on the surface of the earth from every point of which a contact of the sun's and moon's limbs may be observed at the given time. Let

$$T = \text{the given time reckoned at the first meridian,}$$

and let a, d, x, y, l, and $\log i$ be taken from the general tables of the eclipse for this time. Then the co-ordinates ξ, η, ζ of any place at which a contact may be observed at the given time must satisfy the conditions (491),

$$\left.\begin{aligned}(l - i\zeta)\sin Q &= x - \xi \\ (l - i\zeta)\cos Q &= y - \eta\end{aligned}\right\} \quad (493)$$

Let

$$\begin{aligned}\vartheta &= \text{the hour angle of } \textit{the point } Z, \\ \omega &= \text{the west longitude of the place;}\end{aligned}$$

then we have

$$\vartheta = \mu - a = \mu_1 - \omega$$

and the equations (483) become

$$\left.\begin{aligned}\xi &= \rho\cos\varphi'\sin\vartheta \\ \eta &= \rho\sin\varphi'\cos d - \rho\cos\varphi'\sin d\cos\vartheta \\ \zeta &= \rho\sin\varphi'\sin d + \rho\cos\varphi'\cos d\cos\vartheta\end{aligned}\right\} \quad (494)$$

The five equations in (493) and (494) involve the six variables ξ, η, ζ, φ', ϑ, and Q, any one of which may be assumed arbitrarily (excluding, of course, assumed values that give impossible or imaginary results); then for each assumed value of the arbitrary quantity we shall have five equations, which fully determine five unknown quantities, and thereby one point of the required curve. I shall take Q as the arbitrary variable.

In the present form of the equations (494), they involve the unknown quantity ρ, which being dependent upon φ' cannot be determined until the latter is found. This seems to involve the necessity of at first neglecting the compression of the earth, by putting $\rho = 1$, and after an approximate value of φ' has been found, and thereby also the value of ρ, repeating the computation. But, by a simple transformation given by BESSEL, this double computation is rendered unnecessary, and the compression of

the earth is taken into account from the beginning. If φ is the geographical latitude, we have (Art. 82)

$$\rho \cos \varphi' = \frac{\cos \varphi}{\sqrt{(1 - ee \sin^2 \varphi)}} \qquad \rho \sin \varphi' = \frac{\sin \varphi\,(1 - ee)}{\sqrt{(1 - ee \sin^2 \varphi)}}$$

in which

$$\log ee = 7.824409 \qquad \log \sqrt{(1 - ee)} = 9.9985458$$

If we take a new variable φ_1, such that

$$\cos \varphi_1 = \frac{\cos \varphi}{\sqrt{(1 - ee \sin^2 \varphi)}}$$

we shall have

$$\sin \varphi_1 = \sqrt{(1 - \cos^2 \varphi_1)} = \frac{\sin \varphi \sqrt{(1 - ee)}}{\sqrt{(1 - ee \sin^2 \varphi)}}$$

or

$$\cos \varphi_1 = \rho \cos \varphi'$$
$$\sqrt{(1 - ee)} \sin \varphi_1 = \rho \sin \varphi'$$
$$\tan \varphi = \frac{\tan \varphi_1}{\sqrt{(1 - ee)}}$$

Hence the equations (494) become

$$\begin{aligned}
\xi &= \cos \varphi_1 \sin \vartheta \\
\eta &= \sin \varphi_1 \cos d \sqrt{(1 - ee)} - \cos \varphi_1 \sin d \cos \vartheta \\
\zeta &= \sin \varphi_1 \sin d \sqrt{(1 - ee)} + \cos \varphi_1 \cos d \cos \vartheta
\end{aligned}$$

Put

$$\left.\begin{aligned}
\rho_1 \sin d_1 &= \sin d & \rho_2 \sin d_2 &= \sin d \sqrt{(1 - ee)} \\
\rho_1 \cos d_1 &= \cos d \sqrt{(1 - ee)} & \rho_2 \cos d_2 &= \cos d
\end{aligned}\right\} \qquad (495)$$

The quantities ρ_1, d_1, ρ_2, d_2, may be computed for the same times as the other quantities in the tables of the eclipse, and hence obtained by interpolation for the given time. The factors ρ_1 and ρ_2 will be sensibly constant for the whole eclipse. We now have

$$\begin{aligned}
\xi &= \cos \varphi_1 \sin \vartheta \\
\eta &= \rho_1 \sin \varphi_1 \cos d_1 - \rho_1 \cos \varphi_1 \sin d_1 \cos \vartheta \\
\zeta &= \rho_2 \sin \varphi_1 \sin d_2 + \rho_2 \cos \varphi_1 \cos d_2 \cos \vartheta
\end{aligned}$$

Let us put

$$\eta_1 = \frac{\eta}{\rho_1}$$

and assume ζ_1, so that

$$\xi^2 + \eta_1^2 + \zeta_1^2 = 1 \qquad (496)$$

or, which is equivalent, let us take the system

$$\left.\begin{aligned} \xi &= \cos \varphi_1 \sin \vartheta \\ \eta_1 &= \sin \varphi_1 \cos d_1 - \cos \varphi_1 \sin d_1 \cos \vartheta \\ \zeta_1 &= \sin \varphi_1 \sin d_1 + \cos \varphi_1 \cos d_1 \cos \vartheta \end{aligned}\right\} \tag{497}$$

The quantity ζ_1 differs so little from ζ that we may in practice substitute one for the other in the small term $i\zeta$; but if theoretical accuracy is desired we can readily find ζ when ζ is known; for the second and third of (497) give

$$\begin{aligned} \cos \varphi_1 \cos \vartheta &= -\eta_1 \sin d_1 + \zeta_1 \cos d_1 \\ \sin \varphi_1 &= \eta_1 \cos d_1 + \zeta_1 \sin d_1 \end{aligned}$$

which substituted in the value of ζ give

$$\zeta = \rho_2 \zeta_1 \cos (d_1 - d_2) - \rho_2 \eta_1 \sin (d_1 - d_2) \tag{498}$$

Our problem now takes the following form. We have first the three equations

$$\left.\begin{aligned} (l - i\zeta_1) \sin Q &= x - \xi \\ (l - i\zeta_1) \cos Q &= y - \rho_1 \eta_1 \\ \xi^2 + \eta_1^2 + \zeta_1^2 &= 1 \end{aligned}\right\} \tag{499}$$

which for each assumed value of Q determine ξ, η_1, and ζ_1. Then we have

$$\left.\begin{aligned} \cos \varphi_1 \sin \vartheta &= \xi \\ \cos \varphi_1 \cos \vartheta &= -\eta_1 \sin d_1 + \zeta_1 \cos d_1 \\ \sin \varphi_1 &= \eta_1 \cos d_1 + \zeta_1 \sin d_1 \end{aligned}\right\} \tag{500}$$

which determine φ_1 and ϑ. Then the latitude and longitude of a point of the required outline are found by the equations

$$\tan \varphi = \frac{\tan \varphi_1}{\sqrt{(1 - ee)}} \qquad \omega = \mu_1 - \vartheta \tag{501}$$

To solve (499), let β and γ be found by the equations

$$\left.\begin{aligned} \sin \beta \sin \gamma &= x - l \sin Q = a \\ \sin \beta \cos \gamma &= \frac{y}{\rho_1} - \frac{l \cos Q}{\rho_1} = b \end{aligned}\right\} \tag{502}$$

then we have

$$\begin{aligned} \xi &= \sin \beta \sin \gamma + i\zeta_1 \sin Q \\ \eta_1 &= \sin \beta \cos \gamma + i\zeta_1 \cos Q \end{aligned}$$

where we have omitted ρ_1 as a divisor of the small term $i\zeta_1 \cos Q$, since we have very nearly $\rho_1 = 1$. Substituting these values in the last equation of (499), we find

$$\zeta_1^2 = \cos^2 \beta - 2 i\zeta_1 \sin \beta \cos (Q - \gamma) - (i\zeta_1)^2$$

Neglecting the terms involving i^2 as practically insensible, this gives

$$\zeta_1 = \pm [\cos \beta - i \sin \beta \cos (Q - \gamma)]$$

In order to remove the ambiguity of the double sign, let us put

$Z =$ the zenith distance of *the point* Z (Art. 289);

then, since $\vartheta = \mu - a$ is the hour angle of this point, we have

$$\cos Z = \sin \varphi \sin d + \cos \varphi \cos d \cos \vartheta$$

which by means of the preceding equations is reduced to

$$\cos Z = \zeta_1 \rho_1 \frac{\sin \varphi}{\sin \varphi_1} \tag{503}$$

Hence $\cos Z$ and ζ_1 have the same sign.

But, in order that the eclipse may be *visible* from a point on the earth's surface, we must, in general, have Z less than 90°; that is, $\cos Z$ must be positive, and therefore ζ_1 must be taken only with the positive sign. The negative sign would give a second point on the surface of the earth from which, if the earth were not opaque, the same phase of the eclipse would also be observed at the given time. In fact, every element of the cone of shadow which intersects the earth's surface at all, intersects it in two points, and our solution gives both points.

If we put

$$\epsilon = \frac{i \cos (Q - \gamma)}{\sin 1''} \tag{504}$$

we have

$$\zeta_1 = \cos \beta - \sin \beta \sin \epsilon$$

or, with sufficient accuracy,

$$\zeta_1 = \cos (\beta + \epsilon) \tag{505}$$

Thus, β and γ being determined by (502), ζ_1 is determined by (504) and (505): hence also ξ and η_1 by the equations

$$\left.\begin{aligned} \xi &= a + i\zeta_1 \sin Q \\ \eta_1 &= b + i\zeta_1 \cos Q \end{aligned}\right\} \tag{506}$$

The problem is, therefore, fully resolved; but, for the convenience of logarithmic computation, let c and C be determined by the equations

$$\left.\begin{aligned} c \sin C &= \eta_1 \\ c \cos C &= \zeta_1 \end{aligned}\right\} \tag{507}$$

then the equations (500) become

$$\left.\begin{aligned} \cos \varphi_1 \sin \vartheta &= \xi \\ \cos \varphi_1 \cos \vartheta &= c \cos (C + d_1) \\ \sin \varphi_1 &= c \sin (C + d_1) \end{aligned}\right\} \tag{508}$$

The curve thus determined will be the intersection of the penumbral cone, or that of the umbral cone, with the earth's surface, according as we employ the value of l for the one or the other.

299. The above solution is direct, though theoretically but approximate, since we have neglected terms of the order of i^2. It can, however, readily be made quite exact as follows. We have, by substituting the values of ζ_1 and η_1 in (498), and neglecting the term involving the product $i \sin (d_1 - d_2)$, which is of the same order as i^2,

$$\zeta = \rho_2 \cos (\beta + \varepsilon) - \rho_2 \sin \beta \cos \gamma \sin (d_1 - d_2)$$

and, putting

$$\varepsilon' = (d_1 - d_2) \cos \gamma$$

we have, within terms of the order i^2,

$$\zeta = \rho_2 \cos (\beta + \varepsilon + \varepsilon') \tag{509}$$

The substitution of this value of ζ in the term $i\zeta$ involves only an error of the order i^3, which is altogether insensible. The exact solution of the problem is, therefore, as follows. Find β and γ for each assumed value of Q, by the equations

$$\sin \beta \sin \gamma = x - l \sin Q = a$$

$$\sin \beta \cos \gamma = \frac{y}{\rho_1} - \frac{l \cos Q}{\rho_1} = b$$

then ε and ε' by the equations

$$\varepsilon = \frac{i \cos (Q - \gamma)}{\sin 1''} \qquad \varepsilon' = (d_1 - d_2) \cos \gamma$$

Find β' and γ' by the equations

$$\sin\beta'\sin\gamma' = a + i\rho_2\cos(\beta+\varepsilon+\varepsilon')\sin Q = \xi$$

$$\sin\beta'\cos\gamma' = b + \frac{i\rho_2\cos(\beta+\varepsilon+\varepsilon')\cos Q}{\rho_1} = \eta_1$$

then we have, rigorously,

$$\zeta_1 = \cos\beta'$$

and these values of ξ, η_1, and ζ_1 may then be substituted in (500) which can be adapted for logarithmic computation as before.*

300. It remains to be determined whether the eclipse is beginning or ending at the places thus found. A point on the earth's surface which at a given time T is upon the surface of the cone of shadow will at the next consecutive instant $T + dT$ be *within* or *without* the cone according as the eclipse is *beginning* or *ending* at the time T; the former or the latter, according as the distance $\varDelta = \sqrt{[(x-\xi)^2 + (y-\eta)^2]}$ becomes at the time $T + dT$ *less* or *greater* than the radius of the shadow $l - i\zeta$. In the case of total eclipse $l - i\zeta$ is a negative quantity, but by comparing $\varDelta^2$ with $(l - i\zeta)^2$ we shall obtain the required criterion for all cases; and, therefore, the criterion of *beginning* or *ending*, either of partial or of total eclipse, will be the *negative* or *positive* value of the differential coefficient, relatively to the time, of the quantity

$$(x-\xi)^2 + (y-\eta)^2 - (l - i\zeta)^2$$

or the negative or positive value of the quantity

$$(x-\xi)\left(\frac{dx}{dT} - \frac{d\xi}{dT}\right) + (y-\eta)\left(\frac{dy}{dT} - \frac{d\eta}{dT}\right) - (l - i\zeta)\left(\frac{di}{dT} - i\frac{d\zeta}{dT}\right)$$

* In this problem, as well as in most of the subsequent ones, I have not followed BESSEL's methods of solution, which, being mathematically rigorous, though as simple as such methods can possibly be, are too laborious for the practical purposes of mere prediction. As a refined and exhaustive disquisition upon the whole theory, BESSEL's *Analyse der Finsternisse*, in his *Astronomische Untersuchungen*, stands alone. On the other hand, the approximate solutions heretofore in common use are mostly quite imperfect; the compression of the earth, as well as the augmentation of the moon's semidiameter, being neglected, or only taken into account by repeating the *whole* computation, which renders them as laborious as a rigorous and direct method. I have endeavored to remedy this, by so arranging the successive approximations, when these are necessary, that only a small part of the whole computation is to be repeated, and by taking the compression of the earth into account, in all cases, from the commencement of the computation. In this manner, even the first approximations by my method are rendered more accurate than the common methods.

where we omit the insensible variation of i. For brevity, let us write x', y', &c. for $\frac{dx}{dT}$, $\frac{dy}{dT}$, &c. and denote the above quantity by P; then, after substituting the values of $x - \xi = (l - i\zeta) \sin Q$, $y - \eta = (l - i\zeta) \cos Q$, we have

$$P = L\,[(x' - \xi') \sin Q + (y' - \eta') \cos Q - (l' - i\zeta')]$$

in which $L = l - i\zeta$. If we put

$$P' = (x' - \xi') \sin Q + (y' - \eta') \cos Q - (l' - i\zeta') \qquad (510)$$

we shall have

$$P = LP'$$

The quantity P will be positive or negative according as L and P' have like signs or different signs.

For exterior contacts, and for interior contacts in annular eclipse, L is positive (Art. 293), and hence for these cases the eclipse is *beginning* or *ending* according as P' is *negative* or *positive;* but for total eclipse, L being negative, we have *beginning* or *ending* according as P' is *positive* or *negative.*

We must now develop the quantity P'. Taking one hour as the unit of time, x', y', l', ξ', η', ζ', will denote the hourly changes of the several quantities. The first three of these may be derived from the general tables of the eclipse for the given time; but ξ', η', ζ' are obtained by differentiating the equations (494), in which the latitude and longitude of the point on the earth's surface are to be taken as constant. Since $\vartheta = \mu_1 - \omega$, we shall have $\frac{d\vartheta}{dT} = \frac{d\mu_1}{dT}$; and hence, putting

$$\mu' = \frac{d\mu_1}{dT} \sin 1'' \qquad\qquad d' = \frac{dd}{dT} \sin 1''$$

we find

$$\begin{aligned}
\xi' &= \mu'\rho \cos \varphi' \cos \vartheta = \mu' (-\eta \sin d + \zeta \cos d) \\
&= \mu' [-y \sin d + \zeta \cos d + (l - i\zeta) \sin d \cos Q] \\
\eta' &= \mu' \xi \sin d - d'\zeta \\
&= \mu' [x \sin d - (l - i\zeta) \sin d \sin Q] - d'\zeta \\
\zeta' &= -\mu' \xi \cos d + d'\eta \\
&= \mu' [-x \cos d + (l - i\zeta) \cos d \sin Q] + d' [y - (l - i\zeta) \cos Q]
\end{aligned}$$

Substituting these values in (510), and neglecting terms involving i^2 and id' as insensible, we have

$$P' = a' - b' \cos Q + c' \sin Q - \zeta\,(\mu' \cos d \sin Q - d' \cos Q)$$

in which a', b', and c', denote the following quantities:

$$\left.\begin{aligned} a' &= -l' - \mu' ix \cos d \\ b' &= -y' + \mu' x \sin d \\ c' &= x' + \mu' y \sin d + \mu' il \cos d \end{aligned}\right\} \quad (511)$$

The values of these quantities may be computed for the same times as the other quantities in the eclipse tables, and their values for any given time will then be readily found by interpolation. For any assumed value of Q, therefore, and with the value of ζ found by (509), the value of P' may be computed, and its sign will determine whether the eclipse is beginning or ending. In most cases, a mere inspection of the tabulated values of a', b', and c', combined with a consideration of the value of Q, will suffice to determine the sign of P'; but when the place is near the northern or southern limits of the shadow, an accurate computation of P' will be necessary; and, since other applications of this quantity will be made hereafter, it will be proper to give it a more convenient form for logarithmic computation. Put

$$\left.\begin{aligned} e \sin E &= b' \qquad & f \sin F &= d' \\ e \cos E &= c' \qquad & f \cos F &= \mu' \cos d \end{aligned}\right\} \quad (512)$$

then we have

$$P' = a' + e \sin (Q - E) - \zeta f \sin (Q - F) \qquad (513)$$

Since a' and F are both very small quantities, and a very precise computation of P' will seldom be necessary when its algebraic sign is alone required, it will be sufficient in most cases to neglect these quantities, and also to put ζ_1 for ζ, and then we shall have the following simple criterion for the case of partial or annular eclipse:

If $e \sin (Q - E) < \zeta_1 f \sin Q$, the eclipse is beginning.
If $e \sin (Q - E) > \zeta_1 f \sin Q$, the eclipse is ending

For total eclipse, reverse these conditions.

301. In order to facilitate the application of the preceding as well as the subsequent problems, it is expedient to prepare the values of d_1, $\log \rho_1$, d_2, $\log \rho_2$, a', b'; c', e, E, f, F, and to arrange them in tables.

For our example of the eclipse of July 18, 1860, with the values of d given on p. 454, we form the following table by the equations (495):

	d_1	$\log \rho_1$	d_2	$\log \rho_2$
0^h	21° 1′ 39″.5	9.9987324	20° 53′ 58″.0	9.9998143
1	1 14 .0	23	53 32 .6	45
2	0 48 .5	22	53 7 .3	46
3	0 22 .9	21	52 41 .8	47
4	20 59 57 .4	20	52 16 .4	48
5	59 31 .8	19	51 50 .9	50

The values of x', y', and l', required in (511), derived also from the eclipse tables on p. 454, by the method of Art. 75, are as follows:

	x'	y'	l'
0^h	+ 0.545277	— 0.160108	— 0.000038
1	5312	0486	061
2	5310	0846	084
3	5256	1188	107
4	5134	1512	130
5	4928	1818	154

Hence, by (511) we find the values of a', b', c' to be as follows. The values for interior contacts are seldom required.

	For exterior contacts.			For interior contacts.		
	a'	b'	c'	a'	b'	c'
0^h	+ 0.001856	+ 0.050342	+ 0.631779	+ 0.001850	+ 0.050342	+ 0.631165
1	+ 0.000766	+ 0.101816	+ 0.616776	+ 0.000762	+ 0.101816	+ 0.616162
2	+ 0.000175	+ 0.153241	+ 0.601711	+ 0.000175	+ 0.153241	+ 0.601097
3	— 0.000415	+ 0.204612	+ 0.586571	— 0.000413	+ 0.204612	+ 0.585957
4	— 0.001005	+ 0.255925	+ 0.571342	— 0.001000	+ 0.255925	+ 0.570728
5	— 0.001595	+ 0.307171	+ 0.556010	— 0.001586	+ 0.307171	+ 0.555395

The values of e, E, f, F, for *exterior contacts*, deduced from these values of b' and c', and from $d' = -25''.5 \sin 1''$, by (512), are as follows:

	E	log e	I	log f
0^h	4° 33′ 21″	9.801939	— 0° 1′ 44″	9.388244
1	9 22 25	.795965	"	264
2	14 17 17	.793034	"	285
3	19 13 48	.793255	"	305
4	24 7 46	.796604	"	326
5	28 55 7	.802923	"	347

302. To illustrate the preceding formulæ, let us find some points of the outline of the penumbra on the earth's surface at the time $T = 2^h\ 8^m\ 12^s$. For this time, we have

$$x = -0.00672 \qquad \log \rho_1 = 9.99873 \qquad \log i = 7.66287$$
$$y = +0.57409 \qquad d_1 = 21°\ 0'\ 45''$$
$$l = +0.53673 \qquad \mu_1 = 30\ 34\ 13$$

Let us find the points for $Q = 50°$ and $Q = 300°$. The computation may be arranged as follows:

	Q	50°	300°
By (502):	$a = \sin\beta \sin\gamma$	− 0.41788	+ 0.45810
	$b = \sin\beta\cos\gamma$	+ 0.22975	+ 0.30662
	γ	− 61° 11′ 52″	56° 12′ 16″
	β	28 28 52	33 27 7
Hence by (504):	ϵ	— 5 43	— 6 59
	$\beta + \epsilon$	28 23 9	33 20 8
By (505):	$\log \zeta_1 = \log\cos(\beta+\epsilon)$	9 94437	9.92193
	$i\zeta_1 \sin Q$	+ 0.00310	— 0.00333
	$i\zeta_1 \cos Q$	+ 0.00260	+ 0.00192
By (506):	ξ	— 0.41478	+ 0.45477
	η_1	+ 0.23235	+ 0.30854
By (507):	$\log\eta_1 = \log c \sin C$	9.36614	9.48931
	$\log\zeta_1 = \log c\cos C$	9.94437	9.92193
	$\log c$	9.95901	9.94969
	C	14° 47′ 39″	20° 16′ 9″
	$C + d_1$	35 48 24	41 16 54
By (508):	$\log\xi = \log\cos\varphi_1 \sin\vartheta$	n9.61782	9.65779
$\log c\cos(C + d_1) =$	$\log\cos\varphi_1\cos\vartheta$	9.86803	9.82560
	$\log\tan\vartheta$	n9.74979	9.83219
	$\log\cos\varphi_1$	9.92764	9.90803
$\log c\sin(C + d_1) =$	$\log\sin\varphi_1$	9.72620	9.76908
	$\log\tan\varphi_1$	9.79856	9.86105
	$\log\sqrt{(1 - ee)}$	9.99855	9.99855
	$\log\tan\varphi$	9.80001	9.86250
	ϑ	— 29° 20′ 20″	34° 11′ 46″
	$\mu_1 - \vartheta = \omega$	59 54 33	356 22 27
	φ	32 15 3	36 4 40

To find whether the eclipse is beginning or ending at these places, we have, from the table on p. 465, for $T = 2^h\ 8^m\ 12^s$,

$\log e$	9.7931	
E	14° 58′	
$Q - E$	35 2	285° 2′
$\log e \sin (Q - E)$	9.5521	n9.7780
$\log f$	9.3883	
$\log \zeta_1 f \sin Q$	9.2170	n9.2477

At the first point, therefore, we have $e \sin (Q - E) > \zeta_1 f \sin Q$, and the eclipse is ending. At the second point, we have $e \sin (Q - E) < \zeta_1 f \sin Q$, and the eclipse is beginning.

Rising and Setting Limits.

303. *To find the rising and setting limits of the eclipse.*—By these limits we mean the curves upon which are situated all those points of the earth's surface where the eclipse begins or ends with the sun in the horizon. It will be quite sufficient for all practical purposes to determine these limits by the condition that *the point Z* is in the horizon. This gives in (503) $\cos Z = 0$, or $\zeta_1 = 0$, and, consequently, by (496), we have

$$\xi^2 + \eta_1^2 = 1 \tag{514}$$

as the condition which the co-ordinates of the required points must satisfy.

Now, let it be required to find the place where this equation is satisfied at a given time T. Let x and y be taken for this time, then we have, by putting $\zeta_1 = 0$ in (499),

$$l \sin Q = x - \xi$$
$$l \cos Q = y - \eta$$

Let

$$\left.\begin{aligned} m \sin M &= x \qquad & p \sin \gamma &= \xi \\ m \cos M &= y \qquad & p \cos \gamma &= \eta \end{aligned}\right\} \tag{515}$$

then, from the equations

$$\left.\begin{aligned} l \sin Q &= m \sin M - p \sin \gamma \\ l \cos Q &= m \cos M - p \cos \gamma \end{aligned}\right\} \tag{516}$$

we deduce, by adding their squares,

$$l^2 = m^2 - 2mp \cos (M - \gamma) + p^2$$

$$2 \sin^2 \tfrac{1}{2} (M - \gamma) = 1 - \cos (M - \gamma) = \frac{l^2 - (m - p)^2}{2mp}$$

If then we put $\lambda = M - \gamma$, we have

$$\left.\begin{aligned} \sin \tfrac{1}{2} \lambda &= \pm \sqrt{\left[\frac{(l + m - p)(l - m + p)}{4mp}\right]} \\ \gamma &= M \pm \lambda \end{aligned}\right\} \quad (517)$$

in which $\frac{1}{2}\lambda$ may always be taken less than $90°$, but the double sign must be used to obtain the two points on the surface of the earth which satisfy the conditions at the given time.

In this formula, m, M, and l are accurately known for the given time, but p is unknown. It is evident, however, from (514) and (515), that we have nearly $p = 1$, and this value may be used in (517) for a first approximation. To obtain a more correct value of γ, let us put $\xi = \sin \gamma'$; then, by (514), we have $\eta_1 = \cos \gamma'$, and, consequently, since $\eta = \rho_1 \eta_1$,

$$p \sin \gamma = \sin \gamma'$$
$$p \cos \gamma = \rho_1 \cos \gamma'$$

Hence we have

$$\left.\begin{aligned} \tan \gamma' &= \rho_1 \tan \gamma \\ p &= \frac{\sin \gamma'}{\sin \gamma} = \frac{\rho_1 \cos \gamma'}{\cos \gamma} \end{aligned}\right\} \quad (518)$$

and with this value of p the second computation of (517) will give a very exact value of γ. With this second value of γ a still more correct value of p could be found; but the second approximation is always sufficient.

With the second value of γ, therefore, we find the final value of γ' by the formula

$$\tan \gamma' = \rho_1 \tan \gamma$$

and then, substituting the values $\xi = \sin \gamma'$, $\eta_1 = \cos \gamma'$, $\zeta_1 = 0$, in (500), we have, for finding the latitude and longitude of the required points, the formulæ

$$\left.\begin{aligned} \cos \varphi_1 \sin \vartheta &= \sin \gamma' \\ \cos \varphi_1 \cos \vartheta &= - \cos \gamma' \sin d_1 \\ \sin \varphi_1 &= \cos \gamma' \cos d_1 \\ \omega = \mu_1 - \vartheta \qquad \tan \varphi &= \frac{\tan \varphi_1}{\sqrt{(1 - ee)}} \end{aligned}\right\} \quad (519)$$

In the second approximation, we must compute λ and γ by (517) separately for each place.

304. The sun is rising or setting at the given time at the places thus determined, according as ϑ (which is the hour angle of the point Z) is between 180° and 360° or between 0° and 180°.

To determine whether the eclipse is beginning or ending, we may have recourse to the sign of P' (513); and it will usually be sufficient for the present problem to put both a' and $\zeta = 0$ in that expression, and then the eclipse is beginning or ending according as $\sin (Q - E)$ is negative or positive. Now, by (516), we find

$$l \sin (Q - E) = m \sin (M - E) - p \sin (\gamma - E)$$

Hence, for points in the rising or setting limits,

If $m \sin (M - E) < p \sin (\gamma - E)$, the eclipse is beginning,
If $m \sin (M - E) > p \sin (\gamma - E)$, the eclipse is ending.

305. In order to apply the preceding method of determining the rising and setting limits, it is necessary first to find the extreme times between which the time T is to be assumed, or those limits of T between which the solution is possible. The two solutions given by (517) must reduce to a single one when the surface of the cone of shadow has but a single point in common with the earth's surface,—*i.e.* in the case of tangency of the cone and the terrestrial spheroid. Now, the two solutions reduce to one only when $\lambda = 0$, and both values of γ become $= M$; but if $\lambda = 0$, the numerator of the value of $\sin \frac{1}{2}\lambda$ must also be zero; and hence the points of contact are determined by the conditions

$$l + m - p = 0 \qquad \text{and} \qquad l - m + p = 0$$

or by the conditions

$$m = p + l \qquad \text{and} \qquad m = p - l$$

There may be four cases of contact, two of exterior and two of interior contact. The two exterior contacts are the first and last, or *the beginning and the end of the eclipse generally;* the axis of the shadow is then without the earth, and therefore we must have for these cases $m = \sqrt{x^2 + y^2} = p + l$.

The first interior contact corresponds to the last point on the earth's surface where the eclipse ends at sunrise; the second, to the first point where it begins at sunset. But these interior

contacts can occur only when the whole of the shadow on the principal plane falls within the earth, and for these cases, therefore, we must have $m = p - l$.

For the beginning and end generally we have, therefore, by (515),

$$(p + l) \sin M = x$$
$$(p + l) \cos M = y$$

Let T be the time when these conditions are satisfied, and put

$$T = T_0 + \tau$$

in which T_0 is the epoch of the eclipse tables, for which the values of x and y are x_0 and y_0. Then, x' and y' being the mean hourly changes of x and y for the time T, we have

$$x = x_0 + \tau x'$$
$$y = y_0 + \tau y'$$

Putting

$$\left.\begin{array}{ll} m_0 \sin M_0 = x_0 & \qquad n \sin N = x' \\ m_0 \cos M_0 = y_0 & \qquad n \cos N = y' \end{array}\right\} \quad (520)$$

the above conditions become

$$(p + l) \sin M = m_0 \sin M_0 + \tau \,.\, n \sin N$$
$$(p + l) \cos M = m_0 \cos M_0 + \tau \,.\, n \cos N$$

whence

$$(p + l) \sin (M - N) = m_0 \sin (M_0 - N)$$
$$(p + l) \cos (M - N) = m_0 \cos (M_0 - N) + n\tau$$

so that, if we put $M - N = \psi$, we have

$$\left.\begin{array}{l} \sin \psi = \dfrac{m_0 \sin (M_0 - N)}{p + l} \\[2ex] \tau = \dfrac{p + l}{n} \cos \psi - \dfrac{m_0}{n} \cos (M_0 - N) \\[2ex] T = T_0 + \tau \end{array}\right\} \quad (521)$$

in which $\cos \psi$ may be taken with either the negative or the positive sign; and it is evident that the first will give the beginning and the second the end of the eclipse generally.

For the two interior contacts we have

$$\left.\begin{array}{l} \sin \psi = \dfrac{m_0 \sin (M_0 - N)}{p - l} \\[2ex] \tau = \dfrac{p - l}{n} \cos \psi - \dfrac{m_0}{n} \cos (M_0 - N) \end{array}\right\} \quad (522)$$

These interior contacts cannot occur when $p - l$ is less than $m_0 \sin (M_0 - N)$, which would give impossible values of $\sin \psi$.

In these formulæ we at first assume $p = 1$, and, after finding an approximate value of ψ, we have, by (517), in which $\lambda = 0$, $\gamma = M$, and in the present problem $M = N + \psi$: therefore

$$\gamma = N + \psi \tag{523}$$

with which p is found by (518), and the second computation of (521) or (522) will then give the required times. We must employ in (523) the two values of ψ found by taking $\cos \psi$ with the positive and the negative sign; and therefore different values of p will be found for beginning and ending, so that in the second approximation separate computations will be necessary for the two cases.

In the first approximation the mean values of x', y', and l may be used, or those for the middle of the eclipse. With the approximate values of τ thus found, the true values of x', y', and l for the time $T = T_0 + \tau$ may be taken for the second approximation.

After finding the corrected value of ψ, we then have also the true value of $\gamma = N + \psi$ for each point, and hence also the true value of γ' by (518), with which the latitude and longitude of the points will be computed by (519). For the local apparent time of the phenomenon at each place we may take the value of ϑ in time, which is very nearly the sun's hour angle.

306. When the interior contacts exist, the rising and setting limits form two distinct enclosed curves on the earth's surface. If we denote the times of beginning and ending generally, determined by (521), by T_1 and T_2, and the times of interior contact, determined by (522), by T_1' and T_2', a series of points on the rising limit will be found by Art. 303, for a series of times assumed between T_1 and T_1', and points of the setting limit for times assumed between T_2' and T_2.

When the interior contacts do not exist, the rising and setting limits meet and form a single curve extending through the whole eclipse. The form of this curve may be compared to that of the figure 8 much distorted. A series of points upon it will be found by assuming times between T_1 and T_2.

307. Example.—Let us find the rising and setting limits of the eclipse of July 18, 1860.

First.—To find the beginning and ending on the earth gene rally, we have for the assumed epoch $T_0 = 2^h$, page 455,

$$m_0 \sin M_0 = x_0 = -\ 0.081244 \qquad n \sin N = x' = +\ 0.5453$$
$$m_0 \cos M_0 = y_0 = +\ 0.596075 \qquad n \cos N = y' = -\ 0.1608$$

which give

$$\log m_0 = 9.77930 \qquad \log n = 9.75474$$
$$M_0 = 352^\circ\ 14'\ 19'' \qquad N = 106^\circ\ 25'.8$$
$$\log m_0 \sin (M_0 - N) = n9.73938 \qquad \frac{m_0}{n} \cos (M_0 - N) = -\ 0.4336$$

For a first approximation, taking $p = 1$, we find, by (521),

$$p + l = 1.5367 \qquad \log \sin \psi = n9.5528$$
$$\frac{p+l}{n} \cos \psi = \mp\ 2^h.525$$
$$T_0 - \frac{m_0}{n} \cos (M_0 - N) = +\ 2.434$$

Approx. beginning	$T_1 =$	$23^h.909$	(July 17)
" end	$T_2 =$	4 .959	(July 18)

Taking $\cos \psi$ negative for beginning and positive for ending, we have then, by (518) and (523),

	Beginning.	End.
ψ	200° 55′.4	339° 4′.6
$N + \psi = \gamma$	307 21 .2	85 30 .4
$\log \tan \gamma$	n0.11732	1.10466
$\log \rho_1$	9.99873	9.99873
$\log \tan \gamma'$	n0.11605	1.10339
$\log \sin \gamma'$	9.89985	9.99865,5
$\log \sin \gamma$	9 90032	9.99866,3
$\log p$	9.99953	9.99999
p	0.99892	0.99998
l	0.53687	0.53640
$p + l$	1.53579	1.53638

For the above computed times we further find

$\log x' = \log n \sin N$	9.73664	9.73654
$\log y' = \log n \cos N$	n9.20538	n9.20774
$\log n$	9.75467	9.75477
N	106° 23′ 50″	106° 29′ 8″

For a second approximation, therefore, recomputing (521), we now find

	$\log \sin \psi$	$n9.55316$	$n9.55269$
	$\log \cos \psi$	$n9.97032$	9.97039
	T	$23^h.9098$	$4^h.9587$
	ψ	$200°\ 56'\ 27''$	$339°\ 4'\ 58''$
	$N + \psi = \gamma$	$307\ 20\ 17$	$85\ 34\ 6$
and by (518):	$\log \tan \gamma'$	$n0.11629$	1.10942

Then, for the latitude and longitude of the points, we have, by (519),

d_1	$21°\ 1'\ 42''$	$20°\ 59'\ 33''$
μ_1	$357\ 9\ 57$	$72\ 54\ 8$
ϑ	$254\ 38\ 57$	$91\ 35\ 43$
$\mu_1 - \vartheta = \omega$	$102\ 31\ 0$	$341\ 18\ 25$
φ	$34\ 38\ 34$	$4\ 9\ 46$

Therefore the eclipse begins on the earth generally on July 17, $23^h\ 54^m.5$ Greenwich mean time, in west longitude $102°\ 31'\ 0''$ and latitude $34°\ 38'\ 34''$, and ends July 18, $4^h\ 57^m.5$ in longitude $341°\ 18'\ 25''$ and latitude $4°\ 9'\ 46''$.

It is evident that for practical purposes the first approximation, which gives the times within a few seconds, is quite sufficient, especially since the effect of refraction has not yet been taken into account. (See Art. 327.)

Secondly.—We now pass to the computation of the curve which contains all the points where the eclipse begins or ends at sunrise or sunset. In the present example, this curve extends through the whole eclipse, since we have $m_0 \sin (M_0 - N) > 1 - l$: hence the required points will be found for Greenwich times assumed between July 17, $23^h.91$ and July 18, $4^h.96$. Let us take the series

$$T,\ 0^h,\ 0^h.2,\ 0^h.4,\ 0^h.6,\ 0^h.8\ \ldots\ldots\ldots\ 4^h.6,\ 4^h.8$$

The computation being carried on for all the points at once, the regular progression of the corresponding numbers for the successive times furnishes at each step a verification of its correctness. To illustrate the use of the formulæ, I give the computation for $T = 2^h.0$ nearly in full. For this time, we find, from p. 454 and p. 464,

$$x = m \sin M = -0.08124 \qquad l = 0.53675 \qquad d_1 = 21°\ 0'\ 49''$$
$$y = m \cos M = +0.59608 \qquad\qquad \log \rho_1 = 9\ 99873$$

and hence

$$M = 352° \, 14' \, 21'' \qquad \log m = 9.77931 \qquad m = 0.60160$$

Then, by (517), taking $p = 1$, we have

$$\begin{array}{rlr}
 & & \text{ar. co. log } 4mp \;\; 9.61863 \\
l + m - p & = 0.13835 \ldots\ldots\ldots\ldots & \log 9.14098 \\
l - m + p & = 0.93515 \ldots\ldots\ldots\ldots & \log 9.97088 \\
\lambda & = 26° \, 49' & \log \sin^2 \tfrac{1}{2}\lambda \;\; 8.73049
\end{array}$$

With this first approximate value of λ we find the value of p for each of the two points, by (518), as follows:

$M \pm \lambda = \gamma$	19° 3′	325° 25
$\log \tan \gamma$	9.53820	9.83849
$\log \rho_1 \tan \gamma = \log \tan \gamma'$	9.53693	9.83722
$\log \dfrac{\rho_1 \cos \gamma'}{\cos \gamma} = \log p$	9.99887	9.99914
p	0.99740	0.99802

Repeating (517) with these values of p:

ar. co. $\log 4mp$	9.61976	9.61949
$\log (l + m - p)$	9.14907	9.14715
$\log (l - m + p)$	9.96967	9.96996
$\log \sin^2 \frac{1}{2}\lambda$	8.73850	8.73660
$\pm \lambda$	+27° 4′ 4″	−27° 0′ 26″
$M \pm \lambda = \gamma$	19 18 25	325 13 55
$\log \tan \gamma$	9.54448	n9.84148
$\log \tan \gamma'$	9.54321	n9.84021

Hence, by (519),

ϑ	135° 45′ 4″	242° 36′ 45″
For $T = 2^h$. (p. 455), μ_1	28 31 12	28 31 12
$\mu_1 - \vartheta = \omega$	252 46 8	145 54 27
φ	61 52 35	50 13 46
Local app. time $= \vartheta$ in time,	$9^h \; 3^m.0$	$16^h \; 10^m.45$
	Sunset.	Sunrise

To find whether the eclipse is beginning or ending at these points, we have, from p. 465, and by Art. 304,

E	14° 17′	
$\log m \sin (M - E)$	n9.3538	n9.3538
$\log p \sin (\gamma - E)$	8.9406	n9.8772
	Beginning.	Ending.

In the same manner are found the results given in the following table:

SOLAR ECLIPSE, July 18, 1860.—RISING AND SETTING LIMITS.

Greenwich Mean Time.	Latitude. ϕ	Long. W. from Greenwich. ω	Local App. Time. ϑ	
0ʰ.0	+ 44° 27′	110° 35′	16ʰ 31ᵐ.7	Begins at Sunrise.
.2	52 34	121 33	15 59 .8	" "
.4	58 1	132 21	15 28 .7	" "
.6	62 10	144 2	14 53 .9	" "
.8	65 21	157 6	14 13 .7	" "
1 .0	67 36	171 46	13 27 .0	" "
.2	68 49	187 56	12 34 .4	" "
.4	68 58	204 56	11 38 .3	" Sunset.
.6	67 55	221 51	10 42 .7	" "
.8	65 37	237 54	9 50 .5	" "
2 .0	61 53	252 46	9 3 .0	" "
.2	56 16	266 33	8 19 .9	" "
.4	48 5	279 17	7 41 .0	" "
.6	37 15	290 36	7 7 .7	" "
.8	25 6	300 12	6 41 .3	" "
3 .0	13 36	308 12	6 21 .3	" "
.2	+ 3 59	315 0	6 6 .1	" "
.4	— 3 24	320 50	5 54 .8	" "
.6	— 8 43	325 53	5 46 .5	" "
.8	— 12 14	330 17	5 41 .0	" "
4 .0	— 14 11	334 4	5 37 .8	" "
.2	— 14 48	337 19	5 36 .8	" "
.4	— 14 6	340 2	5 38 .0	Ends "
.6	— 11 56	342 9	5 41 .5	" "
.8	— 7 32	343 25	5 48 .4	" "
0 .0	+ 25 45	99 10	17 17 .4	Begins at Sunrise.
.2	20 1	99 33	17 27 .9	" "
.4	17 16	101 22	17 32 .6	" "
.6	16 7	103 52	17 34 .6	Ends "
.8	16 17	106 56	17 34 .3	" "
1 .0	17 46	110 34	17 31 .8	" "
.2	20 42	114 50	17 26 .7	" "
.4	25 17	119 57	17 18 .3	" "
.6	31 45	126 14	17 5 .2	" "
.8	40 0	134 15	16 45 .0	" "
2 .0	50 14	145 54	16 10 .5	" "
.2	60 21	163 47	15 10 .9	" "
.4	67 27	191 43	13 31 .2	" "
.6	68 55	224 18	11 32 .9	" Sunset.
.8	66 27	249 7	10 5 .6	" "

SOLAR ECLIPSE, July 18, 1860.—RISING AND SETTING LIMITS.—(*Continued.*)

Greenwich Mean Time.	Latitude. ϕ	Long. W. from Greenwich. ω	Local App. Time. ϑ	
$3^h.0$	$+62°\ 43'$	$265°\ 37'$	$9^h\ 11^m.6$	Ends at Sunset.
.2	58 44	277 27	8 36 .3	" "
.4	54 42	286 49	8 10 .8	" "
.6	50 35	294 47	7 51 .0	" "
.8	46 21	301 53	7 34 .6	" "
4 .0	41 55	308 26	7 20 .3	" "
2	37 10	314 40	7 7 .4	" "
.4	31 57	320 43	6 55 .2	" "
.6	25 55	326 48	6 42 .9	" "
.8	18 11	333 18	6 28 .9	" "

These points being projected upon a chart (see p. 504), the whole curve may be accurately traced through them. It will be seen that the method of assuming a series of equidistant times gives more points in those portions of the curve where the curvature is greatest than in other portions, thus facilitating the accurate delineation of the curve. This advantage appears to have been overlooked by those who have preferred methods (such, for example, as HANSEN'S) in which a series of equidistant latitudes is assumed.

308. The preceding computations have been made for the penumbra; but we may employ the same method to determine the rising and setting limits of total or annular eclipse by employing in the formulæ the value of l for interior contacts. These limits, however, embrace so small a portion of the earth's surface that they are practically of little interest.

Curve of Maximum in the Horizon.

309. *To find the curve on which the maximum of the eclipse is seen at sunrise or sunset.*—When a point of the earth's surface whose co-ordinates are ξ, η, and ζ is not on the surface of the cone of shadow, but at a distance $\varDelta$ from the axis of the cone, we have the conditions (485),

$$\left.\begin{aligned}\varDelta \sin Q &= x - \xi \\ \varDelta \cos Q &= y - \eta\end{aligned}\right\} \quad (524)$$

The amount of obscuration depends upon the distance by which the place is immersed within the shadow, that is, upon the distance $L - \varDelta$, L being the radius of the shadow on the parallel plane at the distance ζ from the principal plane. For the maximum of the eclipse, therefore, we have the condition

$$\frac{dL}{dT} - \frac{d\varDelta}{dT} = 0$$

Differentiating the above equations relatively to the time, and denoting the derivatives of x, y, &c. by accents, as in Art. 300, we have

$$\frac{d\varDelta}{dT}\sin Q - \varDelta\cos Q \cdot \frac{dQ}{dT} = x' - \xi'$$

$$\frac{d\varDelta}{dT}\cos Q + \varDelta\sin Q \cdot \frac{dQ}{dT} = y' - \eta'$$

which give

$$\frac{d\varDelta}{dT} = (x' - \xi')\sin Q + (y' - \eta')\cos Q$$

The equation $L = l - i\zeta$ gives

$$\frac{dL}{dT} = l' - i\zeta'$$

and, therefore,

$$l' - i\zeta' - (x' - \xi')\sin Q - (y' - \eta')\cos Q = 0 \qquad (525)$$

or, by (510),

$$P' = 0 \qquad (526)$$

This is, therefore, the general condition which characterizes the maximum of the eclipse at a given time. In the present problem we have also the condition that the sun is in the horizon, for which we may, as in Art. 303, substitute the condition $\zeta_1 = 0$. Since, however, the instant of greatest obscuration is not subject to any nice observation, a very precise solution of the problem is quite unimportant, and we may be satisfied with the approximate solution obtained by supposing $\zeta = 0$, and at the same time neglecting the small quantity a' in P'. The condition (526) will then be satisfied when in (513) we have

$$\sin(Q - E) = 0$$

that is, when

$$Q = E \qquad \text{or} \qquad Q = 180° + E$$

Hence, for any given time, the conditions (524) become

$$\pm \Delta \sin E = x - \xi$$
$$\pm \Delta \cos E = y - \eta$$

which with the condition

$$\xi^2 + \eta_1^2 = 1$$

must determine the required points of our curve. The angle E is here known for the given time, being directly obtained from its tabulated values, but Δ is unknown. Putting, as in the preceding problem,

$$m \sin M = x \qquad\qquad p \sin \gamma = \xi$$
$$m \cos M = y \qquad\qquad p \cos \gamma = \eta$$

we have

$$\pm \Delta \sin E = m \sin M - p \sin \gamma$$
$$\pm \Delta \cos E = m \cos M - p \cos \gamma$$

whence

$$0 = m \sin (M - E) - p \sin (\gamma - E)$$
$$\pm \Delta = m \cos (M - E) - p \cos (\gamma - E)$$

Therefore, putting $\psi = \gamma - E$, we have

$$\left.\begin{aligned} \sin \psi &= \frac{m \sin (M - E)}{p} \\ \pm \Delta &= m \cos (M - E) - p \cos \psi \end{aligned}\right\} \qquad (527)$$

The first of these equations will give two values of ψ, since we may take $\cos \psi$ with the positive or the negative sign; but, as only those places satisfy the problem which are actually *within* the shadow, we must have $\Delta < l$, or, at least, Δ not greater than l. That value of ψ which would give $\Delta > l$ must, therefore, be excluded: so that in general we shall have at a given time but one solution.

It will be quite accurate enough, considering the degree of precision above assigned, to employ in (527) a mean value of p, or, since p falls between ρ_1 and unity, to take $\log p = \frac{1}{2} \log \rho_1$. But, if we wish a more correct value, we have only to take

$$\gamma = \psi + E \qquad (528)$$

and then find p as in (518); after which (527) must be recomputed.

Having found the true value of ψ by (527), and of γ by (528), we then have γ' by the equation

$$\tan \gamma' = \rho_1 \tan \gamma$$

and the latitude and longitude of each point of the curve by (519).

The limiting times between which the solution is possible will be known from the computation of the rising and setting limits, in which we have already employed the quantity $m \sin (M - E)$; and the present curve will be computed only for those times for which $m \sin (M - E) < l$. These limiting times are also the same as those for the northern and southern limiting curves, which will be determined in Art. 313.

310. The degree of obscuration is usually expressed by the fraction of the sun's apparent diameter which is covered by the moon's disc. When the place is so far immersed in the penumbra as to be on the edge of the total shadow, the obscuration is total; in this case the distance of the place from the edge of the penumbra is equal to the absolute difference of the radii of the penumbra and the umbra, that is, to the algebraic sum $L + L_1$, L_1 denoting the radius of the umbra (which is, by Art. 293, negative); but in any other case the distance of the place within the penumbra is $L - \Delta$: hence, if D denotes the degree of obscuration expressed as a fraction of the sun's apparent diameter, we shall have, very nearly,

$$D = \frac{L - \Delta}{L + L_1} \qquad (529)$$

This formula may also be used when the eclipse is annular, in which case L_1 is essentially positive; and even when Δ is zero, and the eclipse consequently central, the value of D given by the formula will be less than unity, as it should be, since in that case there is no total obscuration.

In the present problem we have

$$D = \frac{l - \Delta}{l + l_1} \qquad (529^*)$$

in which l and l_1 are the radii of the penumbra and umbra on the principal plane, as found by (488).

Example.—In the eclipse of July 18, 1860, compute the curve on which the maximum of the eclipse is seen in the horizon.

In the computation of the rising and setting limits, the quantity $m \sin(M-E)$ was less than unity only from $T = 0^h.6$ to $T = 4^h.2$: so that the present curve may be computed for the series of times $0^h.6$, $0^h.8 \ldots\ldots 4^h.0$, $4^h.2$. For an approximate computation we may take $\log p = \frac{1}{2}\log \rho_1 = 9.9994$, and employ only four decimal places in the logarithms throughout.

The computation for $T = 2^h$ is as follows. For this time we have already found (p. 473)

	$\log m$	9.7793
	M	352° 14′.4
	E	14 17.3
Hence, by (527),	$M - E$	337 57.1
	$\log m \sin(M-E)$	n9.3538
	$\log p$	9.9994
	$\log \sin \psi$	n9.3544
	$\log \cos \psi$	9.9886
	$\log p \cos \psi$	9.9880
	$\log m \cos(M-E)$	9.7463
	$m \cos(M-E)$	+ 0.5575
	$p \cos \psi$	+ 0.9727
	Δ	0.4152

Here, if $\cos \psi$ were taken with the negative sign we should find $\Delta = 1.5302$, which is greater than l. Taking it, therefore, with the positive sign only, we have

	ψ	− 13° 4′.3
	$\psi + E = \gamma$	+ 1 13.
$\log \rho_1 = 9.9987$	$\log \tan \gamma$	8.3271
	$\log \tan \gamma'$	8.3258

with which we find, by (519),

ϑ	176° 37′.2
μ_1	28 31.2
ω	211 54
φ	69 1
App. time = ϑ in time	$11^h\ 46^m.5$ Sunset.

To express the degree of obscuration according to (529*) we have, taking the mean values of l and l_1 (p. 454),

$$l = 0.5366 \qquad l - \Delta = 0.1214$$
$$l_1 = -0.0092 \qquad D = \frac{0.1214}{0.5274} = 0.23$$
$$l + l_1 = 0.5274$$

In the same manner all the following results are obtained:

SOLAR ECLIPSE, July 18, 1860.—CURVE OF MAXIMUM OF THE ECLIPSE IN THE HORIZON.

Greenwich Mean T.	Latitude. ϕ	Long. W. from Greenwich. ω	App. Local Time. ϑ	Degree of Obscuration. D
$0^h.6$	$+ 24° 44'$	$107° 41'$	$17^h 19^m.3$	0.30
0 .8	37 47	117 47	16 50 .9	.76
1 .0	47 3	127 49	16 22 .8	.97
1 .2	54 31	139 1	15 50 .0	.74
1 .4	60 38	152 24	15 8 .5	.56
1 .6	65 20	169 0	14 14 .1	.41
1 .8	68 16	189 16	13 5 .0	.31
2 .0	69 1	211 54	11 46 .5	.23
2 .2	67 34	233 32	10 31 .9	.18
2 .4	64 20	251 42	9 31 .3	.17
2 .6	59 55	266 11	8 45 .3	.17
2 .8	54 41	277 50	8 10 .8	.21
3 .0	48 52	287 31	7 44 .0	.28
3 .2	42 35	295 56	7 22 .4	.37
3 .4	35 49	303 30	7 4 .1	.50
3 .6	28 28	310 33	6 47 .9	.67
3 .8	20 21	317 22	6 32 .6	.89
4 .0	+ 11 2	324 15	6 17 .2	.87
4 .2	— 0 45	331 14	6 1 .1	.48

Northern and Southern Limiting Curves.

311. *To find the northern and southern limits of the eclipse on the earth's surface.*—These limits are the curves in which are situated all the points of the surface of the earth from which only a single contact of the discs of the sun and moon can be observed, the moon appearing to pass either wholly south or wholly north of the sun. They may also be defined as curves to which the outline of the shadow is at all times in contact during its progress across the earth.

The solution of this problem is derived from the consideration that the simple contact is here the *maximum* of the eclipse, so that we must have, as in (526),

$$P' = 0$$

and consequently, by (513),

$$a' + e \sin (Q - E) = \zeta f \sin (Q - F). \tag{530}$$

For any given time T, therefore, we are to find that point of the outline of the shadow on the surface of the earth for which the value of Q and its corresponding ζ satisfy this equation. This can be effected only indirectly, or by successive approximations. For this purpose, we must know at the outset an approximate value of Q; and therefore, before proceeding any further, we must show how such an approximate value may be found.

We can readily determine sufficiently narrow limits between which Q may be assumed. For this purpose, neglecting a' in (530), as well as F, which are always very small, we have approximately,

$$e \sin (Q - E) = \zeta f \sin Q$$

The extreme values of ζ are $\zeta = 0$ and $\zeta = 1$. The first gives $\sin (Q - E) = 0$, and therefore for a first limit we have

$$Q = E \qquad \text{or} \qquad Q = 180° + E$$

The second gives

$$e \sin (Q - E) = f \sin Q$$

whence

$$\tan (Q - \tfrac{1}{2} E) = \frac{e + f}{e - f} \tan \tfrac{1}{2} E$$

Put

$$\tan \psi = \frac{e + f}{e - f} \tan \tfrac{1}{2} E$$

then the equation $\tan (Q - \frac{1}{2} E) = \tan \psi$ gives for our second limits

$$Q = \tfrac{1}{2} E + \psi \qquad \text{or} \qquad Q = 180° + \tfrac{1}{2} E + \psi$$

To compute ψ readily, put

$$\left.\begin{aligned} \tan \nu &= \frac{f}{e} \\ \text{then} \qquad \tan \psi &= \tan (45° + \nu) \tan \tfrac{1}{2} E \end{aligned}\right\} \quad (531$$

and Q is to be assumed

between E and $\frac{1}{2} E + \psi$
or between $180° + E$ and $180° + \frac{1}{2} E + \psi$

These limits may be computed in advance for the principal hours of the eclipse from the previously tabulated values of E, e, and f, and an approximate value of Q may then be easily inferred for a given time with sufficient precision for a first approximation.

When the shadow passes wholly within the earth, there are two limiting curves, northern and southern. For one of these Q is to be taken between E and $\frac{1}{2}E + \psi$; for the other, between $180° + E$ and $180° + \frac{1}{2}E + \psi$. Since E is always an acute angle, positive or negative, it follows that when Q is taken between E and $\frac{1}{2}E + \psi$, its cosine is in general positive, while it is negative in the other case. The equation $\eta = y - (l - i\zeta)\cos Q$ shows that η will be less in the first case and greater in the second, and hence *the values of Q between E and $\frac{1}{2}E + \psi$ belong to the southern limit*, and *the values of Q between* $180° + E$ *and* $180° + \frac{1}{2}E + \psi$ *belong to the northern limit.*

There is only one limit, northern or southern, when one of the series of values of Q would give impossible values of ζ in the computation of the outline of the shadow by Art. 298. But when the rising and setting limits have been determined, the question of the existence of one or both of the northern and southern limits is already settled; for if the rising and setting limits extend through the whole eclipse in north latitude, only the southern limiting curve of our present problem exists, and *vice versa;* while if the rising and setting limits form two distinct curves, we have both a northern and southern limiting curve; and the latter must evidently connect the extreme northern and southern points respectively of the two enclosed rising and setting curves. In our example of the eclipse of July 18, 1860, there exists only the southern limiting curve of the present problem, the penumbral shadow passing over and beyond the north pole of the earth.

Having assumed a value of Q, we find ζ_1 by the equations (502), (504) and (505), and then ζ by (509). This computed value of ζ and the assumed value of Q being substituted in (530), this equation will be satisfied only when the true value of Q has been assumed. To find the correction of Q, let us suppose that when the equation has been computed logarithmically we find

$$\log \zeta f \sin (Q - F) - \log [a' + e \sin (Q - E)] = x$$

If then dQ and $d\zeta$ are the corrections which Q and ζ require in

order to reduce x to zero, we have, by differentiating this equation,

$$\left[\cot(Q-F) - \frac{e\cos(Q-E)}{a' + e\sin(Q-E)}\right]\frac{dQ}{A} + \frac{d\zeta}{A\zeta} = -x$$

in which A is the reciprocal of the modulus of common logarithms.

In this differential equation we may neglect a' without sensibly affecting the rate of approximation. If then we put

$$g = -\frac{d\zeta}{\zeta dQ}$$

we shall have

$$dQ = \frac{Ax}{\cot(Q-E) - \cot(Q-F) + g}$$

This value of dQ is yet to be reduced to seconds by multiplying it by cosec $1''$ or $206265''$.

To find g, we may take, as a sufficiently exact expression for computing dQ,

$$g = -\frac{d\zeta_1}{\zeta_1\, dQ}$$

and by differentiating (502) (omitting the factor ρ_1, which will not sensibly affect g),

$$\begin{aligned}\cos\beta\sin\gamma\, d\beta + \sin\beta\cos\gamma\, d\gamma &= -l\cos Q\, dQ\\ \cos\beta\cos\gamma\, d\beta - \sin\beta\sin\gamma\, d\gamma &= l\sin Q\, dQ\end{aligned}$$

whence, by eliminating $d\gamma$,

$$\frac{d\beta}{dQ} = -\frac{l\sin(Q-\gamma)}{\cos\beta}$$

By (505) a sufficiently exact value of ζ_1 for our present purpose is

$$\zeta_1 = \cos\beta$$

whence

$$\frac{d\zeta_1}{dQ} = -\sin\beta\frac{d\beta}{dQ}$$

$$g = l\sin\beta\sec^2\beta\sin(Q-\gamma) \tag{532}$$

Putting, finally,

$$G = \cot(Q-E) - \cot(Q-F) = \frac{\sin(E-F)}{\sin(Q-E)\sin(Q-F)} \tag{533}$$

we have

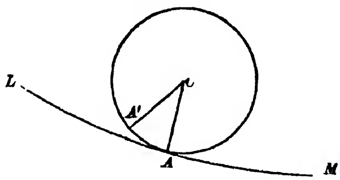
$$dQ = \frac{[5.67664]x}{G + g} \tag{534}$$

in which 5.67664 is the logarithm of $A \times 206265''$.

When the true value of Q has thus been found, the corresponding latitude and longitude on the earth's surface are found as in Art. 298.

312. The preceding solution of this problem (which is commonly regarded as one of the most intricate problems in the theory of eclipses) is very precise, and the successive approximations converge rapidly to the final result. For practical purposes, however, an extremely precise determination of the limiting curves of the penumbra is of little importance, since no valuable observations are made near these limits. I shall, therefore, now show how the process may be abridged without making any important sacrifice of accuracy.

Fig. 43.



In the first place, it is to be observed that great precision in the angle Q is unnecessary. If LM, Fig. 43, is the limiting curve which is tangent at A to the shadow whose axis is at C, and if Q is in error by the quantity ACA', the point determined will be (nearly) A' instead of A. Now, although A' may be at some distance from A, it is evident that it will still be at a proportionally small distance from the limiting curve. In fact, we may admit an error of several minutes in the value of Q without *sensibly* removing the computed point from the curve. The equation (530), which determines Q, may, therefore, without practical error be written under the approximate form

$$e \sin (Q - E) = \zeta_1 f \sin Q$$

and in this we may employ for ζ_1 the value

$$\zeta_1 = \cos \beta$$

Hence, having found β from (502) by employing the first assumed value of Q, we then have

$$\frac{\sin (Q - E)}{\sin Q} = \frac{f \cos \beta}{e}$$

whence

$$\tan\left(Q - \tfrac{1}{2} E\right) = \frac{e + f \cos\beta}{e - f \cos\beta} \tan \tfrac{1}{2} E$$

by which a second and more correct value of Q can be found. This equation will be readily computed under the following form:

$$\left.\begin{aligned} \tan \nu' &= \frac{f}{e} \cos\beta \\ \tan\left(Q - \tfrac{1}{2} E\right) &= \tan\left(45^\circ + \nu'\right) \tan \tfrac{1}{2} E \end{aligned}\right\} \quad (535)$$

The value of Q thus determined may be regarded as final, and we may then proceed to compute the latitude and longitude by the equations (502) to (508). In this approximate method, logarithms of four decimal places will be found quite sufficient.

313. For the computation of a *series* of points by the preceding method, it is necessary first to determine the extreme times between which the solution is possible. It is evident that the first and last points of the curve are those for which $\zeta_1 = 0$, and, consequently, $Q = E$, or $Q = 180^\circ + E$. It is easily seen that these points are also the first and last points of the curve of maximum in the horizon (Art. 309), and, therefore, the limiting times are here the same as for that curve. If, however, we wish to determine these limiting times independently (that is, when the rising and setting limits have not been previously computed), the following approximative process will give them with all the precision necessary.

Since $Q = E$, or $= 180^\circ + E$, we have, at the required time,

$$\left.\begin{aligned} \xi &= x \mp l \sin E \\ \eta &= y \mp l \cos E \end{aligned}\right\} \quad (536)$$

together with the condition (514), for which we may here employ

$$\xi^2 + \eta^2 = 1$$

If we put $\xi = \sin\gamma$, this condition gives $\eta = \cos\gamma$. We have, by (512),

$$\sin E = \frac{b'}{e} \qquad \cos E = \frac{c'}{e}$$

and we may here regard e as constant. Let the required time be denoted by $T = T_0 + \tau$, T_0 being an assumed time near the middle of the eclipse. Let b_0', c_0', be the values of b' and c' for

the time T_0, and denote their hourly changes by b'' and c''; then we have, for the time T,

$$b' = b_0' + b''\tau \qquad\qquad c' = c_0' + c''\tau$$

and hence, E_0 being the tabulated value of E for the time T_0,

$$\sin E = \sin E_0 + \frac{b''}{e}\tau \qquad\qquad \cos E = \cos E_0 + \frac{c''}{e}\tau$$

If, also, x_0, y_0, are the values of x and y for the time T_0, x' and y' their hourly changes, we have

$$x = x_0 + x'\tau \qquad\qquad y = y_0 + y'\tau$$

and the equations (536) become

$$\sin\gamma = x_0 \mp l \sin E_0 + \left(x' \mp \frac{l}{e}\,b''\right)\tau$$

$$\cos\gamma = y_0 \mp l \cos E_0 + \left(y' \mp \frac{l}{e}\,c''\right)\tau$$

Let m, M, n, N, be determined by the equations

$$\left.\begin{aligned} m \sin M &= x_0 \mp l \sin E_0 \\ m \cos M &= y_0 \mp l \cos E_0 \\ n \sin N &= x' \mp \frac{l}{e}\,b'' \\ n \cos N &= y' \mp \frac{l}{e}\,c'' \end{aligned}\right\} \qquad (537)$$

in which the upper sign is to be used for the southern and the lower sign for the northern limit; then, from the equations

$$\sin\gamma = m \sin M + n \sin N\,.\,\tau$$
$$\cos\gamma = m \cos M + n \cos N\,.\,\tau$$

we derive

$$\sin(\gamma - N) = m \sin(M - N)$$
$$\cos(\gamma - N) = m \cos(M - N) + n\tau$$

Hence, putting $\gamma - N = \psi$,

$$\left.\begin{aligned} \sin\psi &= m \sin(M - N) \\ \tau &= \frac{\cos\psi}{n} - \frac{m\cos(M - N)}{n} \\ T &= T_0 + \tau \end{aligned}\right\} \qquad (538)$$

It is evident that $\cos\psi$ is to be taken with the negative sign for the first point and with the positive sign for the last point of the curve.

To find the latitude and longitude of the extreme points, we take $\gamma = N + \psi$, $\tan \gamma' = \rho_1 \tan \gamma$, and proceed by (519).

Example.—To find the southern limit of the eclipse of July 18, 1860.

First. To find the extreme times.—Taking $T_0 = 2^h$, we have, from our tables, pp. 454, 455, and pp. 464, 465,

$$x_0 = -0.0812 \qquad x' = +0.5452$$
$$y_0 = +0.5961 \qquad y' = -0.1610$$
$$l = 0.5367$$
$$E_0 = 14° 17' \qquad b'' = +0.0514$$
$$\log e = 9.7977 \qquad c'' = -0.0151$$

where we take mean values of x', y', &c. From these we find by (537), taking the upper signs in the formulæ,

$$\log m = 9.3555 \qquad M = 289° 35'$$
$$\log n = 9.7182 \qquad N = 106 \;\; 28$$
$$M - N = 183 \;\; 7$$

Hence, by (538),

$$\log \sin (M - N) = n8.7354 \qquad \log \cos (M - N) = n9.9994$$
$$\log \sin \psi = n8.0909 \qquad \frac{m \cos (M - N)}{n} = +0^h.433$$
$$\log \cos \psi = 0.0000 \qquad \frac{\cos \psi}{n} = \mp 1\;.913$$
$$\tau = -1\;.480$$
$$\text{or } \tau = +2\;.346$$

Therefore, for the first and last points of the curve we have, respectively, the times

$$T_1 = 2^h - 1^h.480 = 0^h.520$$
$$T_2 = 2 + 2\;.346 = 4\;.346$$

To find the latitude and longitude of the extreme points corresponding to these times, we have

		First Point.	Last Point.
	ψ	180° 42′	− 0° 42′
	$\gamma = N + \psi$	287 10	105 46
	$\log \tan \gamma$	$n0.5102$	$n0.5492$
$\log \rho_1 = 9.9987$	$\log \tan \gamma'$	$n0.5089$	$n0.5479$
	d_1	21° 1′.4	20° 59′.8
	μ_1	6 19 .2	63 42.7

Hence, by (519),

ω	102° 40′	339° 30′
φ	16 5	− 14 47

Second. To find a series of points on the curve.—We begin by computing the limits of Q for the hours 0^h, 1^h, 2^h, 3^h, 4^h, 5^h. Thus, for 0^h we have, from the table p. 465, and by (531),

T	0^h
$\log f$	9.3882
$\log e$	9.8019
$\log \tan \nu$	9.5863
ν	21° 5′.6
$\frac{1}{2} E$	2 16.7
$\log \tan (45° + \nu)$	0.3533
$\log \tan \frac{1}{2} E$	8.5997
$\log \tan \psi$	8.9530
ψ	5° 7′.7
$\frac{1}{2} E + \psi$	7 24.4

For the southern limiting curve, Q falls between E and $\frac{1}{2} E + \psi$, *i.e.*, for 0^h, between 4° 33′ and 7° 24′. In the same manner we form the other numbers of the following table:

T	Lower limit of Q.	Upper limit of Q.
0^h	4° 33′	7° 24′
1	9 22	15 18
2	14 17	23 13
3	19 14	30 53
4	24 8	38 4
5	28 55	44 36

The points of the curve are to be computed for times between $0^h.520$ and $4^h.346$, and we shall, therefore, assume for T the series $0^h.6$, $0^h.8$, $1^h.0$ . . . . . $4^h.0$, $4^h.2$, which, with the extreme points above computed, will embrace the whole curve.

Instead of determining Q for each of these times by the method of Art. 312, it will be sufficient to determine it for the hours 1^h, 2^h, 3^h, 4^h, and, hence, to infer its values for the intervening times. Thus, for $T = 1^h$, assuming $Q = 12°$, which is a

mean between its two limiting values, we proceed by the equations (502), for which we can here use

$$\sin\beta \sin\gamma = x - l \sin Q$$
$$\sin\beta \cos\gamma = y - l \cos Q$$

as follows:

For $T_0 = 1^h$. x	$-$ 0.6266	$\log\cos\beta$	9.7396
y	$+$ 0.9170	$\log\frac{f}{e}$	9.5923
l	0.5368		
Assume Q	12°	$\log\tan\nu'$	9.3319
$a = x - l\sin Q$	$-$ 0.7382	ν'	12° 7′.1
$b = y - l\cos Q$	$+$ 0.3920	$\frac{1}{2}E$	4 41.2
$\log a = \log\sin\beta\sin\gamma$	n9.8682	$\log\tan(45°+\nu')$	0.1894
$\log b = \log\sin\beta\cos\gamma$	9.5933	$\tan\frac{1}{2}E$	8.9137
$\log\sin\beta$	9.9221	$\tan(Q-\frac{1}{2}E)$	9.1031
		$Q-\frac{1}{2}E$	7° 13′.5
		Q	11 54.7

We thus find,

for $T =$	1^h	2^h	3^h	4^h
$Q =$	11° 55′,	22° 20′,	30° 16′,	32° 17′.

From these numbers we obtain by simple interpolation sufficiently exact values of Q for our whole series of points. And since it is plain from Art. 312, that even an error of half a degree in Q will not remove the computed point from the true curve by any important amount, we may be content to employ the following series of values as final:

T	Q	T	Q	T	Q	T	Q
$0^h.6$	8°	$1^h.6$	18°	$2^h.6$	28°	$3^h.6$	31°
0 .8	10	1 .8	20	2 .8	29	3 .8	32
1 .0	12	2 .0	22	3 .0	30	4 .0	32
1 .2	14	2 .2	24	3 .2	30	4 .2	32 .5
1 .4	16	2 .4	26	3 .4	31		

For each time T we now take x, y, and l, from the tables of the eclipse, and, with the value of Q for the same time, determine the required point on the outline of the shadow by the

complete equations (502) to (508) inclusive, the use of which has already been exemplified in Art. 302. Employing only four decimal places in the logarithms, we shall find that the curve may be traced through the points given in the following table:

SOLAR ECLIPSE, July 18, 1860.—SOUTHERN LIMIT.

Greenwich Mean Time.	Latitude. ϕ	Long. W. from Greenwich ω
$0^h.520$	$+16°\ 5'$	$102°\ 40'$
0 .6	21 32	88 31
0 .8	25 6	76 37
1 .0	26 36	69 2
1 .2	27 17	63 9
1 .4	27 27	58 14
1 .6	27 15	53 57
1 .8	26 47	50 9
2 .0	26 4	46 43
2 .2	25 9	43 33
2 .4	24 3	40 34
2 .6	22 48	37 45
2 .8	21 5	34 33
3 .0	19 9	31 25
3 .2	16 41	27 50
3 .4	14 14	24 39
3 .6	11 9	20 44
3 .8	8 5	16 55
4 .0	+ 4 3	11 46
4 .2	− 0 39	5 17
4 .346	− 14 47	339 30

314. We have applied the preceding method only to the determination of the extreme limits of the penumbra, which may be designated as the extreme limits of *partial eclipse*. The same method will determine the northern and southern limits of total or annular eclipse, by employing the value of l for the total shadow—that is, for interior contacts. The latter are, indeed, more important, practically, than the former, and therefore in

special cases somewhat greater precision might be desired than has been observed in the preceding example. In any such case, recourse may be had to the rigorous method of Art. 311. Since the limits of total or annular eclipse often include but a very narrow belt of the earth's surface, extending nearly equal distances north and south of the curve of central eclipse, they may be derived, with sufficient accuracy for most purposes, from this curve, by a method which will be given in Art. 320.

The curve upon which any given degree of obscuration can be observed may also be computed by the preceding method. It is only necessary to substitute $\varDelta$ for l, and to give $\varDelta$ a value corresponding to D according to the equation (529). All the curves thus found begin and end upon the curve of maximum in the horizon.

Curve of Central Eclipse.

315. *To find the curve of central eclipse upon the surface of the earth.*—This curve contains all those points of the surface of the earth through which the axis of the cone of shadow passes. The problem becomes the same as that of Art. 298 upon the supposition that the shadow is reduced to a point—that is, when $l - i\zeta = 0$, and, consequently, by (493),

$$\xi = x \qquad \eta = y$$

Hence, putting

$$y_1 = \frac{y}{\rho_1}$$

the equations (502) to (508) are reduced to the following extremely simple ones, which are rigorously exact:

$$\left.\begin{aligned}
\sin\beta \sin\gamma &= x \\
\sin\beta \cos\gamma &= y_1 \\
c \sin C &= y_1 \\
c \cos C &= \cos\beta \\
\cos\varphi_1 \sin\vartheta &= x \\
\cos\varphi_1 \cos\vartheta &= c\cos(C + d_1) \\
\sin\varphi_1 &= c\sin(C + d_1) \\
\tan\varphi = \frac{\tan\varphi_1}{\sqrt{(1 - ee)}} & \qquad \omega = \mu_1 - \vartheta
\end{aligned}\right\} \quad (539)$$

It will be convenient to prepare the values of y_1 for the principal hours of the eclipse; and then for any given time T taking the values of x, y_1, d_1, μ_1, from the eclipse tables, these equations determine a point of the curve.

316. The extreme times between which the solution is possible, or the beginning and end of central eclipse upon the earth, are found as follows. At these instants the axis of the shadow is tangent to the earth's surface, and the central eclipse is observed at sunrise and sunset respectively. Hence, Z being the zenith distance of *the point* Z, we have $\cos Z = 0$, or, by (503), $\zeta_1 = 0$, whence, by (499),

$$\xi^2 + \eta_1^2 = 1$$

or

$$x^2 + y_1^2 = 1$$

which is equivalent to putting $\sin\beta = 1$, or $\cos\beta = 0$, in the first two equations of (539), so that we have

$$\sin\gamma = x, \qquad \cos\gamma = y_1$$

Let x' and y_1' denote the mean hourly changes of x and y_1 computed by the method of Art. 296. Let the required time of beginning or ending be denoted by $T = T_0 + \tau$, T_0 being an arbitrarily assumed epoch; then, if (x) and (y_1) are the values of x and y_1 taken for the time T_0, we have for the time T,

$$\sin\gamma = (x) + x'\tau$$
$$\cos\gamma = (y_1) + y_1'\tau$$

Let m, M, n, N, be determined by the equations

$$\left.\begin{array}{ll} m\sin M = (x) & n\sin N = x' \\ m\cos M = (y_1) & n\cos N = y_1' \end{array}\right\}\quad(540)$$

then, from the equations

$$\sin\gamma = m\sin M + n\sin N.\tau$$
$$\cos\gamma = m\cos M + n\cos N.\tau$$

we deduce, in the usual manner,

$$\sin(\gamma - N) = m\sin(M - N)$$
$$\cos(\gamma - N) = m\cos(M - N) + n\tau$$

or, putting $\psi = \gamma - N$, the solution is

$$\left.\begin{array}{l} \sin\psi = m\sin(M - N) \\ \tau = \dfrac{\cos\psi}{n} - \dfrac{m\cos(M - N)}{n} \\ T = T_0 + \tau \end{array}\right\}\quad(541)$$

where $\cos\psi$ is to be taken with the negative sign for the beginning and with the positive sign for the end.

To find the latitude and longitude of the extreme points corresponding to these times, we have, in (539), $\cos\beta = 0$, $\sin\beta = 1$, and, therefore, $C = 90°$, $c = \cos\gamma$: hence, taking $\gamma = N + \psi$,

$$\left.\begin{array}{ll}
\cos\varphi_1 \sin\vartheta = \sin\gamma & \\
\cos\varphi_1 \cos\vartheta = -\cos\gamma \sin d_1 & \\
\sin\varphi_1 = \cos\gamma \cos d_1 & \\
\tan\varphi = \dfrac{\tan\varphi_1}{\sqrt{(1-ee)}} & \omega = \mu_1 - \vartheta
\end{array}\right\} \tag{542}$$

317. *To find the duration of total or annular eclipse at any point of the curve of central eclipse.*—This is readily obtained from numbers which occur in the previous computations. Let T = the time of central eclipse, t = the duration of total or annular eclipse, then $T' = T \mp \frac{1}{2}t$ is the time of beginning or end. Let x and y be the moon's co-ordinates for the time T; ξ and η those of the point on the earth at this time; x', y', ξ', η', the hourly increments of these quantities; then, at the time T' we have, by (491),

$$(l - i\zeta)\sin Q = x \mp \tfrac{1}{2}x't - (\xi \mp \tfrac{1}{2}\xi't)$$
$$(l - i\zeta)\cos Q = y \mp \tfrac{1}{2}y't - (\eta \mp \tfrac{1}{2}\eta't)$$

But we here have $x = \xi$, $y = \eta$, and we may put $\zeta = \zeta_1 = \cos\beta$, whence

$$(l - i\cos\beta)\sin Q = \mp(x' - \xi')\frac{t}{2}$$
$$(l - i\cos\beta)\cos Q = \mp(y' - \eta')\frac{t}{2}$$

For the values of ξ' and η' we have, with sufficient precision, since t is very small,

$$\xi' = \mu'(-y \sin d + \cos\beta\cos d)$$
$$\eta' = \mu' x \sin d$$

Hence, by (511) and (512), we find, very nearly,

$$x' - \xi' = c' - \mu'\cos d\cos\beta = c' - f\cos\beta$$
$$y' - \eta' = -b'$$

If, therefore, we put

$$L = l - i\cos\beta \qquad a = c' - f\cos\beta \tag{543}$$

we have

$$L \sin Q = \frac{at}{2} \qquad\qquad L \cos Q = \frac{b't}{2}$$

where we omit the double sign, since it is only the numerical value of t that is required. Hence, we have, for finding t, the equations

$$\tan Q = \frac{a}{b'} \qquad\qquad t = \frac{7200\, L \sin Q}{a} \tag{544}$$

the last equation being multiplied by 3600, so that it now gives t in seconds.

The value of $\cos \beta$ is to be taken from the computation of the central curve for the given time T, and l, $\log i$, $\log f$, c', b', from our eclipse tables.

318. *To find where the central eclipse occurs at noon.*—In this case we have, evidently, $x = 0$, and hence, in (539),

$$\sin \beta = y_1 \tag{545}$$

by which β is to be found from the value of y_1 which corresponds to the time when $x = 0$. We then have $C = \beta$, $\rho = 1$, $\vartheta = 0$, and therefore the required point is found by the formulæ

$$\varphi_1 = \beta + d_1 \qquad\qquad \omega = \mu_1 \tag{546}$$

in which d_1 and μ_1 are taken for the time when $x = 0$.

319. The formulæ (539), (545), and (546) are not only extremely simple, but also entirely rigorous, and have this advantage over the methods commonly given, that they require no repetition to take into account the true figure of the earth. It may be observed here that the accurate computation of the central curve is of far greater practical importance than that of the limiting curves before treated of.

The formulæ (541) must be computed twice if we wish to obtain the times of beginning and end with the greatest possible precision; for, these times being unknown, we shall have at first to employ the values of x' and y' for the middle of the eclipse, and then to take their values for the times obtained by the first computation of the formulæ. With these new values a second computation will give the exact times.

EXAMPLE.—To compute the curve of central and total eclipse in the eclipse of July 18, 1860.

It is convenient first to prepare the values of $y_1 = \frac{y}{\rho_1}$ for the principal hours of the eclipse, as well as its mean hourly differences. With the value $\log \rho_1 = 9.99873$ we form, from the values of y given in the table p. 454, the following table:

Gr. T.	y_1	y_1'
0^h	+ 0.91972	— 0.16095
1	.75896	114
2	.59782	132
3	.43633	149
4	.27450	166
5	.11237	182

To find the times of beginning and end we may assume $T_0 = 2^h$; and for this time we have

$$(x) = m \sin M = -0.08124 \qquad x' = n \sin N = +0.5453$$
$$(y_1) = m \cos M = +0.59782 \qquad y_1' = n \cos N = -0.1613$$
$$\text{whence} \quad \log m = 9.78054 \qquad \log n = 9.7548$$
$$M = 352^\circ\ 15'\ 40'' \qquad N = 106^\circ\ 28'.7$$

Employing but four decimal places in the logarithms for a first approximation, we find, by (541),

$$-\frac{m \cos (M - N)}{n} = +0^h.435$$
$$\frac{\cos \psi}{n} = \mp 1.468$$
$$\tau_1 = -1.033$$
$$\tau_2 = +1.903$$

Approximate time of beginning $= 2^h - 1^h.033 = 0^h.967$

" " end $= 2 + 1.903 = 3.903$

Taking now x' and y_1' for these times respectively, and repeating the computation, we have

	Beginning.	End.
$x' = n \sin N$	$+ 0.54531$	$+ 0.54525$
$y_1' = n \cos N$	$- 0.16113$	$- 0.16164$
$\log n$	9.75482	9.75489
N	106° 27′ 42″	106° 30′ 45″
$-\frac{m \cos (M - N)}{n}$	$+ 0^h.4349$	$+ 0^h.4357$
$\frac{\cos \psi}{n}$	$- 1\ .4684$	$+ 1\ .4685$
T	0 .9665	3 .9042
ψ	213° 23′ 12″	326° 37′ 40″

For the latitude and longitude of the points of beginning and end, we now take $\gamma = N + \psi$, and with the values of d_1 and μ_1 (pp. 455, 464) for the above computed times, we have

	Beginning.	End.
$\gamma = N + \psi$	319° 50′ 54″	73° 8′ 25″
d_1	21 1 15	21 0 0
μ_1	13 1 1	57 5 3

whence, by (542),

φ	45° 36′ 50″	15° 45′ 34″
ω	126 3 8	320 53 9
Local App. Time $= \vartheta$	$16^h\ 27^m.9$	$6^h\ 24^m.2$

For the series of points on the curve we take the times $1^h.0$, $1^h.2$, $1^h.4$ . . . . $3^h.6$, $3^h.8$, which are embraced within the extreme times above found, and proceed by (539). Thus, for $2^h.0$ we have

T	$2^h.$
$x = \sin \beta \sin \gamma$	$- 0.08124$
$y_1 = \sin \beta \cos \gamma$	$+ 0.59782$
$\log \sin \beta$	9.78054
$\log \cos \beta = \log c \cos C$	9.90173
$\log y_1 = \log c \sin C$	9.77657
$\log c$	9.99856
C	36° 51′ 21″
d_1	21 0 49

$C + d_1$	57° 52′ 10″
$\log x = \log \cos \varphi_1 \sin \vartheta$	n8.90977
$\log c \cos (C + d_1) = \log \cos \varphi_1 \cos \vartheta$	9.72435
$\log c \sin (C + d_1) = \log \sin \varphi_1$	9.92636
ϑ	351° 17′ 13″
μ_1	28 31 12
ω	37 13 59
φ	57 39 20
App. Time = ϑ in time,	23^h 25^m $8^s.8$

For the duration of totality at this point, we take from pp. 454. 464, 465,

$$l = -\,0.009082 \qquad b' = +\,0.1532$$
$$\log i = 7.6608 \qquad c' = +\,0.6011$$
$$\log f = 9.3883$$

and hence, with $\log \cos \beta = 9.9017$ above found, we obtain, by (543),

$$L = -\,0.012734 \qquad a = +\,0.4061$$

and, by (544), disregarding the negative sign of L,

$$t = 211^s.3 = 3^m\, 31^s.3$$

For the place where the central eclipse occurs at noon, we find that $x = 0$ at the time $T = 2^h.149$, at which time we have

$y_1 = \sin \beta$	+ 0.57378
β	35° 0′ 53″
d_1	21 0 45
φ_1	56 1 38
φ	56 6 57
$\mu_1 = \omega$	30 45 18

The whole curve may be traced through the points given in the following table:

SOLAR ECLIPSE, July 18, 1860.—CURVE OF CENTRAL AND TOTAL ECLIPSE

Greenwich Mean Time.	Latitude. ϕ	Long. W. from Greenwich. ω	App. Local Time. ϑ	Duration of Totality.
$0^h.967$	45° 36′.4	126° 3′.1	16^h $27^m.9$	
1 .0	50 37 .8	113 11 .6	17 21 .3	2^m $1^s.5$
1 .2	57 16 .2	89 14 .6	19 9 .1	2 35 .1
1 .4	59 29 .1	72 52 .8	20 26 .6	2 55 .8
1 .6	59 55 .1	59 5 .2	21 33 .7	3 11 .4
1 .8	59 11 .6	47 16 .6	22 33 .0	3 23 .1
2 .0	57 39 .3	37 14 .0	23 25 .1	3 31 .3
2 .149	56 7 .0	30 45 .3	0 0 .0	3 34 .7
2 .2	55 31 .5	28 42 .6	0 11 .2	3 36 .2
2 .4	52 56 .9	21 25 .1	0 52 .4	3 38 .0
2 .6	50 0 .9	15 3 .9	1 29 .8	3 36 .4
2 .8	46 46 3	9 21 .8	2 4 .6	3 32 .0
3 .0	43 13 .6	4 2 .2	2 37 .9	3 24 .6
3 .2	39 20 .7	358 47 .1	3 10 .9	3 14 .4
3 .4	35 1 .6	353 12 .5	3 45 .3	3 1 .1
3 .6	30 1 .5	346 35 .4	4 23 .7	2 43 .5
3 .8	23 28 .5	336 44 .1	5 15 .1	2 18 .5
3 .904	15 45 .6	320 53 .2	6 24 .8	

Northern and Southern Limits of Total or Annular Eclipse.

320. *To find the northern and southern limits of total or annular eclipse.*—As already remarked in Art. 314, these limits may be rigorously determined by the method of Art. 311, by taking $l =$ the radius of the umbra (*i.e.* for interior contacts); but I here propose to deduce them from the previously computed curve of central eclipse. This radius l is assumed to be so small that we may neglect its square, which can seldom exceed .0003, and this degree of approximation will in the greater number of cases suffice to determine points on the limits within 2′ or 3′, which is practically quite accurate enough.

The two limiting curves of total or annular eclipse, then, lie so near to the central curve that the value $\zeta_1 = \cos\beta$, for a given time T, already found in the computation of the latter curve, may be used for the former in the approximate equation which determines Q. We can, therefore, immediately find Q by (535),—*i.e.*

$$\left.\begin{aligned} \tan \nu' &= \frac{f}{e} \cos \beta \\ \tan (Q - \tfrac{1}{2} E) &= \tan (45° + \nu') \tan \tfrac{1}{2} E \end{aligned}\right\} \quad (547)$$

where f, e, and E are to be taken from the eclipse tables for the time T.

The co-ordinates of the point on the central curve corresponding to the time T being $\xi = x$ and $y_1 = \eta_1$ (Art. 315), those for a point on the limiting curve may be denoted by $x + dx$ and $y_1 + dy_1$. These being substituted for ξ and η_1 in the equations (499), we have

$$dx = -(l - i\zeta_1) \sin Q \qquad\qquad dy_1 = -(l - i\zeta_1) \cos Q$$

where in the expression for dy_1 we omit the divisor ρ_1, as not appreciably changing the value of so small a term.

Let φ_1, ϑ, ω be taken from the computation of the central curve for the time T, and let $\varphi_1 + d\varphi_1$, $\omega + d\omega$, be the corresponding values of φ_1 and ω for the point on the limit for the same time. Then, by differentiating (500), observing that $d\vartheta = -d\omega$, we have

$$\begin{aligned} \cos \varphi_1 \cos \vartheta \, d\omega + \sin \varphi_1 \sin \vartheta \, d\varphi_1 &= -dx \\ \cos \varphi_1 \sin \vartheta \, d\omega - \sin \varphi_1 \cos \vartheta \, d\varphi_1 &= -dy_1 \sin d_1 + d\zeta_1 \cos d_1 \\ \cos \varphi_1 d\varphi_1 &= dy_1 \cos d_1 + d\zeta_1 \sin d_1 \end{aligned}$$

whence, by eliminating $d\zeta_1$ and substituting ζ_1 for its value given by the third equation of (497), we find

$$\begin{aligned} \zeta_1 \cos \varphi_1 d\omega &= -dx \,(\cos \varphi_1 \cos d_1 + \sin \varphi_1 \cos \vartheta \sin d_1) \\ &\quad - dy_1 \sin \varphi_1 \sin \vartheta \\ \zeta_1 d\varphi_1 &= -dx \sin \vartheta \sin d_1 + dy_1 \cos \vartheta \end{aligned}$$

Hence, substituting $\cos \beta$ for ζ_1,

$$\begin{aligned} d\omega &= \frac{l - i \cos \beta}{\cos \beta} (\cos \vartheta \sin Q \sin d_1 + \sin \vartheta \cos Q) \tan \varphi_1 \\ &\quad + \frac{l - i \cos \beta}{\cos \beta} \sin Q \cos d_1 \\ d\varphi_1 &= \frac{l - i \cos \beta}{\cos \beta} (\sin \vartheta \sin Q \sin d_1 - \cos \vartheta \cos Q) \end{aligned}$$

These values are yet to be divided by sin 1′ to reduce them to minutes of arc. It will be convenient to put

$$\left.\begin{aligned} l' &= \frac{l}{\sin 1'} \qquad\qquad i' = \frac{i}{\sin 1'} \\ \lambda &= \frac{l - i\cos\beta}{\cos\beta \sin 1'} = \frac{l'}{\cos\beta} - i' \end{aligned}\right\} \qquad (548)$$

in which l', i', and λ will be expressed in minutes.

We may in practice substitute $d\varphi$ for $d\varphi_1$, within the limits of accuracy we have adopted; for we find, from the equations on p. 457,

$$d\varphi = \frac{d\varphi_1}{\sqrt{(1-ee)}} \cdot \frac{\cos^2\varphi}{\cos^2\varphi_1} = d\varphi_1 \frac{1 - ee\sin^2\varphi}{\sqrt{(1-ee)}}$$

where the multiplier of $d\varphi_1$ cannot differ more from unity than $\sqrt{(1-ee)}$ does,—*i.e.* not more than 0.00335: so that the substitution of one for the other will never produce an error of 1′ so long as $d\varphi_1$ is less than 5°.

Finally, adapting the values of $d\omega$ and $d\varphi$ for logarithmic computation, by putting

$$\left.\begin{aligned} h\sin H &= \cos Q \\ h\cos H &= \sin Q \sin d_1 \\ \text{we have} \qquad & \\ d\omega &= \lambda\,[h\cos(\vartheta - H)\tan\varphi_1 + \sin Q\cos d_1] \\ d\varphi &= \lambda\, h\sin(\vartheta - H) \end{aligned}\right\} \qquad (549)$$

The formulæ (547) give two values of Q differing 180°. The second value will evidently give the same numerical values of $d\omega$ and $d\varphi$, but with opposite signs; and therefore we may compute the equations (549) with only the acute value of Q, and then the longitude and latitude of a point on one of the limits are

$$\omega + d\omega, \qquad\qquad \varphi + d\varphi$$

and those of a point on the other limit are

$$\omega - d\omega, \qquad\qquad \varphi - d\varphi$$

The first of these limits will be the northern in the case of total eclipse, but the southern in the case of annular eclipse observing always to take l with the negative sign for total eclipse, as it comes out by the formulæ (487) and (489).

It is evident that this approximate method is not accurate when $\cos\beta$ is very small, that is, near the extreme points of the curves; and it fails wholly for these points themselves, since $\cos\beta$ is then zero and the value of λ becomes infinite. These extreme points, however, are determined directly in a very simple manner by the formulæ (536), (537), (538), combined with (519), by employing in (536) and (537) the value of l for interior contacts; and it is with these formulæ, therefore, that the computation of the limits of total or annular eclipse should be com menced.

Example.—Find the northern and southern limits of total eclipse in the eclipse of July 18, 1860.

First. To find the extreme points.—The values of b' and c' for exterior contacts, from which the values of E on p. 465 are derived, differ so little from those for interior contacts that in practice, unless extreme precision is required, we may dispense with the computation of the latter. For our present example, therefore, taking the value of E for $T_0 = 2^h$ and the mean value of $\log e$, as in the computation of the extreme points of the southern limit for the penumbra, p. 487, together with

$$l = -0.0091$$

we find, by (537), for the northern limit,

$$\log m = 9.7854 \qquad M = 352° \; 33'.6$$
$$\log n = 9.7553 \qquad N = 106 \;\; 27'.0$$

and for the southern limit,

$$\log m = 9.7731 \qquad M = 351° \; 55'.0$$
$$\log n = 9.7542 \qquad N = 106 \;\; 27.0$$

Hence, by (538),

	Northern Limit.		Southern Limit.	
	First Point.	Last Point.	First Point.	Last Point.
ψ	213° 54′.3	326° 5′.7	212° 39′.0	327° 21′.0
T	$0^h.976$	$3^h.892$	$0^h.951$	$3^h.917$

Taking $\gamma = N + \psi$, and the values of d_1 and μ_1 for these times respectively, with $\log \rho_1 = 9.9987$, we find, by (518) and (519),

γ	320° 21′.3	72° 32′.7	319° 6′.0	73° 48 .0
log tan γ'	n9.9170	0.5012	n9.9363	0.5355
d_1	21° 1′.2	21° 0′.0	21° 1′.2	21° 0′.0
ϑ	246 31 .7	96 26 .7	247 26 .7	95 57 .7
μ_1	13 9 .6	56 54 .1	12 47 .1	57 16 .6
ω	126 37 .9	320 27 .4	125 20 .4	321 18 .9
φ	46 7 .7	16 21 .6	45 2 .8	15 11 .4

Second. To find a series of points between these extremes, by the aid of the curve of central eclipse, we assume the same series of times as in the computation of that curve, and proceed by (547), (548), and (549); to illustrate the use of which I add the computation for $T = 2^h$ in full. From the computation, p. 496, we have

For $T = 2^h$

log cos β	9.9017
log tan φ_1	0.1970
ϑ	351° 17′.2
d_1	21 0 .8
ω	37 14 .0
φ	57 39 .3

Then, by (547),

(p. 465) log $\frac{f}{e}$	9.5953
log cos β	9.9017
log tan ν'	9.4970
ν'	17° 26′.0
$\frac{1}{2}E$	7 8 .7
log tan (45° + ν')	0.2823
log tan $\frac{1}{2}E$	9.0982
log tan ($Q - \frac{1}{2}E$)	9.3805
$Q - \frac{1}{2}E$	13° 30′.3
Q	20 39 .0

By (548),

l	— 0.009082
log l	n7.9582
log l'	n1.4945
log i	7.6608
log i'	1.1971
i'	15′.74
l' sec β	— 39 .16
λ	— 54 90

Hence, by (549),

log cos Q = log h sin H	9.9712
log sin Q sin d_1 = log h cos H	9.1020
log h	9.9751
H	82° 18′.2
$\vartheta - H$	268 59 .0

$\log \lambda$	$n1.7396$
$\log h$	9.9751
$\log \cos (\vartheta - H)$	$n8.2490$
$\log \tan \varphi_1$	0.1979
$\log (1)$	0.1607
$\log \lambda$	$n1.7396$
$\log \sin Q \cos d_1$	9.5175
$\log (2)$	$n1.2571$
(1)	$+ \ 1'.45$
(2)	$- 18.08$
$d\omega$	$- 16.63$

$\log \lambda$	$n1.7396$
$\log h$	9.9751
$\log \sin (\vartheta - H)$	$n9.9999$
$\log d\varphi$	1.7146
$d\varphi$	$+ 51'.83$

Hence, for the time $T = 2^h$, we have the two points,

	N. Limit.	S. Limit.
$\omega \pm d\omega$	36° 57'.4	37° 30'.6
$\varphi \pm d\varphi$	58 31.1	56 47.5

SOLAR ECLIPSE, July 18, 1860.

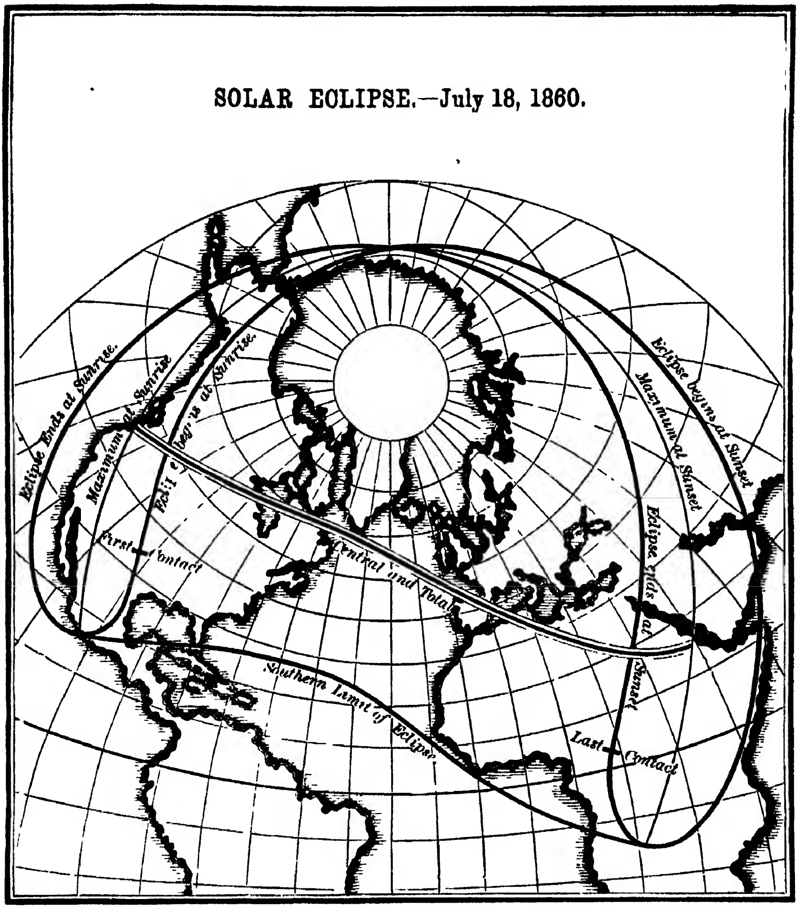
Northern Limit of Total Eclipse.

Gr. T.	Latitude. ϕ	Longitude. ω
$0^h.976$	46° 8'	126° 38'
1 .0	50 18	116 27
1 .2	57 47	90 57
1 .4	60 13	74 0
1 .6	60 46	59 40
1 .8	60 4	47 23
2 .0	58 31	36 57
2 .2	56 21	28 9
2 .4	53 43	20 40
2 .6	50 43	14 12
2 .8	47 24	8 44
3 .0	43 47	3 1
3 .2	39 49	357 43
3 .4	35 25	352 6
3 .6	30 18	345 23
3 .8	23 31	335 8
3 .892	16 22	320 27

Southern Limit of Total Eclipse.

Gr. T.	Latitude. ϕ	Longitude. ω
$0^h.951$	45° 3'	125° 20'
1 .0	50 57	109 56
1 .2	56 45	87 33
1 .4	58 45	71 46
1 .6	59 4	58 31
1 .8	58 19	47 11
2 .0	56 48	37 31
2 .2	54 42	29 16
2 .4	52 11	22 10
2 .6	49 19	15 56
2 .8	46 9	10 39
3 .0	42 41	5 3
3 .2	38 52	359 51
3 .4	34 38	354 20
3 .6	29 45	347 48
3 .8	23 26	338 20
3 .917	15 11	321 19

321. The curves above computed are all exhibited in the following chart.



SOLAR ECLIPSE.—July 18, 1860.

For the construction of such charts, on even a much larger scale, the degree of accuracy with which our computations have been made is far greater than is necessary, and many abridgments may be made which will readily occur to the skilful computer.*

* For a graphic method of constructing eclipse charts, see a paper by Mr Chauncey Wright, Proceedings of the Am. Association for the Adv. of Science, 8th meeting (1854), p. 55.

Prediction of a Solar Eclipse for a Given Place.

322. *To compute the time of the occurrence of a given phase of a solar eclipse for a given place.*—The given phase is expressed by a given value of Δ, and we are to find the time when this value and the co-ordinates of the given place satisfy the conditions (485). This can only be done by successive approximations.

Let it be proposed to find the time of beginning or ending of the eclipse at the place. The phase is then $\Delta = l - i\zeta$, and we must satisfy the equations (491). Let T_0 be an assumed time, and $T = T_0 + \tau$ the required time. Let x, y, x', y', d, l, $\log i$, be taken from the eclipse tables (p. 454) for the time T_0. Assuming that x and y vary uniformly, their values at the time T are $x + x'\tau$ and $y + y'\tau$. The co-ordinates of the place at the time T_0 are found by (483) or (483*), in which μ is the sidereal time at the place. Putting

$$\vartheta = \mu - a = \mu_1 - \omega$$

in which ω is the west longitude of the place and μ_1 may be taken from the table (p. 455) for the time T_0, the formulæ become

$$\left.\begin{array}{ll} A \sin B = \rho \sin \varphi' & \xi = \rho \cos \varphi' \sin \vartheta \\ A \cos B = \rho \cos \varphi' \cos \vartheta & \eta = A \sin (B - d) \\ & \zeta = A \cos (B - d) \end{array}\right\} \quad (550)$$

Let ξ', η' denote the hourly increments of ξ and η; then, assuming that these increments also are uniform, the values of the co-ordinates at the time T are $\xi + \xi'\tau$ and $\eta + \eta'\tau$. The values of ξ' and η' are found by the formulæ (p. 462)

$$\xi' = \mu' \rho \cos \varphi' \cos \vartheta$$
$$\eta' = \mu' \xi \sin d - d'\zeta$$

in which μ' and d' are the hourly changes of μ and d multiplied by $\sin 1''$. The rate of approximation will not be sensibly affected by omitting the small term $d'\zeta$, and the formulæ for ξ' and η' may then be written as follows:

$$\xi' = \mu' A \cos B \qquad\qquad \eta' = \mu' \xi \sin d \qquad (551)$$

Put

$$L = l - i\zeta$$

then, neglecting the variation of this quantity in the first approximation, the conditions (491) become, for the time T,

$$L \sin Q = x - \xi + (x' - \xi')\,\tau$$
$$L \cos Q = y - \eta + (y' - \eta')\,\tau$$

Let the auxiliaries m, M, n, and N be determined by the equations

$$\left.\begin{aligned} m \sin M &= x - \xi \qquad & n \sin N &= x' - \xi' \\ m \cos M &= y - \eta \qquad & n \cos N &= y' - \eta' \end{aligned}\right\} \quad (552)$$

then, from the equations

$$\begin{aligned} L \sin Q &= m \sin M + n \sin N . \tau \\ L \cos Q &= m \cos M + n \cos N . \tau \end{aligned}$$

we deduce

$$\begin{aligned} L \sin (Q - N) &= m \sin (M - N) \\ L \cos (Q - N) &= m \cos (M - N) + n\tau \end{aligned}$$

Hence, putting $\psi = Q - N$, we have

$$\left.\begin{aligned} \sin \psi &= \frac{m \sin (M - N)}{L} \\ \tau &= \frac{L \cos \psi}{n} - \frac{m \cos (M - N)}{n} \end{aligned}\right\} \quad (553)$$

by which τ is found. Since the first of these equations does not determine the sign of $\cos \psi$, the latter may be taken with either the positive or the negative sign. We thus obtain two values of $T = T_0 + \tau$, the first given by the negative sign of $\frac{L \cos \psi}{n}$ being the time of beginning, and the second given by the positive sign being the time of ending of the eclipse at the place.

For a second approximation, let each of the computed times (or two times nearly equal to them) be taken as the assumed time T_0, and compute the equations (550), (551), (552), (553) for beginning and end separately.

The first approximation may be in error several minutes, but the second will always be correct within a few seconds, and, therefore, quite as accurate as can be required; for a *perfect* prediction cannot be attained in the present state of the Ephemerides.

The formula for τ may also be expressed as follows:

$$\tau = \frac{m}{n} \cdot \frac{\sin (M - N - \psi)}{\sin \psi}$$

which in the second approximation will be more convenient than the former expression; but when $\sin \psi$ is very small it will not be so precise.

If we put

$$t = \text{the local mean time of beginning or end,}$$

we have

$$t = T_0 + \tau - \omega.$$

323. The prediction for a given place being made for the purpose of preparing to observe the eclipse, it is necessary also to know the point of the sun's limb at which the first contact is to take place, in order to direct the attention to that point. This is given at once by the value of

$$Q = N + \psi$$

which is the angular distance of the point of contact reckoned from the north point of the sun's limb towards the east (Art. 295).

The simplest method of distinguishing the point of contact on the sun's limb is (as BESSEL suggested) by a thread in the eye-piece of the telescope, arranged so that it can be revolved and made tangent to the sun's limb at the point. The observer then, by a slow motion of the instrument, keeps the limb very nearly in contact with the thread until the eclipse begins. The position of the thread is indicated by a small graduated circle on the rim of the eye-piece, as in the common position micrometer.

This method is applicable whatever may be the kind of mounting of the telescope. Nevertheless, if the instrument is arranged with motion in altitude and azimuth, it will be convenient to know the angle of the point of contact from the *vertex* of the sun's limb, which is that point of the limb which is nearest to the zenith. The distance of the vertex from the north point of the limb is equal to the parallactic angle which being here denoted by γ, is found, according to Art. 15, by the formulæ

$$p \sin \gamma = \cos \varphi \sin \vartheta$$
$$p \cos \gamma = \sin \varphi \cos d - \cos \varphi \sin d \cos \vartheta$$

(where we have put p for $\sin \zeta$ and ϑ for the sun's hour angle). As γ is not required with very great accuracy, we may here take [see (494)]

$$p \sin \gamma = \xi \qquad\qquad p \cos \gamma = \eta$$

in which ξ and η are the values of the co-ordinates of the place at the instant of contact. But, if ξ and η denote the values at the time T_0, we must take

$$p \sin \gamma = \xi + \xi'\tau \qquad\qquad p \cos \gamma = \eta + \eta'\tau \qquad (554)$$

in which we employ the values of ξ, η, ξ', η', and τ furnished by the last approximation. We then have

$$\left.\begin{array}{l}\text{Angular distance of the point of contact from}\\ \text{the vertex towards the east,}\end{array}\right\} \begin{array}{l}= Q - \gamma \\ = N + \psi - \gamma\end{array} \qquad (555)$$

324. *To find the instant of maximum obscuration for a given place, and the degree of obscuration.*—At the instant of greatest obscuration the distance $\varDelta$ of the axis of the shadow from the place of observation is a minimum.* If we denote the required time by $T_1 = T_0 + \tau_1$, the equations of Art. 322 determine τ_1 for a given value of $\varDelta$ if we substitute $\varDelta$ for L. Denoting the value of $Q - N$ for this case by ψ_1, we have, therefore,

$$\varDelta \sin \psi_1 = m \sin (M - N)$$
$$\varDelta \cos \psi_1 = m \cos (M - N) + n\tau_1$$

the sum of the squares of which gives

$$\varDelta^2 = m^2 \sin^2 (M - N) + [m \cos (M - N) + n\tau_1]^2$$

Since m and M are computed for the time T_0, and N is sensibly constant, the term $m^2 \sin^2 (M - N)$ is constant, and therefore $\varDelta$ is a minimum when the last term is zero, that is, when

$$\tau_1 = -\frac{m \cos (M - N)}{n} \qquad (556)$$

which quantity is already known from the computation of (553).

We have, also,

$$\varDelta = \pm m \sin (M - N) = \pm L \sin \psi \qquad (557)$$

in which the sign is to be so taken as to make $\varDelta$ positive. The degree of obscuration is then given by the formula (Art. 310),

$$D = \frac{L - \varDelta}{L + L_1}$$

in which D is expressed in fractional parts of the sun's diameter, and L and L_1 are the radii of the penumbra and umbra (the

* More strictly, $L - \varDelta$ is a maximum, as in Art. 309; but we here neglect the small variation of L. The rigorous solution of the problem may be obtained from the condition (526) $P' = 0$; but the above approximation is sufficient in practice.

latter being negative) for the place of observation. From (488) we find, by putting $\sec f = 1$,

$$L + L_1 = 2(L - k)$$

and hence

$$D = \frac{L - \Delta}{2(L - k)} \tag{558}$$

in which $k = 0.2723$.

If we neglect the augmentation of the moon's diameter, or, which is equivalent, the small difference between L and l, and put

$$\left.\begin{aligned} \varepsilon &= \frac{l}{2(l - k)} \\ D &= \varepsilon \mp \varepsilon \sin \psi \end{aligned}\right\} \tag{559}$$

we have

where the lower sign is to be used when $\sin \psi$ is negative, so that D is always the *numerical* difference of ε and $\varepsilon \sin \psi$. In this form ε may be computed for the eclipse generally, and ψ will be derived from the computation for the penumbra for the given place. A preference should be given to the value of ψ found from the computation for the time nearest to that of greatest obscuration, which is usually that used in the first approximation of Art. 322.

EXAMPLE.—Find the time of beginning and end, &c., of the eclipse of July 18, 1860, at Cambridge, Mass.

The latitude and longitude are

$$\varphi = 42° \, 22' \, 49'' \qquad \omega = 71° \, 7' \, 25''$$

For this latitude we find, by the aid of Table III., or by the formulæ (87),

$$\log \rho \sin \varphi' = 9.82644 \qquad \log \rho \cos \varphi' = 9.86912$$

With the aid of the chart, p. 504, we estimate the time of the middle of the eclipse at Cambridge to be not far from 1^h. Hence, taking $T_0 = 1^h$ for our first approximation, we take for this time, from the eclipse tables, p. 454,

$x = -\ 0.6266$	$x' = +\ 0.5453$	$l = 0.5368$
$y = +\ 0.7567$	$y' = -\ 0.1605$	$\log i = 7.66287$
$d = 20° \, 57'.4$	$\mu_1 = 13° \, 31'.2$	$\log \mu' = 9.41799$

Hence, by (550) and (551),

$$\mu_1 \sim \omega = \vartheta = 302^\circ 23'.8 \qquad \log \zeta = 9.7853$$
$$B = 59\ 24.6 \qquad i\zeta = 0.0028$$
$$\xi = -0.6246 \qquad L = l - i\zeta = 0.5340$$
$$\eta = +0.4844$$
$$\xi' = +0.1038$$
$$\eta' = -0.0585$$

and, by (552) and (553),

$$m \sin M = x - \xi = -0.0020 \qquad n \sin N = x' - \xi' = +0.4415$$
$$m \cos M = y - \eta = +0.2723 \qquad n \cos N = y' - \eta' = \cdot\ 0.1020$$
$$\log m = 9.4350 \qquad \log n = 9.6562$$
$$M = 359^\circ 34'.7 \qquad N = 103^\circ 0'.6$$
$$M - N = 256\ 34.1 \qquad -\frac{m \cos (M - N)}{n} = +0^h.140$$
$$\log \sin \psi = n9.6955 \qquad \frac{L \cos \psi}{n} = \mp 1.023$$
$$\log \cos \psi = 9.9387$$
$$\tau = \begin{cases} -\ 0.883 \\ \text{or} +\ 1.163 \end{cases}$$

Approximate time of beginning $= 0^h.117$

" " end $= 2.163$

Taking then for a second approximation $T_0 = 0^h.12$ for beginning, and $T_0 = 2^h.16$ for end, we shall find*

	Beginning.	End.
T_0	$0^h.12$	$2^h.16$
x	$-$ 1.10642	$+$ 0.00601
y	$+$ 0.89783	$+$ 0.57034
x'	$+$ 0.54528	$+$ 0.54530
y'	$-$ 0.16015	$-$ 0.16090
d	20° 57′ 45″	20° 56′ 53″
μ_1	0 19 8	30 55 13
l	0.53686	0.53673
ϑ	289° 11′ 43″	319° 47′ 48″
ξ	$-$ 0.69868	$-$ 0.47755
η	$+$ 0.53915	$+$ 0.42423
$\log \zeta$	9.66935	9.88504
ξ'	$+$ 0.06368	$+$ 0.14793
η'	$-$ 0.06544	$-$ 0.04470

* The values of x' and y' here employed are not those given in the table p. 455, but their actual values for the time T_0, as given in the table of x' and y' on p. 464.

	Beginning.	End.
$i\zeta$	0.00215	0.00353
L	0.53471	0.53320
$m \sin M$	— 0.40774	+ 0.48356
$m \cos M$	+ 0.35868	+ 0.14611
$\log m$	9.73484	9.70342
M	311° 20′ 16″	73° 11′ 15″
$n \sin N$	+ 0.48160	+ 0.39737
$n \cos N$	— 0.09471	— 0.11620
$\log n$	9.69093	9.61702
N	101° 7′ 32″	106° 18′ 0″
$M - N$	210 12 44	326 53 15
ψ	210 44 0	328 49 56
$M - N - \psi$	— 31′ 16″	— 1° 56′ 41″
τ	+ $0^h.0197$	+ $0^h.0800$
T	$0^h.1397$	2 .2400
	0^h 8^m 23^s	2^h 14^m 24^s
ω	4 44 30	4 44 30
Local time, t	19 23 53	21 29 54
	July 17.	July 17.
Angle of Pt. of Contact from North Pt. of the sun = $Q = N + \psi$	311° 51′ 32″	75° 7′ 56″

A third approximation, commencing with the last computed times, changes them by only a fraction of a second.

To find the angular distance of the point of contact from the *vertex* of the sun's limb, we have from the second approximation, by (554) and (555),

	Beginning.	End.
$\xi + \xi'\tau = p \sin \gamma$	— 0.6974	— 0.4658
$\eta + \eta'\tau = p \cos \gamma$	+ 0.5379	+ 0.4206
γ	307° 38′.8	312° 4′.5
Angle from vertex $= Q - \gamma$	4 12 .7	123 3 .4

The time of greatest obscuration is best found from the first approximation, which gives, by (556),

$$
\begin{aligned}
T_0 &= \quad 1^h. \\
-\frac{m\cos(M-N)}{n} = \tau_1 &= +\,0.140 \\
T_1 &= \quad 1^h.140 \\
&= \quad 1^h\ 8^m\ 24^s. \\
\omega &= \quad 4\ 44\ \ 30 \\
\text{Local time of max. obscur.} = t &= \quad 2_h\ 23\ \ 54
\end{aligned}
$$

For the amount of greatest obscuration we have, also, from the first approximation, by (557) and (558),

$$
\begin{aligned}
L &= 0.5340 \\
k &= 0.2723 \\
L - k &= 0.2617 \\
2(L-k) &= 0.5234
\end{aligned}
\qquad\qquad
\begin{aligned}
\log I &= \ \ 9.7275 \\
\log \sin \psi &= n9.6955 \\
\log \Delta &= \ \ 9.4230 \\
\Delta &= \ \ 0.2649 \\
L - \Delta &= \ \ 0.2691
\end{aligned}
$$

$$D = \frac{0.2691}{0.5234} = 0.514$$

Or, by (559), taking as constant the value of ε found by employing the mean value $l = 0.5367$, *i.e.*

$$\varepsilon = 1.015$$

we have

$$
\begin{aligned}
\varepsilon \sin \psi &= -\,0.503 \\
D &= \quad 0.512
\end{aligned}
$$

which is quite accurate enough.

325. *Prediction for a given place by the method of the American Ephemeris.*—This method is based upon a transformation of Bessel's formula suggested by T. Henry Safford, Jr., and, with the aid of the extended tables in the Ephemeris, is somewhat more convenient than the preceding. The fundamental equation (490) gives, by transposition,

$$(x - \xi)^2 = (l - \zeta \tan f)^2 - (y - \eta)^2$$

the second member of which may be resolved into the factors

$$
\begin{aligned}
b &= (l - \zeta \tan f) + (y - \eta) \\
c &= (l - \zeta \tan f) - (y - \eta)
\end{aligned}
$$

or, by (494),

$$
\begin{aligned}
b = l + y &- \rho \sin \varphi' (\cos d + \sin d \tan f) \\
&+ \rho \cos \varphi' (\sin d - \cos d \tan f) \cos \vartheta \\
c = l - y &+ \rho \sin \varphi' (\cos d - \sin d \tan f) \\
&- \rho \cos \varphi' (\sin d + \cos d \tan f) \cos \vartheta
\end{aligned}
$$

If we put

$$A = x \qquad B = l + y \qquad C = -l + y$$

$$\begin{aligned} E &= \cos d + \sin d \tan f = \cos(d - f) \sec f \\ F &= \cos d - \sin d \tan f = \cos(d + f) \sec f \\ G &= \sin d - \cos d \tan f = \sin(d - f) \sec f \\ H &= \sin d + \cos d \tan f = \sin(d + f) \sec f \end{aligned}$$

all of which are independent of the place of observation and are given in the Ephemeris for each solar eclipse, for successive times at the Washington meridian, we shall then have to compute for the place

$$\left.\begin{aligned} a &= x - \xi = A - \rho \cos \varphi' \sin \vartheta \\ b &= B - E\rho \sin \varphi' + G\rho \cos \varphi' \cos \vartheta \\ c &= -C + F\rho \sin \varphi' - H\rho \cos \varphi' \cos \vartheta \end{aligned}\right\} \quad (560)$$

and the fundamental equation becomes

$$a = \sqrt{bc}$$

We have here, as before, $\vartheta = \mu_1 - \omega$; and the value of μ_1 is also given in the Ephemeris for the Washington meridian.

If now for any assumed time T_0 we take from the Ephemeris the values of these auxiliaries, and, after computing a, b, and c by (560), find that a differs from $\sqrt{bc}$, the assumed time requires to be corrected; and the correction is found by the following process. Put

$m = \sqrt{bc}$,
$a', b', m' =$ the changes of a, b, m, in one second,
$\tau =$ the required correction of the assumed time;

then at the time of beginning or ending of the eclipse we must have

$$a + a'\tau = m + m'\tau$$

whence

$$\tau = \frac{m - a}{a' - m'}$$

To find a' we have, by differentiating the value of a and denoting the derivatives by accents,

$$a' = A' - \mu' \rho \cos \varphi' \cos \vartheta \quad (561)$$

in which μ' denotes the change of μ_1 in one second, and is the same as the μ' of our former method divided by 3600.

To find m' we have, following the same notation, and neglecting the small changes of E, F, G, H, l, and f,

$$\begin{aligned} B' &= \quad y' = C' \\ b' &= \quad B' - \mu' G \rho \cos \varphi' \sin \vartheta \\ c' &= - C' + \mu' H \rho \cos \varphi' \sin \vartheta \end{aligned}$$

Since f is small, we may in these approximate expressions put $G = H$, and hence

$$b' = - c' = B' - \mu' G \rho \cos \varphi' \sin \vartheta \tag{561*}$$

Now, from the formula $m^2 = bc$, we derive

$$2\, mm' = cb' + bc' = (c - b)\, b'$$

$$m' = \tfrac{1}{2}\left(\sqrt{\frac{c}{b}} - \sqrt{\frac{b}{c}}\right) b'$$

which, if we assume

$$\tan \tfrac{1}{2} Q = \sqrt{\frac{c}{b}} = \frac{c}{m} = \frac{m}{b} \tag{562}$$

becomes

$$m' = - b' \cot Q$$

and therefore τ is found by the formula

$$\tau = \frac{m - a}{a' + b' \cot Q} \tag{563}$$

The Ephemeris gives also the values of A', B', and C', which are the changes of A, B, and C in one second. These changes being very small, the unit adopted in expressing them is .000001; so that the above value of τ, as also the value of μ' in (561), must be multiplied by 10^6. The formulæ (560–563) then agree with those given in the explanation appended to the Ephemeris.

It is easily seen that Q here denotes the same angle as in the preceding articles; for we have at the instant of contact

$$\tan Q = - \frac{b'}{m'} = \frac{2\, m}{b - c} = \frac{x - \xi}{y - \eta}$$

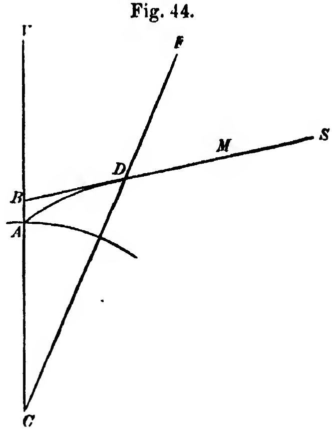
Examples of the application of this method are given in every volume of the American Ephemeris.

326. The preceding articles embrace all that is important in relation to the prediction of solar eclipses. Since absolute rigor is not required in mere predictions, I have thus far said nothing of the effect of refraction, which, though extremely small, must be treated of before we proceed to the application of *observed* eclipses, where the greatest possible degree of precision is to be sought.

CORRECTION FOR ATMOSPHERIC REFRACTION IN ECLIPSES.

327. That the refraction varies for bodies at different distances from the earth has already been noticed in Art. 106; but the difference is so small that it is disregarded in all problems in which the absolute position of a single body is considered. Here, however, where two points at very different distances from the earth are observed in apparent contact, it is worth while to inquire how far the difference in question may affect our results.

Fig. 44.



Let $SMDA$, Fig. 44, be the path of the ray of light from the sun's limb to the observer at A, which touches the moon's limb at M; SMB the straight line which coincides with this path between S and M, but when produced intersects the vertical line of the observer in B. It is evident that the observer at A sees an apparent contact of the limbs at the instant when an observer at B would see a true contact if there were no refraction. Hence, if we substitute the point B for the point A in the formula of the eclipse, we shall fully take into account the effect of refraction.

For the purpose of determining the position of the point B, whose distance from A is very small, it will suffice to regard the earth as a sphere with the radius $\rho = CA$. It is one of the properties of the path of a ray of light in the atmosphere that the product $q\mu \sin i$ is constant (Art. 108), q denoting the normal to any infinitesimal stratum of the atmosphere at the point in which the ray intersects the stratum, μ the index of refraction of that stratum, and i the angle which the ray makes with the normal.

If, then, ρ, μ_0, Z' denote the values of q, μ, and i for the point A, we have, as in the equation (149),

$$q\mu \sin \iota = \rho\mu_0 \sin Z'$$

in which Z' is the apparent zenith distance of the point M, and μ_0 is the index of refraction of the air at the observer.

Now, let us consider the normal q to be drawn to a point D of the ray where the refractive power of the air is zero, that is, to a point in the rectilinear portion of the path where $\mu = 1$. Then our equation becomes

$$q \sin \iota = \rho\mu_0 \sin Z'$$

in which $q = CD$, $\iota = MDF = CDB$. Putting Z = the true zenith distance of $M = MBV$, and s = the height of B above the surface of the earth $= AB$, the triangle CDB gives

$$(\rho + s) \sin Z = q \sin \iota$$

which with the preceding equation gives

$$1 + \frac{s}{\rho} = \frac{\mu_0 \sin Z'}{\sin Z} \tag{564}$$

In order to substitute the point B for the point A in our computation of an eclipse, we have only to write $\rho + s$ for ρ in the equations (483), or $\rho\left(1 + \frac{s}{\rho}\right)$ for ρ. Therefore, when we have computed the values of $\log \xi$, $\log \eta$, and $\log \zeta$ by those equations in their present form, we shall merely have to correct them by adding to each the value of $\log\left(1 + \frac{s}{\rho}\right)$. This logarithm may be computed by (564) for a mean value of μ_0 ($= 1.0002800$) and for given values of Z. For Z we may take the true zenith distance of *the point* Z (Art. 289), determined by a and d. But by the last equation of (483) we have so nearly $\cos Z = \zeta$ that in the table computed by (564) we may make $\log \zeta$ the argument, as in the following table, which I have deduced from that of Bessel (*Astron. Untersuchungen*, Vol. II. p. 240).

log ζ	Correction of logs. of ξ, η, ζ.	log ζ	Correction of logs. of ξ, η, ζ.
0.0	0.0000000	8.5	0.0000446
9.9	.0000001	8.4	.0000525
9.8	.0000002	8.3	.0000602
9.7	.0000005	8.2	.0000672
9.6	.0000008	8.1	.0000734
9.5	0.0000014	8.0	0.0000788
9.4	.0000023	7.9	.0000835
9.3	.0000035	7.8	.0000875
9.2	.0000054	7.7	.0000909
9.1	.0000081	7.6	.0000937
9.0	0.0000119	7.4	0.0000978
8.9	.0000167	7.2	.0001006
8.8	.0000225	7.0	.0001023
8.7	.0000292	6.5	.0001044
8.6	.0000367	6.0	.0001051
8.5	0.0000446	$-\infty$	0.0001054

The numbers in this table correspond to that state of the atmosphere for which the refraction table (Table II.) is computed; that is, for the case in which the factors β and γ of that table are each $= 1$. For any other case the tabular logarithm is to be varied in proportion to β and γ.

It is evident from this table that the effect of refraction will mostly be very small, for so long as the zenith distance of the moon is less than 70° we have log $\zeta > 9.53$, and the tabular correction less than .000001. From the zenith distance 70° to 90° the correction increases rapidly, and should not be neglected.

CORRECTION FOR THE HEIGHT OF THE OBSERVER ABOVE THE LEVEL OF THE SEA.

328. If s' is the height of the observer above the level of the sea, it is only necessary to put $\rho + s'$ for ρ in the general formulæ of the eclipse; and this will be accomplished by adding to log ξ, log η, and log ζ the value of log $\left(1 + \frac{s'}{\rho}\right)$, which is ($M$ being the modulus of common logarithms)

$$M\left[\frac{s'}{\rho} - \tfrac{1}{2}\left(\frac{s'}{\rho}\right)^2 + \&c.\right]$$

But s' is always so small in comparison with ρ that we may

neglect all but the first term of this formula; and hence, by taking a mean value of ρ (for latitude 45°) and supposing s' to be expressed in *English feet*, we find

$$\text{Correction of } \log \xi, \log \eta, \log \zeta = 0.00000002079\, s' \tag{565}$$

For example, if the point of observation is 1000 feet above the level of the sea, we must increase the logarithms of ξ, η, and ζ by 0.0000208.

If s' is expressed in *metres*, the correction becomes $0.000000064\, s'$.

APPLICATION OF OBSERVED ECLIPSES TO THE DETERMINATION OF TERRESTRIAL LONGITUDES AND THE CORRECTION OF THE ELEMENTS OF THE COMPUTATION.

329. *To find the longitude of a place from the observation of an eclipse of the sun.*—The observation gives simply the local times of the contacts of the discs of the sun and moon: in the case of partial eclipse, two exterior contacts only; in the case of total or annular eclipse, also two interior contacts.

Let

ω = the west longitude of the place,
t = the local mean time of an observed contact,
μ = the corresponding local sidereal time.

The conversion of t into μ requires an approximate knowledge of the longitude, which we may always suppose the observer to possess, at least with sufficient precision for this purpose.

Let T_0 be the adopted epoch from which the values of x and y are computed (Art. 296), and let

x_0, y_0 = the values of x and y at the time T_0,
x', y' = their *mean* hourly changes for the time $t + \omega$;

then, if we also put

$$\tau = t + \omega - T_0 \tag{566}$$

the values of x and y at the time $t + \omega$ (which is the time at the first meridian when the contact was observed) are

$$x_0 + x'\tau, \qquad y_0 + y'\tau$$

The values of x' and y' to be employed in these expressions may be taken for the time $t + \omega$ obtained by employing the

approximate value of ω, and will be sufficiently precise unless the longitude is very greatly in error.

The quantities l and i change so slowly that their values taken for the approximate time $t + \omega$ will not differ sensibly from the true ones. For the same reason, the quantities a and d taken for this time will be sufficiently precise: so that, the latitude being given, the co-ordinates ξ, η, ζ of the place of observation may be correctly found by the formulæ (483). Since, then, at the instant of contact the equation (490) or (491) must be exactly satisfied, we have, putting $L = l - i\zeta$,

$$\left.\begin{aligned} L \sin Q &= x_0 - \xi + x'\tau \\ L \cos Q &= y_0 - \eta + y'\tau \end{aligned}\right\} \qquad (567)$$

in which τ is the only unknown quantity. Let the auxiliaries m, M, n, N be determined by the equations

$$\left.\begin{aligned} m \sin M &= x_0 - \xi \qquad & n \sin N &= x' \\ m \cos M &= y_0 - \eta \qquad & n \cos N &= y' \end{aligned}\right\} \qquad (568)$$

then, from the equations

$$\begin{aligned} L \sin Q &= m \sin M + n \sin N \,.\, \tau \\ L \cos Q &= m \cos M + n \cos N \,.\, \tau \end{aligned}$$

by putting $\psi = Q - N$, we obtain

$$\left.\begin{aligned} \sin \psi &= \frac{m \sin (M - N)}{L} \\ \tau &= \frac{L \cos \psi}{n} - \frac{m \cos (M - N)}{n} \\ &= \frac{m}{n} \cdot \frac{\sin (M - N - \psi)}{\sin \psi} \end{aligned}\right\} \qquad (569)$$

where the second form for τ will be the more convenient except when $\sin \psi$ is very small. As in the similar formulæ (553), the angle ψ must be so taken that $L \cos \psi$ shall be negative for first contacts and positive for last contacts, remembering that in the case of total eclipse L is a negative quantity.

Having found τ, the longitude becomes known by (566), which gives

$$\omega = T_0 - t + \tau \qquad (570)$$

If the observed local time is sidereal, let μ_0 be the sidereal time at the first meridian, corresponding to T_0; then, τ being reduced to sidereal seconds, we shall have

$$\omega = \mu_0 - \mu + \tau$$

and this process will be free from the theoretical inaccuracy arising from employing an approximate longitude in converting μ into t.

The unit of τ in (569) is one mean hour; but, if we write

$$\tau = \frac{h\, L \cos \psi}{n} - \frac{h\, m \cos (M - N)}{n}$$

$$= h \cdot \frac{m}{n} \cdot \frac{\sin (M - N - \psi)}{\sin \psi}$$

we shall find τ in mean or sidereal seconds, according as we take $h = 3600$, or $h = 3609.856$.

330. The rule given in the preceding article for determining the sign of $\cos \psi$ (which is that usually given by writers on this subject) is not without exception in theory, although in practice it will be applicable in all cases where the observations are suitable for finding the longitude with precision; and, were an exceptional case to occur in practice, a knowledge of the approximate longitude would remove all doubt as to the sign of the term $\frac{L \cos \psi}{n}$. But it is is easy to deduce the mathematical condition for this case.

At the instant of contact, the quantity

$$(x_0 - \xi + x'\tau)^2 + (y_0 - \eta + y'\tau)^2$$

is equal to L^2. At the next following instant, when τ becomes $\tau + d\tau$, it is less or greater than L^2 according as the eclipse is beginning or ending. If then we regard L^2 as sensibly constant, the differential coefficient of this quantity relatively to the time must be negative for first and positive for last contacts. The half of this coefficient is

$$(x_0 - \xi + x'\tau)(x' - \xi') + (y_0 - \eta + y'\tau)(y' - \eta')$$

(where the derivatives of ξ and η are denoted by ξ' and η'), or, by (567), putting $N + \psi$ for Q,

$$L\,[\sin (N + \psi)\,(x' - \xi') + \cos (N + \psi)\,(y' - \eta')]$$

Computing ξ' and η' by the formulæ (551), or, in this case, by

$$\xi' = \mu'\rho \cos\varphi' \cos(\mu - a) \qquad \eta' = \mu'\xi \sin d$$

and putting

$$n' \sin N' = x' - \xi' \qquad n' \cos N' = y' - \eta'$$

the above expression becomes

$$L\, n' \cos(N - N' + \psi)$$

Hence, when L is positive, that is, for exterior contacts and interior contacts in annular eclipse, ψ must be so taken that $\cos(N - N' + \psi)$ shall be negative for first and positive for last contact. That is, *for first contact ψ must be taken between $N' - N + 90°$ and $N' - N + 270°$; and for last contact between $N' - N + 90°$ and $N' - N - 90°$.* For total eclipse, invert these conditions.

In Art. 322, we have $N = N'$, and hence the rule given for the case there considered is always correct.

331. *To investigate the correction of the longitude found from an observed solar eclipse, for errors in the elements of the computation.*

Let

Δx, Δy, ΔL = the corrections of x, y, and L, respectively, for errors of the Ephemeris,

$\Delta\xi$, $\Delta\eta$ = the corrections of ξ and η for errors in ρ and φ',

$\Delta\tau$ = the resulting correction of τ.

The relation between these corrections, supposing them very small, will be obtained by differentiating the values of $L \sin Q$ and $L \cos Q$ of the preceding article, by which we obtain

$$\Delta L \sin Q + L \cos Q\, \Delta Q = \Delta x - \Delta\xi + x'\,\Delta\tau$$
$$\Delta L \cos Q - L \sin Q\, \Delta Q = \Delta y - \Delta\eta + y'\,\Delta\tau$$

where Δx and Δy, being taken to denote the corrections of $x = x_0 + x'\tau$ and $y = y_0 + y'\tau$, include the corrections of x' and y'. Substituting in these equations $n \sin N$ for x' and $n \cos N$ for y', and eliminating ΔQ, we find

$$\Delta L = (\Delta x - \Delta\xi) \sin Q + (\Delta y - \Delta\eta) \cos Q + n \cos(Q - N) \,.\, \Delta\tau$$

and substituting for Q its value $N + \psi$,

$$\Delta\tau = -(\Delta x - \Delta\xi)\frac{\sin(N + \psi)}{n \cos\psi} - (\Delta y - \Delta\eta)\frac{\cos(N + \psi)}{n \cos\psi} + \frac{\Delta L}{n \cos\psi}$$

or

$$\Delta\tau = -\frac{1}{n}(\Delta x \sin N + \Delta y \cos N) + \frac{1}{n}(-\Delta x \cos N + \Delta y \sin N)\tan\psi$$
$$+\frac{1}{n}(\Delta\xi \sin N + \Delta\eta \cos N) - \frac{1}{n}(-\Delta\xi \cos N + \Delta\eta \sin N)\tan\psi$$
$$+\frac{\Delta L \sec\psi}{n} \tag{571}$$

which is at once the correction of τ and of the longitude, since we have, by (570), $\Delta\omega = \Delta\tau$.

332. In this expression for $\Delta\tau$, the corrections Δx, Δy, &c. have particular values belonging to the given instant of observation or to the given place. In order to render it available for determining the corrections of the original elements of computation, we must endeavor to reduce it to a function of quantities which are constant during the whole eclipse and independent of the place of observation. For this purpose, let us first consider those parts of $\Delta\tau$ which involve Δx and Δy. For any time T_1, at the first meridian, we have

$$x = x_0 + n \sin N(T_1 - T_0)$$
$$y = y_0 + n \cos N(T_1 - T_0)$$

whence

$$x \sin N + y \cos N = \quad x_0 \sin N + y_0 \cos N + n(T_1 - T_0)$$
$$-x \cos N + y \sin N = -x_0 \cos N + y_0 \sin N$$

The last of these expressions, being independent of the time, is constant. If we denote it by $\varkappa$; that is, put

$$\varkappa = -x_0 \cos N + y_0 \sin N = -x \cos N + y \sin N \tag{572}$$

we shall find from the two expressions

$$xx + yy = \varkappa\varkappa + [x_0 \sin N + y_0 \cos N + n(T_1 - T_0)]^2 \tag{573}$$

This equation shows that the quantity $\sqrt{xx + yy}$, which is the distance of the axis of the shadow from the centre of the earth, can never be less than the constant $\varkappa$, and it attains this minimum value when the second term vanishes, that is, when

$$x_0 \sin N + y_0 \cos N + n(T_1 - T_0) = 0$$

and hence when

$$T_1 = T_0 - \frac{1}{n}(x_0 \sin N + y_0 \cos N) \tag{574}$$

which formula, therefore, gives *the time* T_1 *of nearest approach* of the axis of the shadow to the centre of the earth, while (572) gives the value of the distance of the axis from the centre of the earth at this time. By the introduction of the auxiliary quantities T_1 and $\varkappa$, we can express the corrections involving Δx and Δy in their simplest form; for we have now, for the time of observation $t + \omega$,

$$x \sin N + y \cos N = x_0 \sin N + y_0 \cos N + n\,(t + \omega - T_0)$$
$$= n\,(t + \omega - T_1)$$

and if Δn, ΔT_1 and $\Delta \varkappa$ are the corrections of n, T_1, and $\varkappa$ on account of errors in the elements, we have

$$\left.\begin{aligned} \Delta x \sin N + \Delta y \cos N &= -n\,\Delta T_1 + (t + \omega - T_1)\,\Delta n \\ -\Delta x \cos N + \Delta y \sin N &= \Delta \varkappa \end{aligned}\right\} \quad (575)$$

These expressions reduce those parts of $\Delta\tau$ which involve Δx and Δy to functions of ΔT_1, Δn, and $\Delta\varkappa$, which may be regarded as constant quantities for the same eclipse.

We proceed to consider those parts of $\Delta\tau$ which involve $\Delta\xi$ and $\Delta\eta$. These corrections we shall regard as depending only upon the correction of the eccentricity of the terrestrial meridian; for the latitude itself may always be supposed to be correct, since it is easily obtained with all the precision required for the calculation of an eclipse; the values of a and d depend chiefly on the sun's place, which we assume to be correctly given in the Ephemeris; and μ is derived directly from observation. Now, we have (Art. 82), e being the eccentricity of the meridian,

$$\rho \cos \varphi' = \frac{\cos \varphi}{\sqrt{(1 - ee \sin^2 \varphi)}} \qquad \rho \sin \varphi' = \frac{(1 - ee) \sin \varphi}{\sqrt{(1 - ee \sin^2 \varphi)}}$$

whence, by differentiation,

$$\frac{\Delta \,.\, \rho \cos \varphi'}{\Delta ee} = \rho \cos \varphi' \cdot \frac{\rho\rho \sin^2 \varphi'}{2(1 - ee)^2}$$

$$\frac{\Delta \,.\, \rho \sin \varphi'}{\Delta ee} = \rho \sin \varphi' \cdot \frac{\rho\rho \sin^2 \varphi'}{2(1 - ee)^2} - \frac{\rho \sin \varphi'}{1 - ee}$$

or, putting

$$\beta = \frac{\rho \sin \varphi'}{1 - ee}$$

$$\left.\begin{aligned}\frac{\Delta \cdot \rho \cos \varphi'}{\Delta ee} &= \tfrac{1}{2}\beta\beta\rho \cos \varphi' \\ \frac{\Delta \cdot \rho \sin \varphi'}{\Delta ee} &= \tfrac{1}{2}\beta\beta\rho \sin \varphi' - \beta\end{aligned}\right\} \quad (576)$$

From the values

$$\xi = \rho \cos \varphi' \sin (\mu - a)$$
$$\eta = \rho \sin \varphi' \cos d - \rho \cos \varphi' \sin d \cos (\mu - a)$$

we deduce

$$\frac{\Delta\xi}{\Delta ee} = \tfrac{1}{2}\beta\beta\xi \qquad \frac{\Delta\eta}{\Delta ee} = \tfrac{1}{2}\beta\beta\eta - \beta \cos d$$

and hence

$$\Delta\xi \sin N + \Delta\eta \cos N = \tfrac{1}{2}\beta\beta\,(\xi \sin N + \eta \cos N)\,\Delta ee - \beta \cos d \cos N\,\Delta ee$$
$$-\Delta\xi \cos N + \Delta\eta \sin N = \tfrac{1}{2}\beta\beta\,(-\xi \cos N + \eta \sin N)\,\Delta ee - \beta \cos d \sin N\,\Delta ee$$

The values of ξ and η may be put under the forms

$$\xi = x_0 - (x_0 - \xi) = x_0 - m \sin M$$
$$\eta = y_0 - (y_0 - \eta) = y_0 - m \cos M$$

by which the second members of the preceding expressions are changed respectively into

$$\tfrac{1}{2}\beta\beta\,[\;x_0 \sin N + y_0 \cos N - m \cos (M - N)]\,\Delta ee - \beta \cos d \cos N\,\Delta ee$$
and $$\tfrac{1}{2}\beta\beta\,[-x_0 \cos N + y_0 \sin N + m \sin (M - N)]\,\Delta ee - \beta \cos d \sin N\,\Delta ee$$

or, by (574) and (572), into

$$\tfrac{1}{2}\beta\beta\,[n\,(T_0 - T_1) - m \cos (M - N)]\,\Delta ee - \beta \cos d \cos N\,\Delta ee$$
and $$\tfrac{1}{2}\beta\beta\,[\qquad\qquad x + m \sin (M - N)]\,\Delta ee - \beta \cos d \sin N\,\Delta ee$$

or, again, by (569) and (570), into

$$\tfrac{1}{2}\beta\beta\,[n\,(t + \omega - T_1) - L \cos \psi]\,\Delta ee - \beta \cos d \cos N\,\Delta ee$$
and $$\tfrac{1}{2}\beta\beta\,[\qquad\qquad\qquad x + L \sin \psi]\,\Delta ee - \beta \cos d \sin N\,\Delta ee$$

Hence, that part of $\Delta\tau$ which depends upon Δee is equal to

$$\frac{\beta\beta}{2n}\,[n\,(t + \omega - T_1) - x \tan \psi - L \sec \psi]\,\Delta ee - \frac{\beta \cos d \cos (N + \psi)}{n \cos \psi}\,\Delta ee$$

When these substitutions are made in (571), we have

$$\Delta\tau = \Delta\omega - h\Delta T_1 + h \tan \psi \cdot \frac{\Delta x}{n} - h\,(t + \omega - T_1)\,\frac{\Delta n}{n} + h \sec \psi \cdot \frac{\Delta L}{n}$$
$$+ \frac{h}{n}\left[\tfrac{1}{2}\beta\beta\,[n\,(t + \omega - T_1) - x \tan \psi - L \sec \psi] - \frac{\beta \cos d \cos (N + \psi)}{\cos \psi}\right]\Delta ee \quad (577)$$

where we have multiplied by h to reduce to seconds. The unit is either one second of mean or one second of sidereal time, according as τ is in mean or sidereal time. If the former, we take $h = 3600$; if the latter, $h = 3610$.

333. The transformations of the preceding article have led us to an expression in which the corrections ΔT_1, $\Delta \varkappa$, Δn, and Δee are all constants for the earth generally, and which, therefore, have the same values in all the equations of condition formed from the observations in various places. But a still further transformation is necessary if we wish the equation to express the relation between the longitude and the corrections of the Ephemeris. so that we may finally be enabled not only to correct the longitudes, but also the Ephemeris.

Since ΔT_1, $\Delta \varkappa$, Δn are constant for the whole eclipse, we can determine them for any assumed time, as the time T_1 itself. For this time we have

$$\left.\begin{aligned} x \sin N + y \cos N &= 0 \\ -x \cos N + y \sin N &= \varkappa \\ \Delta x \sin N + \Delta y \cos N &= -n \Delta T_1 \\ -\Delta x \cos N + \Delta y \sin N &= \Delta \varkappa \end{aligned}\right\} \tag{578}$$

The general values of x and y (482) may be thus expressed:

$$x = \frac{X}{\sin \pi} \qquad y = \frac{Y}{\sin \pi}$$

where

$$X = \cos \delta \sin (\alpha - a) \qquad Y = \sin \delta \cos d - \cos \delta \sin d \cos (\alpha - a)$$

From these we deduce

$$\Delta x = \frac{\Delta X}{\sin \pi} - x \frac{\Delta \pi}{\tan \pi} \qquad \Delta y = \frac{\Delta Y}{\sin \pi} - y \frac{\Delta \pi}{\tan \pi}$$

whence

$$\Delta x \sin N + \Delta y \cos N = \frac{\Delta X \sin N + \Delta Y \cos N}{\sin \pi} - (x \sin N + y \cos N) \frac{\Delta \pi}{\tan \pi}$$

$$-\Delta x \cos N + \Delta y \sin N = \frac{-\Delta X \cos N + \Delta Y \sin N}{\sin \pi} + (x \cos N - y \sin N) \frac{\Delta \pi}{\tan \pi}$$

and for the time T_1 these become, according to (578),

$$-n \Delta T_1 = \frac{\Delta X \sin N + \Delta Y \cos N}{\sin \pi}$$

$$\Delta \varkappa = \frac{-\Delta X \cos N + \Delta Y \sin N}{\sin \pi} - \varkappa \frac{\Delta \pi}{\tan \pi}$$

Again, by differentiating the values of X and Y, we have

$$\begin{aligned}
\Delta X &= \cos\delta\cos(\alpha - a)\Delta(\alpha - a) - \sin\delta\sin(\alpha - a)\Delta\delta \\
\Delta Y &= [\cos\delta\cos d + \sin\delta\sin d\cos(\alpha - a)]\,\Delta\delta \\
&\quad - [\sin\delta\sin d + \cos\delta\cos d\cos(\alpha - a)]\,\Delta d \\
&\quad + \cos\delta\sin d\sin(\alpha - a)\Delta(\alpha - a)
\end{aligned}$$

But for the time of nearest approach we may take $\alpha = a$ and put $\cos(\delta - d) = 1$, whence

$$\Delta X = \cos\delta\,.\,\Delta(\alpha - a) \qquad\qquad \Delta Y = \Delta(\delta - d)$$

so that

$$\left.\begin{aligned}
-n\Delta T_1 &= \frac{\sin N\cos\delta\,.\,\Delta(\alpha - a) + \cos N\,.\,\Delta(\delta - d)}{\sin\pi} \\
\Delta x &= \frac{-\cos N\cos\delta\,.\,\Delta(\alpha - a) + \sin N\,.\,\Delta(\delta - d)}{\sin\pi} - x\,\frac{\Delta\pi}{\tan\pi}
\end{aligned}\right\}\quad(579)$$

To find Δn, which depends upon the corrections of x' and y', we observe that x' and y', regarded as derivatives of x and y, are of the form

$$x' = \frac{dX}{dT}\cdot\frac{1}{\sin\pi} \qquad\qquad y' = \frac{dY}{dT}\cdot\frac{1}{\sin\pi}$$

But $\frac{dX}{dT}$ and $\frac{dY}{dT}$ depend upon the changes of the moon's right ascension and declination, which for the brief duration of an eclipse are correctly given in the Ephemeris. The errors of x' and y', therefore, depend upon those of π: so that if we write

$$x' = \frac{a}{\sin\pi} \qquad\qquad y' = \frac{b}{\sin\pi}$$

and regard a and b as correct, we find

$$\Delta x' = -x'\,\frac{\Delta\pi}{\tan\pi} \qquad\qquad \Delta y' = -y'\,\frac{\Delta\pi}{\tan\pi}$$

From the equations $n\sin N = x'$, $n\cos N = y'$, we have

$$\Delta n\sin N + n\,\Delta N\cos N = \Delta x' = -n\sin N\,.\,\frac{\Delta\pi}{\tan\pi}$$

$$\Delta n\cos N - n\,\Delta N\sin N = \Delta y' = -n\cos N\,.\,\frac{\Delta\pi}{\tan\pi}$$

whence, by eliminating ΔN,*

$$\frac{\Delta n}{n} = -\frac{\Delta \pi}{\tan \pi} \tag{580}$$

Since $\Delta(\alpha - a)$, $\Delta(\delta - d)$, $\Delta\pi$ will in practice be expressed in seconds of arc, we should substitute for them $\Delta(\alpha - a)\sin 1''$, $\Delta(\delta - d)\sin 1''$, $\Delta\pi \sin 1''$ in the above expressions; but if we at the same time put $\pi \sin 1''$ for $\sin \pi$ and $\tan \pi$, the factor $\sin 1''$ will disappear.

To develop ΔL, we may neglect the error of the small term $i\zeta$ and assume $\Delta L = \Delta l$. We have from (486) and (488), by neglecting the small term $k \sin \pi_0$ and putting $g = 1$, $z = \frac{1}{\sin \pi}$, the following approximate expression for l:

$$l = \frac{\sin H}{r' \sin \pi} \pm k$$

which gives

$$\Delta L = \Delta l = \frac{\Delta H}{r'\pi} \pm \Delta k - \frac{H}{r'\pi} \cdot \frac{\Delta\pi}{\pi} \tag{581}$$

Substituting the values of ΔT_1, Δx, Δn, and Δl given by (579), (580), and (581), in (577), and putting

$$\nu = \frac{h}{n\pi}$$

the formula becomes, finally,

$$\begin{aligned}
\Delta\omega = & -\nu\,[\quad \sin N \cos \delta.\Delta(\alpha - a) + \cos N.\Delta(\delta - d)] \\
& + \nu\,[-\cos N \cos \delta\, \Delta(\alpha - a) + \sin N.\Delta(\delta - d)] \tan \psi \\
& + \nu \left[\frac{\Delta H}{r'} \pm \pi \Delta k\right] \sec \psi \\
& + \nu \left[n\,(t + \omega - T_1) - x \tan \psi - \frac{H}{r'\pi} \sec \psi\right] \Delta\pi \\
& + \nu \left[\tfrac{1}{2}\beta\beta\,[n\,(t + \omega - T_1) - x \tan \psi - L \sec \psi] - \frac{\beta \cos d \cos (N + \psi)}{\cos \psi}\right] \pi \Delta e
\end{aligned} \tag{582}$$

where the negative sign of $\pi\Delta k$ is to be used for interior contacts.

It is easily seen that $\pi\Delta k$ represents very nearly the correction

* The angle N is independent of errors in π, since $\tan N = \frac{a}{b}$: so that we might have taken $\Delta N = 0$.

of the moon's apparent semidiameter, and $\frac{\Delta H}{r'}$ that of the sun's semidiameter; and that $\pi\Delta ee$ is the correction of the assumed reduction of the parallax for the latitude 90°.

334. *Discussion of the equations of condition for the correction of the longitude and of the elements of the computation.*—The longitude ω found by the equation (570), (Art. 329), requires the correction $\Delta\omega$ of (582). If, for brevity, we put

$$\left.\begin{aligned} \gamma &= \quad \sin N \cos \delta \, \Delta(\alpha - a) + \cos N \, \Delta(\delta - d) \\ \vartheta &= -\cos N \cos \delta \, \Delta(\alpha - a) + \sin N \, \Delta(\delta - d) \end{aligned}\right\} \quad (583)$$

and

$$\omega' = \text{the true longitude,}$$

we have the equation of condition

$$\omega' = \omega + \Delta\omega = \omega - \nu\gamma + \nu \tan \psi \, . \, \vartheta + \&c. \quad (584)$$

If the eclipse has been observed at several places, we can form as many such equations as there are contacts observed. If the observations are complete at all the places, we can, for the most part, eliminate from these equations the unknown corrections of the elements, and determine the *relative* longitudes of the several places; and if the absolute longitude of one of the places is known, that of each place will also be determined.

I shall at first consider only the terms involving γ and ϑ. The quantity $\nu\gamma$ is a constant for all the places of observation, and combines with ω, so that it cannot be determined unless the longitude of at least one of the places is known. If then we put

$$\Omega = \omega' + \nu\gamma \qquad\qquad a = \nu \tan \psi$$

the equations of condition will assume the form

$$\Omega - a\vartheta - \omega = 0$$

Suppose, for the sake of completeness, that the four contacts of a total or annular eclipse have been observed at any one place, and that the values of the longitude found from the several contacts by Art. 329 are ω_1, ω_2, ω_3, ω_4. We then have the four equations

$$\begin{aligned} &[1] \quad \Omega - a_1 \vartheta - \omega_1 = 0 \\ &[2] \quad \Omega - a_2 \vartheta - \omega_2 = 0 \\ &[3] \quad \Omega - a_3 \vartheta - \omega_3 = 0 \\ &[4] \quad \Omega - a_4 \vartheta - \omega_4 = 0 \end{aligned}$$

where the numerals may be assumed to express the order in which the contacts are observed; [1] and [4] being exterior, and [2] and [3] interior. In a partial eclipse we should have but the 1st and 4th of these equations.

Since exterior contacts cannot (in most cases) be observed with as much precision as interior ones, let us assign different weights to the observations, and denote them by p_1, p_2, p_3, p_4, respectively. Combining the four equations according to the method of least squares, we form the two normal equations

$$[p\ ]\,\Omega - [pa]\,\vartheta - [p\omega\ ] = 0$$
$$[pa]\,\Omega - [paa]\,\vartheta - [pa\omega] = 0$$

where the rectangular brackets are used as symbols of summation. From these, by eliminating Ω, and putting

$$[paa] - \frac{[pa]}{[p]}[pa] = P$$

$$[pa\omega] - \frac{[p\omega]}{[p]}[pa] = Q$$

we find

$$P\vartheta + Q = 0 \qquad (585)$$

from which the value of ϑ would be determined with the weight P. But the computation of Q under this form is inconvenient. By developing the quantities P and Q, observing that $[paa] = p_1a_1^2 + p_2a_2^2 + p_3a_3^2 + p_4a_4^2$, &c., we shall find

$$P = \frac{p_1p_2(a_1 - a_2)^2 + p_1p_3(a_1 - a_3)^2 + p_1p_4(a_1 - a_4)^2}{p_1 + p_2 + p_3 + p_4}$$
$$+ \frac{p_2p_3(a_2 - a_3)^2 + p_2p_4(a_2 - a_4)^2 + p_3p_4(a_3 - a_4)^2}{p_1 + p_2 + p_3 + p_4}$$

$$Q = \frac{p_1p_2(a_1 - a_2)(\omega_1 - \omega_2) + p_1p_3(a_1 - a_3)(\omega_1 - \omega_3) + p_1p_4(a_1 - a_4)(\omega_1 - \omega_4)}{p_1 + p_2 + p_3 + p_4}$$
$$+ \frac{p_2p_3(a_2 - a_3)(\omega_2 - \omega_3) + p_2p_4(a_2 - a_4)(\omega_2 - \omega_4) + p_3p_4(a_3 - a_4)(\omega_3 - \omega_4)}{p_1 + p_2 + p_3 + p_4}$$

These forms show that if we subtract each of the equations [1], [2], [3] from each of those that follow it in the group, whereby we obtain the six equations

$$(a_1 - a_2)\,\vartheta + \omega_1 - \omega_2 = 0$$
$$(a_1 - a_3)\,\vartheta + \omega_1 - \omega_3 = 0$$
$$(a_1 - a_4)\,\vartheta + \omega_1 - \omega_4 = 0$$
$$(a_2 - a_3)\,\vartheta + \omega_2 - \omega_3 = 0$$
$$(a_2 - a_4)\,\vartheta + \omega_2 - \omega_4 = 0$$
$$(a_3 - a_4)\,\vartheta + \omega_3 - \omega_4 = 0$$

and combine these six equations according to the method of least squares, taking their weights to be respectively

$$\frac{p_1 p_2}{p_1 + p_2 + p_3 + p_4} \qquad \frac{p_1 p_3}{p_1 + p_2 + p_3 + p_4}, \text{ \&c.}$$

we shall arrive at the same final equation (585) as by the direct process, with the advantage of avoiding the multiplication of the large numbers ω_1, ω_2, &c.

Suppose that at another place but three contacts have been observed, the true longitude being ω'', and the computed longitudes ω_5, ω_6, ω_7, and that, having put $\Omega' = \omega'' + \nu\gamma$, we have formed the three equations

$$\begin{array}{lll}
[5] & \Omega' - a_5\,\vartheta - \omega_5 = 0 & \text{with the weight } p_5 \\
[6] & \Omega' - a_6\,\vartheta - \omega_6 = 0 & \quad\text{"}\qquad\text{"}\quad p_6 \\
[7] & \Omega' - a_7\,\vartheta - \omega_7 = 0 & \quad\text{"}\qquad\text{"}\quad p_7
\end{array}$$

The subtraction of each of the first two from those which follow gives the three equations

$$\begin{aligned}
(a_5 - a_6)\,\vartheta + \omega_5 - \omega_6 &= 0 \\
(a_5 - a_7)\,\vartheta + \omega_5 - \omega_7 &= 0 \\
(a_6 - a_7)\,\vartheta + \omega_6 - \omega_7 &= 0
\end{aligned}$$

of which the weights will be respectively, according to the above forms,

$$\frac{p_5 p_6}{p_5 + p_6 + p_7} \qquad \frac{p_5 p_7}{p_5 + p_6 + p_7} \qquad \frac{p_6 p_7}{p_5 + p_6 + p_7}$$

and the combination of these three equations, according to weights, will give a normal equation of the form

$$P'\vartheta + Q' = 0$$

which gives a value of ϑ with the weight P'.

Now, suppose that this method applied to all the observations at all the places has given us the series of equations in ϑ,

$$\begin{aligned}
P\vartheta + Q &= 0 \\
P'\vartheta + Q' &= 0 \\
P''\vartheta + Q'' &= 0, \text{ \&c.};
\end{aligned}$$

then, since P, P', P'', &c. are the weights of these several determinations, the final normal equation for determining ϑ, derived from all the observations, is

$$[P]\,\vartheta + [Q] = 0$$

that is, it is simply the sum of all the individual equations in ϑ formed for the places severally.

The same reasoning is applicable to any of the terms which follow the term in ϑ in (584); so that if we suppose all the terms to be retained, this process gives an equation in ϑ for each place, in which besides the term $P\vartheta$ there will be terms in Δk, ΔH, &c., and from all the equations, by addition, a final normal equation (still called the equation in ϑ) as before. In the same manner, final normal equations in Δk, ΔH, &c. will be formed. Thus we shall obtain five normal equations involving the five unknown quantities ϑ, Δk, ΔH, $\Delta \pi$, Δee, which are then determined by solving the equations in the usual manner. But, unless the eclipse has been observed at places widely distant in longitude, it will not be possible to determine satisfactorily the value of $\Delta\pi$, much less that of Δee. It will be advisable to retain these terms in our equations, however, in order to show what effect an error in π or ee may produce upon the resulting longitudes.

When ϑ, &c. have been found, we find Ω, Ω', &c. from the equations [1], [2] . . . . [5], [6] . . . . The final value of Ω will be the mean of its values [1 — 4] taken with regard to the weights; and so of Ω', &c. Hence we shall know the several *differences* of longitude

$$\omega' - \omega'' = \Omega - \Omega', \qquad \omega' - \omega''' = \Omega - \Omega'', \text{ \&c.}$$

If one of the longitudes, as for instance ω', is previously known, we have

$$\nu\gamma = \Omega - \omega'$$

and hence all the longitudes become known.

Finally, from the values of γ and ϑ the corrections of the Ephemeris in right ascension and declination are obtained by the formulæ

$$\left.\begin{aligned} \cos\delta\,\Delta(\alpha - a) &= \sin N\,.\,\gamma - \cos N\,.\,\vartheta \\ \Delta(\delta - d) &= \cos N\,.\,\gamma + \sin N\,.\,\vartheta \end{aligned}\right\} \quad (586)$$

335. When only two places of observation are considered, one of which is known, it will be sufficiently accurate to deduce γ and ϑ from the observations at the known place (disregarding the other corrections), and to employ their values in finding the longitude of the other place.

336. When good meridian observations of the moon are available, taken near the time of the eclipse, the quantities $\Delta(\alpha - a)$, $\Delta(\delta - d)$ [for which we may take $\Delta(\alpha - \alpha')$, $\Delta(\delta - \delta')$], may be found from them. The terms in γ and ϑ may then be directly computed by (583) and applied to the computed longitude; after which the discussion of the equations of condition may with advantage be extended to the remaining terms.

337. Before proceeding to give an example of the computation by the preceding method, it will be well to recapitulate the necessary formulæ, and to give the equations of condition a practical form.

I. The general elements of the eclipse, a, d, l, $\log i$, x, y, x', y', are supposed to have been computed and tabulated as in Art. 297.

II. The latitude of the place being φ, the logarithms of $\rho \cos \varphi'$ and $\rho \sin \varphi'$ are found by the aid of our Table III., or by the formulæ (87).

The mean local time t of an observed contact being given, find the corresponding local sidereal time μ; also the time $t + \omega$ at the first meridian, employing the approximate value of the longitude ω.

[If the observed time is the sidereal time μ, the time $\mu + \omega$ at the first meridian, converted into mean time, will give the approximate value of $t + \omega$.]

For the time $t + \omega$ take a, d, l, and $\log i$ from the eclipse tables, and compute the co-ordinates of the place and the radius of the shadow by the formulæ

$$
\begin{aligned}
A \sin B &= \rho \sin \varphi' & \xi &= \rho \cos \varphi' \sin (\mu - a) \\
A \cos B &= \rho \cos \varphi' \cos (\mu - a) & \eta &= A \sin (B - d) \\
L &= l - i\zeta & \zeta &= A \cos (B - d)
\end{aligned}
$$

When $\log \zeta$ is small, add to $\log \xi$, $\log \eta$, and $\log \zeta$ the correction for refraction, from the table on p. 517.

III. For the assumed epoch T_0 at the first meridian (being the epoch from which the mean hourly changes x' and y' are reckoned), take the values of x and y from the eclipse tables, denoting them by x_0 and y_0. Also the mean hourly changes x'

and y' for the time $t + \omega$. Compute the auxiliaries m, M, &c. by the formulæ*

$$m \sin M = x_0 - \xi \qquad n \sin N = x'$$
$$m \cos M = y_0 - \eta \qquad n \cos N = y'$$
$$\sin \psi = \frac{m \sin (M - N)}{L}$$

where ψ is (in general) to be so taken that $L \cos \psi$ shall be negative for a first and positive for a last contact (but in certain exceptional cases of rare occurrence see Art. 330).

Then

$$\tau = \frac{h L \cos \psi}{n} - \frac{h m \cos (M - N)}{n}$$

or, when $\sin \psi$ is not very small,

$$\tau = \frac{hm}{n} \cdot \frac{\sin (M - N - \psi)}{\sin \psi}$$

If the local mean time t was observed, take $h = 3600$ in these formulæ, and then the (uncorrected) longitude is found by the equation

$$\omega = T_0 - t + \tau$$

If the local sidereal time μ was observed, take $h = 3609.856$, in the preceding formulæ; then, μ_0 being the sidereal time at the first meridian corresponding to T_0, we have

$$\omega = \mu_0 - \mu + \tau$$

The longitudes thus found will be the true ones only when all the elements of the computation are correct.

IV. To form the equations of condition for the correction of these longitudes, when the eclipse has been observed at a sufficient number of places, compute the time T_1 of nearest approach, and the minimum distance $\varkappa$, by the formulæ

$$T_1 = T_0 - \frac{1}{n} (x_0 \sin N + y_0 \cos N)$$
$$\varkappa = - x_0 \cos N + y_0 \sin N$$

* The values of N and $\log n$ being nearly constant, it will be expedient, where many observations are to be reduced, to compute them for the several integral hours at the first meridian, and to deduce their values for any given time by simple interpolation.

Take π for the time T_1, and compute the logarithm of

$$\nu = \frac{h}{n\pi}$$

the same value of h being used here as before.

For each observation at each place compute the coefficients $\nu \tan \psi$, $\nu \sec \psi$, and

$$E = \nu n(t + \omega - T_1) - x\nu \tan \psi - \frac{H}{r'\pi}\nu \sec \psi$$

where the unit of $t + \omega - T_1$ is one mean hour,

$$F = \tfrac{1}{2}\beta\beta\,[\nu n(t + \omega - T_1) - x\nu \tan \psi - L \sec \psi] - \frac{\nu\beta \cos d \cos (N + \psi)}{\cos \psi}$$

in which

$$H = 959''.788 \qquad \log H = 2.98218$$

$$\beta = \frac{\rho \sin \varphi'}{1 - ee} \qquad \log (1 - ee) = 9.99709$$

Then, ω' denoting the true longitude, the equation of condition is

$$\omega' = \omega - \nu . \gamma + \nu \tan \psi . \vartheta \pm \nu \sec \psi . \pi\Delta k + \nu \sec \psi \frac{\Delta H}{r'} + E\Delta\pi + F\pi\Delta ee$$

where the negative sign of the term $\nu \sec \psi . \pi\Delta k$ is to be used for interior contacts.

The discussion of the equations thus formed may then be carried out by Art. 334; taking as the unknown quantities γ, ϑ, $\pi\Delta k$, $\frac{\Delta H}{r'}$, $\Delta\pi$, and $\pi\Delta ee$.

EXAMPLE.—Find the longitude of Washington from the following observations of the solar eclipse of July 28, 1851:

At Washington (partial eclipse):

Beginning of eclipse,	July	27,	19^h	21^m	$31^s.2$	M.T
End "	"	"	20	50	38 .0	"

At Königsberg (total eclipse):

Beginning of eclipse,	July	28,	3	38	10 .8	"
Beginning of total obsc.,	"	"	4	38	57 .6	"
End of total obscuration,	"	"	4	41	54 .2	"
End of eclipse,	"	"	5	38	32 .9	"

For these places we have given—

	Lat. ϕ	Long. ω
Washington,	$+\ 38°\ 53'\ 39''.25$	$+\ 5^h\ 8^m\ 11^s.2$
Konigsberg,	$+\ 54\ \ 42\ \ 50\ .4$	$-\ 1\ \ 22\ \ \ 0\ .4$

The longitudes are reckoned from Greenwich. That of Königsberg will be assumed as correct, while that of Washington will be regarded as an approximate value which it is proposed to correct by these observations.

I. The mean Greenwich time of conjunction of the sun and moon in right ascension being, July 28, $2^h\ 21^m\ 2^s.6$, the general eclipse tables will be constructed for the Greenwich hours 0^h, 1^h, 2^h, 3^h, 4^h, and 5^h of July 28. For these times we find the following quantities from the *Nautical Almanac:*

For the Moon.*

Greenwich mean time.	α	δ	π
July 28, 0^h	125° 40′ 6″.75	+ 20° 3′ 30″.00	60′ 27″.30
1	126 19 9 .41	19 58 9 .36	28 .41
2	126 58 10 .80	19 52 39 .99	29 .49
3	127 37 10 .82	19 47 1 .92	30 .54
4	128 16 9 .37	19 41 15 .21	31 .56
5	128 55 6 .36	19 35 19 .89	32 .56

For the Sun.

Greenwich mean time.	α'	δ'	log r'
July 28, 0^h	127° 6′ 5″.25	+ 19° 5′ 24″.70	0.006578
1	8 32 .63	4 50 .23	76
2	10 59 .99	4 15 .74	74
3	13 27 .34	3 41 .21	72
4	15 54 .67	3 6 .64	70
5	18 21 .99	2 32 .05	67

* The moon's α and δ in the *Naut. Alm.* are directly computed only for every noon and midnight and interpolated for each hour. I have not used these interpolated values, but have interpolated anew to fifth differences. The moon's parallax has been diminished by 0″.3 according to Mr. ADAMS's Table in the Appendix to the *Naut. Alm.* for 1856.

With these values we form the following tables, as in Art. 297:

	a	d	Exterior Contacts.		Interior Contacts.	
			l	log i	l	log i
0^h	127° 6′ 17″.22	19° 5′ 16″.56	0.534046	7.663244	— 0.011771	7.661131
1	8 39 .51	4 42 .76	4023	45	11795	82
2	11 1 .78	4 8 .96	3973	47	11844	34
3	13 24 .03	3 35 .14	3899	49	11917	36
4	15 46 .27	3 1 .30	3801	51	12015	38
5	18 8 .50	2 27 .46	3679	53	12137	40

	x	Δ_1	Δ_2	Δ_3	y	Δ_1	Δ_2	Δ_3
0^h	— 1.338900	+ 0.569534			+ 0.968589	— 0.083020		
1	— 0.769366	.569591	+ 57		.885569	.083384	—364	
2	— 0.199775	.569590	— 1	—58	802185	.083736	—352	+12
3	+ 0.369815	.569535	— 55	—54	718449	.084079	—343	+ 9
4	+ 0.939350	.569416	—119	—64	.634370	.084420	—341	+ 2
5	+ 1.508766				.549950			

Hence the mean changes x' and y', for the epoch $T_0 = 2^h$ (according to the method of Art. 296), and the corresponding values of N and log n, are as follows:

	x'	y'	N	log n
0^h	+ 0.569563	— 0.083202	98° 18′ 39″.7	9.760126
1	591	3384	19 42 .7	168
$T_0 = 2$	600	3562	20 45 .3	194
3	590	3736	21 47 .5	205
4	563	3908	22 50 .0	203
5	514	4078	23 52 .7	186

II. The full computation for Königsberg, where both exterior and interior contacts were observed, will serve to illustrate the use of the preceding formulæ in every practical case.

For $\varphi = 54° \, 42' \, 50''.4$ we find

$$\log \rho \sin \varphi' = 9.909898 \qquad \log \rho \cos \varphi' = 9.762639$$

The sidereal time at Greenwich mean noon, July 28, was $8^h \, 22^m \, 13^s.27$, with which μ is found as given below. The computation of ξ, η, and L will be as follows:

	1st Ext. Cont.	1st Int. Cont.	2d Int. Cont.	2d Ext. Cont.
t	$3^h\ 38^m\ 10^s.8$	$4^h\ 38^m\ 57^s.6$	$4^h\ 41^m\ 54^s.2$	$5^h\ 38^m\ 32^s.9$
$t + \omega$	2 16 10 .4	3 16 57 .2	3 19 53 .8	4 16 32 .5
μ	12 0 46 .44	13 1 43 .22	13 4 40 .31	14 1 28 .31
μ (in arc)	180° 11′ 36″.6	195° 25′ 48″.3	196° 10′ 4″.7	210° 22′ 4″.7
For $t + \omega$, α	127 11 40 .1	127 14 4 .2	127 14 11 .2	127 16 25 .6
" δ	19 3 59 .8	19 3 25 .6	19 3 23 .9	19 2 52 .0
$\mu - \alpha$	52 59 56 .5	68 11 44 .1	68 55 53 .5	83 5 39 .1
$\log \sin (\mu - \alpha)$	9.902343	9.967762	9.969952	9.996838
$\log \cos (\mu - \alpha)$	9.779473	9.569889	9.555679	9.080040
$\log \xi$	9.664982	9.730401	9.732591	9.759477
ξ	+ 0.462362	+ 0.537528	+ 0.540244	+ 0.574748
$\log A \sin B$	9.909898	9.909898	9.909898	9.909898
$\log A \cos B$	9.542112	9.332528	9.318318	8.842679
B	66° 47′ 32″.2	75° 10′ 40″.4	75° 38′ 5″.9	85° 6′ 14″.3
$B - \delta$	47 43 32 .4	56 7 14 .8	56 34 42 .0	66 3 22 .3
$\log A$	9.946544	9 924595	9.923693	9.911486
$\log \sin (B - \delta)$	9.869192	9.919191	9.921499	9.960919
$\log \cos (B - \delta)$	9.827809	9.746201	9.740991	9.608355
$\log \eta$	9.815736	9 843786	9.845192	9.872405
η	+ 0.654239	+ 0.697888	+ 0.700152	+ 0.745427
$\log \zeta$	9.774353	9.670796	9.664684	9.519841
For $t + \omega$, $\log i$	7.663248	7.661137	7.661137	7.663252
" " l	+ 0.533956	— 0.011940	— 0.011944	+ 0.533772
$i\zeta$	+ 0.002739	+ 0.002148	+ 0.002117	+ 0.001524
L	+ 0.531217	— 0.014088	— 0.014061	+ 0.532248

III. The epoch of the table of x' and y' being $T_0 = 2^h$, we have for this time

$$x_0 = -\ 0.199775 \qquad\qquad y_0 = +\ 0.802185$$

with which we proceed to find the values of ω.

$m \sin M = x_0 - \xi$	— 0.662137	— 0.737303	— 0.740019	— 0.774523
$m \cos M = y_0 - \eta$	+ 0.147946	+ 0.104297	+ 0.102033	+ 0.056758
$\log m \sin M$	n9.820948	n9.867646	n9.869242	n9.889035
$\log m \cos M$	9.170107	9.018272	9.008741	8.754027
M	282° 35′ 42″.8	278° 3′ 5″.4	277° 51′ 1″.5	274° 11′ 28″.3
$\log m$	9.831527	9.871949	9.873331	9.890198
For $t + \omega$, N	98° 21′ 2″.1	98° 22′ 5″.1	98° 22′ 8″.2	98° 23′ 7″.3
" " $\log n$	9.760198	9.760206	9.760206	9.760200
$M - N$	184° 14′ 40″.7	179° 41′ 0″.3	179° 28′ 53″.3	175° 48′ 21″.0
$\log \sin (M - N)$	n8.869321	7.742368	7.956643	8.864135

$\log L$	9.725272	n8.148849	n8.148016	9.726114
$\log \sin \psi$	n8.975576	n9.465463	n9.681958	9.028219
ψ	185° 25′ 27″.7	343° 1′ 8″.6	208° 44′ 14″.0	6° 7′ 33″.2
$M - N - \psi$	358 49 13 .0	196 39 51 .7	330 44 39 .3	169 40 47 .8
$\log \sin (M - N - \psi)$	n8.313626	n9.457526	n9.689051	9.253208
$h = 3600$, $\log h$	3.556303			
$\log \tau$	2.965682	3.660109	3.676521	3.911290
τ	+ 0^h 15^m 24^s.0	+ 1^h 16^m 12^s.0	+ 1^h 19^m 8^s.1	+ 2^h 15^m 52^s.5
$T_0 - t$	− 1 38 10 .8	− 2 38 57 6	− 2 41 54 .2	− 3 38 32 9
ω	− 1 22 46 .8	− 1 22 45 .6	− 1 22 46 .1	− 1 22 40 .4

IV. *Equations of condition.*—To find T_1 and $\varkappa$, we have for $T_0 = 2^h$,

$$\log x_0 = n9.3006 \qquad N = 98° \; 20'.7$$
$$\log y_0 = 9.9043 \qquad \log n = 9.7602$$

whence

$$-\frac{x_0 \sin N}{n} = +0.3434 \qquad -x_0 \cos N = -0.0290 \qquad \log H = 2.9822$$

$$-\frac{y_0 \cos N}{n} = +0.2023 \qquad +y_0 \sin N = +0.7938 \qquad \log r' = 0.0066$$

$$T_1 = 2^h.5457 \qquad \varkappa = +0.7648 \qquad \log \pi = 3.5599$$

$$\pi = 3630'' \qquad \log \varkappa = 9.8835 \qquad \log \frac{h}{r'\pi} = 9.4157$$

$$\log \beta = \log \frac{\rho \sin \varphi'}{1 - ee} = 9.9128 \qquad \log \nu = \log \frac{h}{n\pi} = 0.2362$$

With these constants prepared, we readily form the coefficients of the equations of condition as follows:

	1st Ext. Cont.	1st Int. Cont.	2d Int. Cont.	2d Ext. Cont.
$\log \tan \psi$	8.9775	n9.4848	9.7390	9.0307
$\log \sec \psi$	n0.0019	0.0194	n0.0571	0.0025
$\nu \tan \psi$	+ 0.163	− 0.526	+ 0.944	+ 0.185
$\nu \sec \psi$	− 1.730	+ 1.801	− 1.964	+ 1.733
$t + \omega - T_1$	0^h.2762	+ 0^h.7355	+ 0^h.7860	+ 1^h.7300
$\log (t + \omega - T_1)$	n9.4412	9.8666	9 8954	0.2380
$\nu n (t + \omega - T_1)$	− 0.2739	+ 0.7295	+ 0.7795	+ 1.7155
$-\varkappa \nu \tan \psi$	− 0.1251	+ 0.4023	− 8.7223	− 0.1414
$-\frac{H}{r'\pi} \nu \sec \psi$	+ 0.4506	− 0.4691	+ 0.5117	− 0.4512
E	+ 0 0516	+ 0.6627	+ 0.5689	+ 1.1229

	1st Ext. Cont.	1st Int. Cont.	2d Int. Cont.	2d Ext. Cont.
$\nu n (t + \omega - T_1)$	$-$ 0.2739	$+$ 0.7295	$+$ 0.7795	$+$ 1.7155
$\varkappa \nu \tan \psi$	$-$ 0.1251	$+$ 0.4023	$-$ 0.7223	$-$ 0.1414
$- L \nu \sec \psi$	$+$ 0.9192	$+$ 0.0254	0.0276	$-$ 0.9222
	$+$ 0.5202	$+$ 1.1572	$+$ 0.0296	$+$ 0.6519
log	9.7162	0.0634	8.4713	9.8142
$\log \frac{1}{2} \beta\beta$	9.5246			
log 1st part of F	9.2408	9.5880	7.9959	9.3388
$N + \psi$	283° 46′.2	81° 21′.8	307° 4′.9	104° 28′.3
$\log \cos (N + \psi)$	9.3766	9.1766	9.7803	n9.3978
$\log (- \nu\beta \cos d \sec \psi)$	0.1264	n0.1439	0.1816	n0.1270
log 2d part of F	9.5030	n9.3205	9.9619	9.5248
1st part of F	$+$ 0.1741	$+$ 0.3873	$+$ 0.0099	$+$ 0.2182
2d " " F	$+$ 0.3184	$-$ 0.2092	$+$ 0.9160	$+$ 0.3349
F	$+$ 0.4925	$+$ 0.1781	$+$ 0.9259	$+$ 0.5531

Putting $\omega' + \nu\gamma = \Omega$, we have, therefore, for the four Königsberg observations, the equations

$$
(A)\quad
\begin{array}{c|llllll}
p & & & & & & \\
1 & \Omega = -1^h\,22^m\,46^s.8 + 0.163\,\vartheta & -1.730\,\pi\Delta k & -1.730\,\dfrac{\Delta\Pi}{r'} & +0.052\,\Delta\pi & +0.493\,\pi\Delta e \\
2 & \Omega = -1\;\;22\;\;45\,.6 - 0.526 & -1.801 & +1.801 & +0.663 & +0.178 \\
2 & \Omega = -1\;\;22\;\;46\,.1 + 0.944 & +1.964 & -1.964 & +0.569 & +0.926 \\
1 & \Omega = -1\;\;22\;\;40\,.4 + 0.185 & +1.733 & +1.733 & +1.123 & +0.553
\end{array}
$$

where we have annexed a column for the weight p, giving interior contacts double weight.

A similar computation for the two observations at Washington gives the following equations, in which $\Omega' = \omega'' + \nu\gamma$, ω'' denoting the true longitude of Washington:

$$
(B)\quad
\begin{array}{c|llllll}
p & & & & & & \\
1 & \Omega' = 5^h\,7^m\,29^s.9 + 1.660\,\vartheta & -2.392\pi\Delta k & -2.392\,\dfrac{\Delta\Pi}{r'} & -2.681\,\Delta\pi & +0.722\,\pi\Delta e \\
1 & \Omega' = 5\;\;7\;\;21\,.9 - 2.406 & +2.959 & +2.959 & +0.509 & -1.323
\end{array}
$$

More observations would be necessary in order to determine all the corrections; but I shall retain all the terms in order to illustrate the general method. Subtracting each of the Königsberg equations from each of those which follow it, we obtain the six equations,

$$
(A')\quad
\begin{array}{c|rrrrrrr}
p & & & & & & & \\
\frac{1}{3} & 0 = +1''.2 & -0.689\,\vartheta & -0.071\,\pi\Delta k & +3.531\,\frac{\Delta H}{r'} & +0.611\,\Delta\pi & -0.315\,\pi\Delta ee \\
\frac{1}{3} & 0 = +0\ .7 & +0.781 & +3.694 & -0.234 & +0.517 & +0.433 \\
\frac{1}{6} & 0 = +6\ .4 & +0.022 & +3.463 & +3.463 & +1.071 & +0.060 \\
\frac{2}{3} & 0 = -0\ .5 & +1.470 & +3.765 & -3.765 & -0.094 & +0.748 \\
\frac{1}{3} & 0 = +5\ .2 & +0.711 & +3.534 & -0.068 & +0.460 & +0.375 \\
\frac{1}{3} & 0 = +5\ .7 & -0.759 & -0.231 & +3.697 & +0.554 & -0.373
\end{array}
$$

where the weight in each case is the quotient obtained by dividing the product of the two weights of the equations whose difference is taken, by the sum of the weights of the four original equations (Art. 334).

The same method, applied in the case of the two Washington equations, gives the single equation

$$
(B')\quad
\begin{array}{c|l}
p & \\
\frac{1}{2} & 0 = -8''.0 - 4.066\,\vartheta + 5.351\,\pi\Delta k + 5.351\,\frac{\Delta H}{r'} + 3.190\,\Delta\pi - 2.055\,\pi\Delta ee
\end{array}
$$

From the equations (A′) and (B′) are formed the following final equations, having regard to their weights, in the usual manner:

$$
\begin{array}{lrrrrrr}
0 = & +15.495 & +10.426\,\vartheta & -5.300\,\pi\Delta k & -16.377\,\frac{\Delta H}{r'} & -6.609\,\Delta\pi & +5.281\,\pi\Delta ee \\
0 = & -12.445 & -5.300 & +34.506 & +6.135 & +10.040 & -2.575 \\
0 = & -8.191 & -16.377 & +6.135 & +34.505 & +10.740 & -8.214 \\
0 = & -9.371 & -6.609 & +10.040 & +10.740 & +5.672 & -3.316 \\
0 = & +7.951 & +5.281 & -2.575 & -8.214 & -3.316 & +2.675
\end{array}
$$

As we cannot expect a satisfactory determination of $\Delta\pi$ and $\pi\Delta ee$ from these observations, we disregard the last two equations, and then, solving the first three, we obtain ϑ, $\pi\Delta k$, and $\frac{\Delta H}{r'}$ in terms of $\Delta\pi$ and $\pi\Delta ee$, as follows:

$$
\begin{aligned}
\vartheta &= -4''.36 + 0.375\,\Delta\pi - 0.525\,\pi\Delta ee \\
\pi\Delta k &= +0\ .02 - 0.216\,\Delta\pi - 0.004\,\pi\Delta ee \\
\frac{\Delta H}{r'} &= -1\ .83 - 0.095\,\Delta\pi - 0.010\,\pi\Delta ee
\end{aligned}
$$

These values substituted in the equations (A) give

$$
\begin{array}{lrrrr}
\Omega = & -1^h\ 22^m\ 44^s.38 & +0.651\,\Delta\pi & +0.432\,\pi\Delta ee \\
\Omega = & -1\ \ 22\ \ 46\ .64 & +0.684 & +0.443 \\
\Omega = & -1\ \ 22\ \ 46\ .58 & +0.685 & +0.442 \\
\Omega = & -1\ \ 22\ \ 44\ .34 & +0.653 & +0.432
\end{array}
$$

the mean of which, giving the second and third double weight, is

(A″) $$\Omega = -1^h\ 22^m\ 45^s.86 + 0.674\ \Delta\pi + 0.439\ \pi\Delta ee$$

The equations (B) become

$$\Omega' = 5^h\ 7^m\ 26^s.99 - 1.314\ \Delta\pi - 0.116\ \pi\Delta ee$$
$$\Omega' = 5\ \ 7\ \ 27.03 - 1.314\ \ \ \ \ \ - 0.101$$

the mean of which is

(B″) $$\Omega' = 5^h\ 7^m\ 27^s.01 - 1.314\ \Delta\pi - 0.109\ \pi\Delta ee$$

Now, if we assume the longitude of Königsberg to be well determined, we have

$$\Omega = \omega' + \nu\gamma = \quad 1^h\ 22^m\ 0^s.4 + \nu\gamma$$

which, with the equation (A″), gives

$$\nu\gamma = -\ \ 45^s.46 + 0.674\ \Delta\pi + 0.439\ \pi\Delta ee$$

Hence, by (B″), the true longitude of Washington is

$$\omega'' = \Omega' - \nu\gamma = 5^h\ 8^m\ 12^s.47 - 1.988\ \Delta\pi - 0.548\ \pi\Delta ee$$

If the longitude of Washington were also previously well established, this last equation would give us a condition for determining the correction of the moon's parallax. Thus, if we adopt $\omega'' = 5^h\ 8^m\ 12^s.34$, which results from the U. S. Coast Survey Chronometric Expeditions of 1849, '50, '51, and '55, this equation gives

$$0 = +\ 0.13 - 1.988\ \Delta\pi - 0.548\ \pi\Delta ee$$

whence

$$\Delta\pi = +\ 0''.07 - 0.276\ \pi\Delta ee$$

The probable value of Δee, according to Bessel, is within $\pm\ 0.0001$, so that the last term cannot here exceed $0''.10$. If, therefore, the above observations are reliable and the supposed longitudes exact, the probable correction of the parallax indicated scarcely exceeds $0''.1$, a quantity too small to be established by so small a number of observations. Nevertheless, the example proves both that the adopted parallax is very nearly perfect, and that a large number of observations at various well determined places in the two hemispheres may furnish a good determination of the correction which it yet requires.

Finally, the corrections of the Ephemeris in right ascension and declination, according to the above determination of γ and ϑ, are found by (586) to be (putting α' for a and δ' for d)

$$\Delta(\alpha - \alpha') = -28''.42 + 0.469\,\Delta\pi + 0.187\,\pi\Delta ee$$
$$\Delta(\delta - \delta') = -\ \ 0\ .48 + 0.314\,\Delta\pi - 0.556\,\pi\Delta ee$$

This large correction in right ascension agrees with the results of the best meridian observations on and near the date of this eclipse. Since that time the Ephemerides have been greatly improved.

LUNAR ECLIPSES.

338. *To find whether near a given opposition of the moon and sun a lunar eclipse will occur.*—The solution of this problem is similar to that of Art. 287, except that for the sun's semidiameter there must be substituted the apparent semidiameter of the earth's shadow at the distance of the moon; and also that the apparent distance of the centres of the moon and the shadow will not be affected by parallax, since when the earth's shadow falls upon the moon an eclipse occurs for all observers who have the moon above their horizon. If S, Fig. 45, is the sun's centre, E that of the earth, LM the semidiameter of the earth's shadow at the moon, we have

Fig. 45.

$$\begin{aligned}\text{Apparent semidiameter of the total}\\ \text{shadow} &= LEM\\ &= BLE - EVL\\ &= BLE - (AES - EAV)\\ &= \pi - s' + \pi'\end{aligned}$$

where we employ the same notation as in Art. 287.

But observation has shown that the earth's atmosphere increases the apparent breadth of the shadow by about its one-fiftieth part:* so that we take

* This fractional increase of the breadth of the shadow was given by Lambert as $\frac{1}{40}$, and by Mayer as $\frac{1}{60}$. Beer and Madler found $\frac{1}{50}$ from a number of observations of eclipses of lunar spots in the very favorable eclipse of December 26, 1833. See "*Der Mond nach seinen kosmischen und individuellen Verhältnissen, oder allgemeine vergleichende Selenographie, von* Wilhelm Beer *und* Dr. Johann Heinrich Madler," § 98.

$$\text{App. semid. of shadow} = \frac{51}{50}(\pi - s' + \pi') \tag{587}$$

In order that a lunar eclipse may happen, we must have, therefore, instead of (477),

$$\beta \cos I' < \frac{51}{50}(\pi - s' + \pi') + s \tag{588}$$

or, taking a mean value of I', as in Art. 287,

$$\beta < \left[\frac{51}{50}(\pi - s' + \pi') + s\right] \times 1.00472$$

Employing mean values in the small fractional part, we have

$$\left[\frac{51}{50}(\pi - s' + \pi') + s\right] \times .00472 = 16''$$

and the condition becomes

$$\beta < \frac{51}{50}(\pi - s' + \pi') + s + 16'' \tag{589}$$

If in this we substitute the greatest values of π, π', and s, and the least value of s', the limit

$$\beta < 63' \; 53''$$

is the greatest limit of the moon's latitude at the time of opposition for which an eclipse can occur.

If we substitute the least values of π, π', and s, and the greatest value of s', the limit

$$\beta < 52' \; 4''$$

is the least limit for which an eclipse can fail to occur.

Hence, a lunar eclipse is *certain* if at full moon $\beta < 52' \; 4''$, *impossible* if $\beta > 63' \; 53''$, and doubtful between these limits. The doubtful cases can be examined by (589), or still more exactly by (588), employing the actual values of π, π', s, s', at the time, and computing I' by (475).

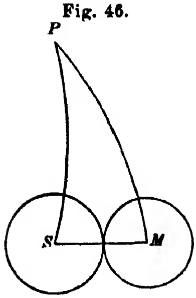
These limits are for the total shadow. For the penumbra we have

$$\text{App. semid. of penumbra} = \frac{51}{50}(\pi + s' + \pi') \tag{590}$$

so that the condition (588) may be employed to determine whether any portion of the penumbra will pass over the moon, by substituting $+ s'$ for $- s'$. It will be worth while to make this examination only when it has been found that the total shadow does not fall upon the moon.

339. *To find the time when a given phase of a lunar eclipse will occur.*—The solution of this problem may be derived from the general formulæ given for solar eclipses, by interchanging the moon and earth and regarding the lunar eclipse as an eclipse of the sun seen from the moon; but the following direct investigation is even more simple.

Fig. 46.



Let S, Fig. 46, be the point of the celestial sphere which is opposite the sun, or the apparent geocentric position of the centre of the earth's shadow; M, the geocentric place of the centre of the moon; P, the north pole. If we put

$$\begin{aligned}
\alpha &= \text{the right ascension of the moon,} \\
\alpha' &= \text{the right ascension of the point } S, \\
&= \text{R. A. of the sun} + 180^\circ, \\
\delta &= \text{the declination of the moon,} \\
\delta' &= \text{the declination of the sun,} \\
Q &= \text{the angle } PSM, \\
L &= SM,
\end{aligned}$$

we have

$$-\delta' = \text{the declination of } S,$$

and the triangle PSM gives

$$\left.\begin{aligned}
\sin L \sin Q &= \cos \delta \sin (\alpha - \alpha') \\
\sin L \cos Q &= \cos \delta' \sin \delta + \sin \delta' \cos \delta \cos (\alpha - \alpha')
\end{aligned}\right\} \quad (591)$$

The eclipse begins or ends when the arc SM is exactly equal to the sum of the apparent semidiameters of the moon and the shadow. The figure of the shadow will differ a little from a circle, as the earth is a spheroid; but it will be sufficiently accurate to regard the earth as a sphere with a mean radius, or that for the latitude 45°. This is equivalent to substituting for π in (587) and (590) the parallax reduced to the latitude 45°, which may be found by the formula

$$\pi_1 = [9.99929]\,\pi \tag{592}$$

where the factor in brackets is given by its logarithm.

Hence the first and last contacts of the moon with the penumbra occur when we have

$$L = \frac{51}{50}(\pi_1 + s' + \pi') + s \tag{593}$$

For the first and last contacts with the total shadow,

$$L = \frac{51}{50}(\pi_1 - s' + \pi') + s \tag{594}$$

For the first and second internal contacts with the penumbra,

$$L = \frac{51}{50}(\pi_1 + s' + \pi') - s \tag{595}$$

For the first and second internal contacts with the total shadow, or the beginning and end of total eclipse,

$$L = \frac{51}{50}(\pi_1 - s' + \pi') - s \tag{596}$$

The solution of our problem consists in finding the time at which the equations (591) are satisfied when the proper value of L is substituted in them. A very precise computation would, however, be superfluous, as the contacts cannot be observed with accuracy, on account of the indefinite character of the outline both of the penumbra and of the total shadow. It will be sufficient to write for (591) the following approximate formulæ, easily deduced from them:

$$\left.\begin{aligned} L\sin Q &= (\alpha - \alpha')\cos\delta \\ L\cos Q &= \delta + \delta' - \frac{\sin 2\delta' \sin^2 \frac{1}{2}(\alpha - \alpha')}{\sin 1''} \end{aligned}\right\} \tag{597}$$

Let us put

$$\left.\begin{aligned} \varepsilon &= \frac{\sin 2\delta' \sin^2 \frac{1}{2}(\alpha - \alpha')}{\sin 1''} \\ x &= (\alpha - \alpha')\cos\delta \\ y &= \delta + \delta' - \varepsilon \\ x', y' &= \text{the hourly increase of } x \text{ and } y\,, \end{aligned}\right\} \tag{598}$$

then, if the values of x and y are computed for several successive

hours near the time of full moon, we shall also have x' and y' from their differences; and if x_0 and y_0 denote the values of x and y for an assumed epoch T_0, near the time of opposition, we shall have for the required time of contact $T = T_0 + \tau$ the equations

$$L \sin Q = x_0 + x'\tau$$
$$L \cos Q = y_0 + y'\tau$$

from which τ is obtained by the process already frequently employed in the preceding problems. Thus, computing the auxiliaries m, M, n, N, by the equations

$$\left.\begin{array}{ll} m \sin M = x_0 & n \sin N = x' \\ m \cos M = y_0 & n \cos N = y' \end{array}\right\} \quad (599)$$

we shall have

$$\left.\begin{aligned} \sin \psi &= \frac{m \sin (M - N)}{L} \\ \tau &= \frac{L \cos \psi}{n} - \frac{m \cos (M - N)}{n} \\ T &= T_0 + \tau \end{aligned}\right\} \quad (600)$$

in which we take $\cos \psi$ with the negative sign for the first contact and with the positive sign for the last contact.

The angle $Q = N + \psi$ is very nearly the supplement of the angle PMS, Fig. 46; from which we infer that *the angle of position of the point of contact reckoned on the moon's limb from the north point of the limb towards the east* $= 180° + N + \psi$.

The time of greatest obscuration is found, as in Art. 324, to be

$$T_1 = T_0 - \frac{m \cos (M - N)}{n} \quad (601)$$

which is also the middle of the eclipse.

The least distance of the centres of the shadow and of the moon being denoted by Δ, we have, as in Art. 324,

$$\Delta = \pm\, m \sin (M - N) \quad (602)$$

the sign being taken so that Δ shall be positive. If then we put

$D =$ the magnitude of the eclipse, the moon's diameter being unity,

we evidently have

$$D = \frac{L - \Delta}{2s} \tag{603}$$

in which the value of L for total shadow from (594) is to be employed.

The small correction ε in (598) may usually be omitted, but its value may be taken at once from the following table:

Value of ε.

δ'	$\alpha - \alpha'$						
	0″	1000″	2000″	3000″	4000″	5000″	6000″
0°	0″	0″	0″	0″	0″	0″	0″
5	0	0	1	2	3	5	8
10	0	0	2	4	7	10	15
15	0	1	2	6	10	15	22
20	0	1	3	7	13	19	28
25	0	1	4	8	15	23	33
30	0	1	4	9	17	26	38

The quantity ε has the same sign as δ', and is to be subtracted algebraically from $\delta + \delta'$.

EXAMPLE.—Compute the lunar eclipse of April 19, 1856. The Greenwich mean time of full moon is April 19, $21^h\ 5^m.5$. We therefore compute the co-ordinates x and y for the Greenwich times April 19, 18^h, 21^h, 24^h.

	18^h	21^h	24^h
☽ R. A. $= \alpha$	$13^h\ 46^m\ 36^s.62$	$13^h\ 52^m\ 9^s.81$	$13^h\ 57^m\ 45^s.12$
☉ R.A. + 180° $= \alpha'$	13 52 52 .98	13 53 20 .93	13 53 48 .88
$\alpha - \alpha'$	— 6 16 .36	— 1 11 .12	+ 3 56 .24
$\alpha - \alpha'$ (in arc)	— 5645″	— 1067″	+ 3544″
☽ Decl $= \delta$	—11° 27′ 0″.2	—12° 6′ 43″.7	—12° 46′ 5″.5
☉ " $= \delta'$	+11 35 49 .4	+11 38′ 22 .8	+11 40 56 .6
$- \varepsilon$	— 15 .	0 .	— 6 .
y	+ 514″	— 1701″	— 3915″
log $(\alpha - \alpha')$	n3.75166	n3.02816	3.54949
log cos δ	9.99127	9.99022	9.98913
log x	n3.74293	n3.01838	3.53862

Hence we have the following table:

	x	Diff. $= 3x'$		y	Diff. $= 3y'$	
18^h	$-5533''$	$+4490$	$x' = +1498$	$+514''$	-2215	$y' = -738$
21	-1043	$+4499$		-1701	-2214	
24	$+3456$			-3915		

To find L, we have, by (593) and (594),

$$\pi = 54' \, 32'' \qquad \pi_1 = 3267''$$
$$s' = 957$$
$$\pi' = 9$$

$$\pi_1 - s' + \pi' = 2319 \qquad\qquad \pi_1 + s' + \pi' = 4233''$$
$$\tfrac{1}{50}(\pi_1 - s' + \pi') = 46 \qquad\qquad \tfrac{1}{50}(\pi_1 + s' + \pi') = 85$$
$$s = 891 \qquad\qquad s = 891$$
$$L \text{ for shadow} = 3256 \qquad\qquad L \text{ for penumbra} = 5209$$

Assuming the time $T_0 = 21^h$, we proceed by (599) and (600):

$x_0 = m \sin M$	-1043	$x' = n \sin N$	$+1498$
$y_0 = m \cos M$	-1701	$y' = n \cos N$	-738
M	$210^\circ \, 31'.0$	N	$116^\circ \, 13'.7$
$\log m$	3.3000	$\log n$	3.2227

$$-\frac{m}{n}\cos(M - N) = +0^h.089$$
$$T_0 = 21$$
$$T_1 = \text{Time of middle of eclipse} = 21\,.089$$

	Shadow.	Penumbra.
$\log \sin \psi$	9.7861	9.5820
$\dfrac{L \cos \psi}{n}$	$\mp 1^h.543$	$\mp 2^h.883$
T_1	21 .089	21 .089
Beginning	19 .546	18 .20~~7~~6
End	22 .63~~3~~2	23 .972

For the magnitude of the eclipse, we have, by (602) and (603):

$$m \sin (M - N) = \Delta = 1990''$$
$$L = 3256$$
$$L - \Delta = 1266 \qquad D = \frac{1266}{1782} = 0.710$$
$$2s = 1782$$

For the position of the points of contact with the shadow, we have, from the above value of log sin ψ for shadow, taking cos ψ as negative for first and positive for second contact,

	1st Contact.	2nd Contact.
ψ	142° 20′	37° 40′
N	116 14	116 14
$180° + N + \psi$	78 34	333 54

and hence

1st contact is 79° from north point of limb towards the east,
2d " 26° " " " " " west.

The times of the several contacts for any meridian are obtained from the times above found by subtracting the west longitude of that meridian.

OCCULTATIONS OF FIXED STARS.

340. The occultation of a fixed star by the moon may be treated as a simple case of a solar eclipse, in which the sun is removed to so great a distance that its parallax and semidiameter may be put equal to zero. The cone of shadow then becomes a cylinder, and *the point* Z of Art. 289 is nothing more than the position of the star, so that the co-ordinates of the moon at any time are found by the formulæ (482) by regarding a and d as the right ascension and declination of the star. In like manner the co-ordinates of the place of observation will be found by (483). The radius of the shadow is constant and equal to k, which is, therefore, to be substituted for $L = l - i\zeta$ in (490) and (491). The co-ordinates z and ζ will not be required unless we compute the latter for the purpose of taking into account the effect of refraction according to Art. 327.

For the convenience of the computer I shall here recapitulate the formulæ required in the practical applications, making the modifications just indicated.

341. *To find the longitude from an observed occultation of a star by the moon.*—According to the method of Art. 329, we proceed as follows:

I. Find, approximately, the time of conjunction of the moon and star in right ascension, reckoned at the first meridian. Take from the Ephemeris, for four consecutive integral hours, two preceding and two following the time of conjunction, the moon's right ascension (α), declination (δ), and horizontal parallax (π). Take also from the most reliable source the star's right ascension (α') and declination (δ').

For each of these hours compute the co-ordinates x and y by the formulæ

$$x = \frac{\cos\delta\sin(\alpha - \alpha')}{\sin\pi}$$

$$y = \frac{\sin(\delta - \delta')\cos^2\tfrac{1}{2}(\alpha - \alpha') + \sin(\delta + \delta')\sin^2\tfrac{1}{2}(\alpha - \alpha')}{\sin\pi}$$

and, arranging their values in a table, deduce their hourly variations x' and y' for the same instants for which x and y have been computed.

II. Let μ be the local sidereal time of an observed immersion or emersion of the star at a place whose latitude is φ, and west longitude ω; t the corresponding local mean time. The co-ordinates of the place are to be computed by the formulæ

$$A\sin B = \rho\sin\varphi' \qquad \xi = \rho\cos\varphi'\sin(\mu - \alpha')$$

$$A\cos B = \rho\cos\varphi'\cos(\mu - \alpha') \qquad \eta = A\sin(B - \delta')$$

$$\zeta = A\cos(B - \delta')$$

When log ζ is small, add to logs ξ and η the correction for refraction from the table on p. 517.

III. Assume any convenient time T_0 reckoned at the first meridian, so near to $t + \omega$ that x and y may be considered to vary proportionally with the time in the interval $t + \omega - T_0$. For the assumed time, take the values of x and y (denoting them by x_0 and y_0), and also those of x' and y', and compute the auxiliaries m, M, &c. by the formulæ

$$m \sin M = x_0 - \xi \qquad n \sin N = x'$$
$$m \cos M = y_0 - \eta \qquad n \cos N = y'$$
$$\sin \psi = \frac{m \sin (M - N)}{k} \qquad \log k = 9.435000^*$$

where ψ is (in general) to be so taken that $\cos \psi$ shall be negative for immersion and positive for emersion (but in certain exceptional cases of rare occurrence, and of but little use in finding the longitude, see Art. 330). Then

$$\tau = \frac{hk \cos \psi}{n} - \frac{hm \cos (M - N)}{n}$$

or, when $\sin \psi$ is not very small,

$$\tau = \frac{hm}{n} \cdot \frac{\sin (M - N - \psi)}{\sin \psi}$$

If the local mean time t was observed, take $h = 3600$ in these formulæ, and then the longitude will be found by

$$\omega = T_0 - t + \tau$$

But if the local sidereal time μ was observed, take $h = 3609.856$ in the preceding formulæ; then, μ_0 being the sidereal time at the first meridian corresponding to T_0,

$$\omega = \mu_0 - \mu + \tau$$

The longitude thus found will be affected by the errors of the Ephemeris.

IV. To form the equations of condition for correcting the longitude for errors of the Ephemeris when the occultation has been observed at more than one place, compute the auxiliaries

$$T_1 = T_0 - \frac{1}{n} (x_0 \sin N + y_0 \cos N)$$
$$\varkappa = - x_0 \cos N + y_0 \sin N$$
$$\nu = \frac{h}{n \pi}$$

the same value of h being used as before.

* According to OUDEMANS (*Astron. Nach.*, Vol. LI., p. 30), we should use for occultations $k = 0.27264$, or $\log k = 9.435590$. which amounts to taking the moon's apparent semidiameter about 1″.25 greater in occultations than in solar eclipses. But it is only for the reduction of isolated observations that we need an exact value, since, when we have a number of observations, the correction of whatever value of k we may use will be obtained by the solution of our equations of condition.

Then, for each observation at each place, compute the coefficients $\nu \tan \psi$, $\nu \sec \psi$, and

$$E = \nu n (t + \omega - T_1) - x\nu \tan \psi$$

where ω is the approximate longitude and the unit of $t - \omega - T$ is one mean hour, and also

$$F = \tfrac{1}{2} \beta\beta\, [\nu n (t + \omega - T_1 - x\nu \tan \psi - k\nu \sec \psi] - \frac{\nu\beta \cos \delta' \cos (N + \psi)}{\cos \psi}$$

in which

$$\beta = \frac{\rho \sin \varphi'}{1 - ee} \qquad \log (1 - ee) = 9.99709$$

Then, ω' denoting the true longitude,

$$\omega' = \omega - \nu\gamma + \nu \tan \psi \,.\, \vartheta + \nu \sec \psi \,.\, \pi\Delta k + E \,.\, \Delta\pi + F \,.\, \pi\Delta ee$$

in which γ and ϑ have the signification

$$\begin{aligned} \gamma &= \quad \sin N \cos \delta \,\Delta(\alpha - \alpha') + \cos N \,\Delta(\delta - \delta') \\ \vartheta &= - \cos N \cos \delta \,\Delta(\alpha - \alpha') + \sin N \,\Delta(\delta - \delta') \end{aligned}$$

The discussion of the equations of condition thus formed may then be carried out precisely as in Art. 334, taking γ, ϑ, $\pi\Delta k$, $\Delta\pi$, and $\pi\Delta ee$ as the unknown quantities.

EXAMPLE.—The occultation of *Aldebaran*, April 15, 1850, was observed at Cambridge, Mass., and Königsberg, as follows:*

At Cambridge, $\varphi = 42° 22' 48''.6$, $\omega = + 4^h 44^m 30^s$.
Immersion, $2^h 1^m 52^s.45$ Mean time.
Emersion, 3 1 38 .35 " "

At Konigsberg, $\varphi = 54° 42' 50''.4$, $\omega = - 1^h 22^m 0^s.4$
Immersion, $10^h 57^m 43^s.66$ Sidereal time.
Emersion, 11 47 47 .60 " "

I. The Greenwich mean time of conjunction of the moon and star was about $7^h 30^m$, and hence we take our data from the Nautical Almanac as follows:

---

* Astronomical Journal, Vol. I, pp. 139 and 175.

1850 April 15.	α	δ	π
6^h	65° 56′ 21″.16	+ 16° 40′ 0″.05	58′ 55″.22
7	66 32 32 .06	16 46 30 .53	58 55 .87
8	67 8 46 .02	16 52 54 .77	58 56 .49
9	67 45 3 .02	16 59 12 .76	58 57 .10

The position of *Aldebaran* for the same date was

$$\alpha' = 66° \; 49' \; 33''.9 \qquad\qquad \delta' = +\, 16° \; 12' \; 1''.7$$

Hence, by I. of the preceding article, we form the following table:

Gr. T.	x	x'	y	y'
6^h	− 0.86519	+ 0.58849	+ 0.47664	+ 0.10871
7	− 0.27671	47	.58531	63
8	+ 0.31176	42	.69390	56
9	+ 0.90014	32	.80243	48

II. The sidereal time of Greenwich Mean Noon, April 15, 1860, was $1^h \; 33^m \; 8^s.96$. With this number, converting the Königsberg times into mean times for the sake of uniformity, we find

	Cambridge.		Königsberg	
	Immersion	Emersion.	Immersion	Emersion
t	2^h 1^m $52^s.45$	3^h 1^m $38^s.35$	9^h 23^m $15^s.64$	10^h 13^m $11^s.38$
$t + \omega$	6 46 22 .45	7 46 8 .35	8 1 15 .24	8 51 10 .98
μ	54° 2′ 2″.55	69° 0′ 58″.35	164° 25′ 54″.90	176° 56′ 54″.00
$\mu - \alpha'$	347 12 28 .65	2 11 24 .45	97 36 21 .00	110 7 20 .10
$\log \rho \sin \phi'$	9.826441		9.909898	
$\log \rho \cos \phi'$	9.869121		9.762639	
$\log \xi$	n9.214324	8.451362	9.758801	9.735287
$\log \eta$	9.646065	9 641159	9.904038	9.922175
$\log \zeta$	9.944427	9.952794	9.185091	8.549725

The value of log ζ has been found in order to find the correction for refraction. This correction is here quite sensible in the case of the Königsberg observations which were made at a great

zenith distance. By the table on p. 517, we find that the logarithms of ξ and η must be increased by 0.000006 for immersion, and by 0.000041 for emersion. Applying these corrections, the values of the co-ordinates are as follows:

ξ	-0.16380	$+0.02827$	$+0.57386$	$+0.54366$
η	$+0.44266$	$+0.43768$	$+0.80175$	$+0.83602$

III. Assuming convenient times not far from $t+\omega$, we have

Assumed T_0	$6^h.8$	$7^h.8$	$8^h.0$	$8^h.85$
x_0	-0.39440	$+0.19406$	$+0.31176$	$+0.81188$
y_0	$+0.56358$	$+0.67218$	$+0.69390$	$+0.78615$
$x_0-\xi=m\sin M$	-0.23460	$+0.16579$	-0.26210	$+0.26822$
$y_0-\eta=m\cos M$	$+0.12092$	$+0.23450$	-0.10785	-0.04987
M	297° 40′ 16″.5	35° 15′ 36″.1	247° 38′ 1″.0	100° 31′ 57″.7
$\log m$	9.415608	9.458164	9.452433	9.435871
$x'=n\sin N$	$+0.58847$	$+0.58843$	$+0.58842$	$+0.58836$
$y'=n\cos N$	$+0.10865$	$+0.10857$	$+0.10856$	$+0.10849$
N	79° 32′ 21″.1	79° 32′ 45″.8	79° 32′ 48″.5	79° 33′ 8″.5
ψ	216 11 35 .9	312 33 59 .0	167 35 28 .5	21 1 28 .1
($h=3600$) τ	$-89^s.74$	$-128^s.82$	$-68^s.63$	$-3^s.52$
ω	$4^h\ 44^m\ 37^s.81$	$4^h\ 44^m\ 12^s.83$	$-1^h\ 22^m\ 7^s.01$	$-1^h\ 22^m\ 4^s.90$

IV. For the equations of condition, taking $T_0=7^h.8$,

$$T_1 = 7^h.2772 \qquad \pi = 3536''$$
$$\varkappa = +0.6258 \qquad \log\nu = 0.2308$$

and putting

$$\omega_1 = \text{the true longitude of Cambridge,}$$
$$\omega_1' = \text{" " Konigsberg,}$$

we find, neglecting terms in Δee,

$$\begin{aligned}
\omega_1 &= 4^h\ 44^m\ 37^s.81 - \nu\gamma + 1.245\,\vartheta - 2.108\,\pi\Delta k - 1.293\,\Delta\pi\\
\omega_1 &= 4\ \ 44\ \ 12\ .83 - \nu\gamma - 1.852\,\vartheta + 2.515\,\pi\Delta k + 1.660\,\Delta\pi\\
\omega_1' &= -1\ \ 22\ \ \ 7\ .01 - \nu\gamma - 0.374\,\vartheta - 1.742\,\pi\Delta k + 0.991\,\Delta\pi\\
\omega_1' &= -1\ \ 22\ \ 14\ .90 - \nu\gamma + 0.654\,\vartheta + 1.822\,\pi\Delta k + 1.195\,\Delta\pi
\end{aligned}$$

whence the two equations

$$0 = +24^s.98 + 3.097\,\vartheta - 4.623\,\pi\Delta k - 2.953\,\Delta\pi$$
$$0 = +\ 7\ .89 - 1.028\,\vartheta - 3.564\,\pi\Delta k - 0.204\,\Delta\pi$$

If we determine ϑ and $\pi\Delta k$ in terms of $\Delta\pi$, these equations give

$$\vartheta = -3''.33 + 0.607\,\Delta\pi$$
$$\pi\Delta k = +3\ .17 - 0.232\,\Delta\pi$$

and then we find

$$\omega_1 = \quad 4^h\,44^m\,26^s.98 - \nu\gamma - 0.048\,\Delta\pi$$
$$\omega_1' = -1\;\;22\;\;11\,.29 - \nu\gamma + 1.169\,\Delta\pi$$

Assuming $\omega_1' = -1^h\,22^m\,0^s.4$ as well determined, the last equation gives

$$\nu\gamma = -10^s.89 + 1.169\,\Delta\pi$$

which substituted in the value of ω_1 gives

$$\omega_1 = 4^h\,44^m\,37^s.87 - 1.217\,\Delta\pi$$

Finally, adopting the correction of the parallax for this date as given in Mr. ADAMS's table (Appendix to the Nautical Almanac for 1856), $\Delta\pi = +5''.1$, this last value becomes

$$\omega_1 = 4^h\,44^m\,31^s.66$$

which agrees almost perfectly with the longitude of Cambridge found by the chronometric expeditions, which is $4^h\,44^m\,31^s.95$.

With the same value of $\Delta\pi$ we find

$$\gamma = -2''.90 \qquad \vartheta = -0''.23 \qquad \pi\Delta k = +1''.99$$

and hence, by (586), the corrections of the Ephemeris on this date, according to these observations, are

$$\Delta(\alpha - \alpha') = -2''.93 \qquad \Delta(\delta - \delta') = -0''.77$$

The value $\pi\Delta k = +1''.99$ gives $\Delta k = 0.00056$, and hence the corrected value $k = 0.27227 + 0.00056 = 0.27283$, which is not very different from OUDEMANS's value. (See p. 551).

342. When a number of occultations have been observed at a place for the determination of its longitude, it will usually be found that but few of the same occultations have been observed at other places. If, then, we were to depend altogether upon *corresponding* observations at other places, we should lose the greater part of our own. In order to employ all our data, we may in such case find for each date the corrections of the moon's place from meridian observations (see Art. 235), and, employing the corrected right ascension and declination in the computation of x and y, our equations of condition will involve only terms in $\pi\Delta k$ and $\Delta\pi$. The value of $\Delta\pi$ will, however, be different on

different dates, and, therefore, if we wish to retain this term, we must introduce in its stead the correction of the *mean* parallax which is the constant of parallax in the lunar tables. If this constant is denoted by π_0, we shall have, very nearly,

$$\Delta\pi = \frac{\pi}{\pi_0}\Delta\pi_0$$

where π is the parallax for the given date. The equations of condition will then be of the form

$$\omega_1 = \omega + a\,.\,\pi\Delta k + b\,.\,\Delta\pi_0$$

where

$$a = \nu \sec \psi \qquad\qquad b = \frac{\pi}{\pi_0} E$$

In PEIRCE'S Lunar Tables, now employed in the construction of our Ephemeris, $\pi_0 = 3422''.06$.

343. The passage of the moon through a well determined group of stars, such as the *Pleiades*, affords a peculiarly favorable opportunity for determining the correction of the moon's semidiameter as well as of the moon's relative place, of the relative positions of the stars themselves, and also (if observations are made at distant but well determined places) of the parallax. Prof. PEIRCE has arranged the formulæ of computation, with a view to this special application, for the use of the U. S. Coast Survey. See Proceedings of the American Association for the Adv. of Science, 9th meeting, p. 97.

344. When an isolated observation of either an immersion or an emersion is to be computed, with no corresponding observations at other places, it will not be necessary to compute the values of x and y for a number of hours. It will be sufficient to compute them for the time $t + \omega$ (t being the observed local mean time, and ω the assumed longitude); and, as the correction of this time will always be small, the hourly changes may be found with sufficient precision by the approximate formulæ, easily deduced from (482),

$$x' = \frac{d\alpha}{\pi}\cos\delta \qquad\qquad y' = \frac{d\delta}{\pi}$$

where $d\alpha$ and $d\delta$ denote the hourly increase of α and δ respectively.

345. *To predict an occultation of a given star by the moon for a given place on the earth.*—We here suppose that it is already known that the star is to be occulted at the given place on a certain date, and that we wish to determine approximately the time of immersion and emersion in order to be prepared to observe it. The limiting parallels of latitude between which the occultation can be observed will be determined in the next article.

For a precise computation we proceed by Art. 322, making the modifications indicated in Art. 340.

But, for a sufficient approximation in preparing for the observation, the process may be abridged by assuming that the moon's right ascension and declination vary uniformly during the time of occultation, and neglecting the small variation of the parallax. It is then no longer necessary to compute the co-ordinates x and y directly for several different times at the first meridian, but only for any one assumed time, and then to deduce their values for any other time by means of their uniform changes. It will be most simple to find them for the time of true conjunction of the moon and star in right ascension, which is readily obtained by the aid of the hourly Ephemeris of the moon. Let this time be denoted by T_0. We have at this time $x = 0$, and the value of y will be found with sufficient accuracy by the formula

$$y_0 = \frac{\delta - \delta'}{\pi}$$

in which δ, π, are the moon's declination and horizontal parallax at the time T_0, and δ' is the star's declination.

Let $\Delta\alpha$ (in seconds of arc) and $\Delta\delta$ here denote the hourly changes of the moon's right ascension and declination for the time T_0. Then we have, nearly,

$$x' = \frac{\Delta\alpha}{\pi}\cos\delta \qquad\qquad y' = \frac{\Delta\delta}{\pi}$$

Let T_1 be any assumed time (which, in a first approximation, may be the time T_0 itself). Then the values of the co-ordinates at this time are

$$x = x'(T_1 - T_0) \qquad\qquad y = y_0 + y'(T_1 - T_0)$$

and to find the time (T) of contact of the star and the moon's limb, we shall, according to Art. 322, have the following formulæ:

$$\vartheta = \mu_1 - \alpha' - \omega$$

in which μ_1 is the sidereal time at the first meridian corresponding to T_1, α' is the star's right ascension, and ω is the longitude·

$$A \sin B = \rho \sin \varphi' \qquad \xi = \rho \cos \varphi' \sin \vartheta$$
$$A \cos B = \rho \cos \varphi' \cos \vartheta \qquad \eta = A \sin (B - \delta')$$

$$\mu' = 54148 \sin 1'' \qquad \xi' = \mu' A \cos B$$
$$\log \mu' = 9.41916 \qquad \eta' = \mu' \xi \sin \delta'$$

$$m \sin M = x - \xi \qquad n \sin N = x' - \xi'$$
$$m \cos M = y - \eta \qquad n \cos N = y' - \eta'$$

$$\sin \psi = \frac{m \sin (M - N)}{k} \qquad \log k = 9.43500$$

$$\tau = \frac{k \cos \psi}{n} - \frac{m \cos (M - N)}{n}$$

$$T = T_1 + \tau$$

where ψ is to be taken so that $\cos \psi$ shall be negative for immersion and positive for emersion.

For a second approximation, we take T as the assumed time T_1 and repeat the computation for immersion and emersion separately. The new value of ϑ for this second approximation will be most readily found by adding the sidereal equivalent of τ (converted into arc) to its former value.

It is more especially desirable to know the true time of emersion, and the angle of position of the point of reappearance of the star. Since this angle in solar eclipses was reckoned on the sun's limb, while here it must be reckoned on the moon's, it will be equal to $180 + Q$: so that, taking the value of ψ from the last approximation, we shall have

$$\left.\begin{array}{l}\text{Angle of pt. of contact from the}\\ \text{north pt. of the moon's limb}\end{array}\right\} = 180^\circ + N + \psi$$

For the angle from the *vertex* of the moon's limb, we find γ by the equations

$$p \sin \gamma = \xi + \xi'\tau \qquad p \cos \gamma = \eta + \eta'\tau$$

where ξ, η, ξ', η', τ are to be taken from the last approximation; and then

$$\left.\begin{array}{l}\text{Angle of pt. of contact from}\\ \text{the vertex of the moon's limb}\end{array}\right\} = 180^\circ + N + \psi - \gamma$$

If the computation in any case gives $m \sin(M - N) > k$, we have the impossible value $\sin\psi > 1$, which shows that the star is not occulted at the given place. If we wish to know how far the star is from the moon's limb at the time of nearest approach, we have (Art. 324)

$$\varDelta = \pm\, m \sin(M - N)$$

the sign being taken so that $\varDelta$ shall be positive. This is the linear distance of the moon's centre from the line drawn from the place of observation to the star, and therefore the angular distance as seen from the earth is $\pi\varDelta$. The apparent semidiameter of the moon is πk, and hence the apparent distance of the star from the moon's limb is $\pi(\varDelta - k)$.*

EXAMPLE.—Find the times of immersion and emersion in the occultation of *Aldebaran*, April 15, 1850, at Cambridge, Mass.

The elements of this occultation have been found on p. 553, with which an accurate computation may be made by the method of Art. 322; but, according to the preceding approximate method, we proceed as follows. The Greenwich time when the moon's right ascension was equal to that of the star is found, from the values of α on p. 553, to be

$$T_0 = 7^h.47 = 7^h\ 28^m\ 12^s.$$

For this time we have

$$\begin{array}{rlrl} \Delta\alpha = & +\ 2174'' & \delta = & +\ 16^\circ\ 49'\ 31''.1 \\ \Delta\delta = & +\ \ \ 384 & \delta' = & \ \ \ 16\ \ \ 12\ \ \ 1\ .7 \\ \pi = & \ \ \ \ 3536 & \delta - \delta' = & +\ \ \ \ 2249'' \end{array}$$

whence, by the above formulæ,

$$y_0 = +\ 0.6360 \qquad x' = +\ 0.5886 \qquad y' = +\ 0.1086$$

Then the computation for Cambridge, $\varphi = 42^\circ\ 22'\ 49''$, $\omega = 4^h\ 44^m\ 30^s$, will be as follows. For the first approximation, we assume $T_1 = T_0$, and hence we have

* More exactly, allowing for the augmentation of the moon's semidiameter, it is $\pi(\varDelta - k)(1 + \zeta \sin\pi)$, where we have $\zeta = A \cos(B - \delta')$.

$$
\begin{aligned}
T_1 &= 7^h\ 28^m\ 12^s. \\
\text{Sid. time Gr. noon} &= 1\ \ 33\ \ \ 9\ 0 \\
\text{Reduction for } T_1 &= \quad\ \ 1\ \ 13\ .6 \\
\mu_1 &= 9\ \ \ 2\ \ 34\ .6 \\
\alpha' &= 4\ \ 27\ \ 18\ .3 \\
\omega &= 4\ \ 44\ \ 30 \\
\mu_1 - \alpha' - \omega = \vartheta &= 23\ \ 50\ \ 46\ .3 \\
&= 357^\circ\ 41'.6
\end{aligned}
$$

with which we find the following results:

$$
\begin{aligned}
x &= 0. & y &= +\ 0.6360 \\
\xi &= -\ 0.0298 & \eta &= +\ 0.4377 \\
m \sin M &= +\ 0.0298 & m \cos M &= +\ 0.1983 \\
M &= 8^\circ\ 32'.5 & \log m &= 9.3021 \\
x' &= +\ 0.5886 & y' &= +\ 0.1086 \\
\xi' &= +\ 0.1940 & \eta' &= -\ 0.0022 \\
n \sin N &= +\ 0.3946 & n \cos N &= +\ 0.1108 \\
N &= 74^\circ\ 19'.1 & \log n &= 9.6127 \\
\log \sin \psi &= n9.8395 & \log \cos \psi &= 9.8590 \\
-\frac{m \cos (M - N)}{n} &= -\ 0^h.1690 & \frac{k \cos \psi}{n} &= \mp\ 0^h.4801
\end{aligned}
$$

For immersion.		For emersion.	
$\tau =$	$-\ 0^h.6491$	$\tau =$	$+\ 0^h.3111$
$T_1 =$	$7\ .4700$	$T_1 =$	$7\ .4700$
$T =$	$6\ .8209$	$T =$	$7\ .7811$
$T =$	$6^h\ 49^m\ 15^s$	$T =$	$7^h\ 46^m\ 52^s$
$\omega =$	$4\ \ 44\ \ 30$	$\omega =$	$4\ \ 44\ \ 30$
Local time $=$	$2\ \ \ 4\ \ 45$	Local time $=$	$3\ \ \ 2\ \ 22$

These times are nearly correct enough; but, for a more accurate time of emersion, we now assume $T_1 = 7^h.7811$, with which we find

$$
x = x'(T_1 - T_0) = +\ 0.1831 \qquad
\begin{aligned}
y'(T_1 - T_0) &= +\ 0.0338 \\
y_0 &= +\ 0.6360 \\
y &= +\ 0.6698
\end{aligned}
$$

and to find the new value of ϑ we have $\tau = +\ 0^h.3111 = 18^m\ 40^s$, the sidereal equivalent of which is $18^m\ 43^s.1$, or in arc $4^\circ\ 40'.8$. This, added to the above value of ϑ, gives the corrected value $\vartheta = 2^\circ\ 22'.4$. Repeating the computation with these new values of x, y, and ϑ, we find

$$
\begin{aligned}
-\frac{m\cos(M-N)}{n} &= -0^{h}.5082 \\
\frac{k\cos\psi}{n} &= +0\,.4901 \\
\tau &= -0\,.0181 \\
T_1 &= 7\,.7811 \\
T &= 7\,.7630 \\
&= 7^{h}\,45^{m}\,47^{s} \\
\text{Local time} &= 3\;\;\;1\;\;17
\end{aligned}
\qquad\qquad
\begin{aligned}
\psi &= 317^\circ\;22' \\
N &= 74\;\;\;55 \\
&\;180 \\
&\;212\;\;\;17 \\
\gamma &= 3\;\;\;33 \\
&\;208\;\;\;34
\end{aligned}
$$

The star reappears at 212° 17′ from the north point, or 208° 34′ from the vertex, of the moon's limb.

This time agrees within 21^s with the actually observed time of emersion (given on p. 552). The principal part of the difference is due to the error of the Ephemeris on this date.

346. *To find the limiting parallels of latitude on the earth for a given occultation.*—The limiting curves within which the occultation of a given star is visible may be found by the general method given for solar eclipses, Art. 311, which, of course, may be much abridged in such an application. But, on account of the great number of stars which may be occulted, it is not possible to make even this abridged computation for all of them. The extreme parallels of latitude are, however, found by very simple formulæ, and may be used for each star.

For a point on the limiting curve, the least value of Δ in Art. 324 is in a solar eclipse $= L$, but in an occultation it is $= k$. Hence we have, by (557), the condition

$$\pm\, m\sin(M-N) = k$$

or, restoring the values of $m\sin M = x - \xi$, $m\cos M = y - \eta$,

$$(x-\xi)\cos N - (y-\eta)\sin N = \pm\, k$$

The angle N is here determined by the equations (552); but, for an approximate determination of the limits quite sufficient for our present purpose, we may neglect the changes of ξ and η, and take

$$n\sin N = x' \qquad\qquad n\cos N = y'$$

Let x_0 and y_0 be the values of x and y for the assumed epoch T_0; then for any time $T = T_0 + \tau$ we have

$$x = x_0 + n\sin N\,.\,\tau \qquad\qquad y = y_0 + n\cos N\,.\,\tau$$

which reduce the above condition to

$$(x_0 - \xi) \cos N - (y_0 - \eta) \sin N = \pm k$$

By the last equation of (500), we have, by neglecting the compression of the earth,

$$\sin \varphi = \eta \cos \delta' + \zeta \sin \delta'$$

in which

$$\zeta = \sqrt{(1 - \xi^2 - \eta^2)}$$

and we are now to determine the maximum and minimum values of φ, which fulfil these conditions. Let us put

$$\begin{aligned} a &= - \xi \cos N + \eta \sin N \\ b &= \quad \xi \sin N + \eta \cos N \end{aligned}$$

from which follow

$$\begin{aligned} \xi &= - a \cos N + b \sin N \\ \eta &= \quad a \sin N + b \cos N \\ \zeta &= \quad \sqrt{(1 - a^2 - b^2)} \end{aligned}$$

Then we also have, by our first condition,

$$a = - x_0 \cos N + y_0 \sin N \pm k$$

which is a constant quantity, since we may here assume x' and y' to be constant.

Since we have $a^2 + b^2 + \zeta^2 = 1$, we can assume γ and ε so as to satisfy the equations

$$\begin{aligned} \cos \gamma &= a \\ \sin \gamma \cos \varepsilon &= b \\ \sin \gamma \sin \varepsilon &= \zeta \end{aligned}$$

in which $\sin \gamma$ may be restricted to positive values. The formula for φ thus becomes

$$\sin \varphi = \cos \gamma \sin N \cos \delta' + \sin \gamma \cos \varepsilon \cos N \cos \delta' + \sin \gamma \sin \varepsilon \sin \delta'$$

which may be put under a more simple form by assuming β and λ, so as to satisfy the conditions

$$\begin{aligned} \sin \beta &= \sin N \cos \delta' \\ \cos \beta \cos \lambda &= \cos N \cos \delta' \\ \cos \beta \sin \lambda &= \sin \delta' \end{aligned}$$

in which $\cos \beta$ may be restricted to positive values.

We thus obtain

$$\sin \varphi = \sin \beta \cos \gamma + \cos \beta \sin \gamma \cos (\lambda - \varepsilon)$$

in which φ and ε are the only variables. Since $\cos \beta \sin \gamma$ is positive, this value of $\sin \varphi$ is a maximum when $\cos (\lambda - \varepsilon) = 1$ or $\lambda - \varepsilon = 0$; and a minimum when $\cos (\lambda - \varepsilon) = -1$, or $\lambda - \varepsilon = 180°$. Hence we have, for the limits, $\sin \varphi = \sin (\beta \pm \gamma)$, that is

$$\text{for the northern limit,} \quad \varphi = \beta + \gamma$$
$$\text{for the southern limit,} \quad \varphi = \beta - \gamma$$

One of the points thus determined may, however, be upon that side of the earth which is farthest from the moon, since we have not restricted the sign of ζ, and our general equations express the condition that the point of observation lies in a line drawn from the star tangent to the moon's limb, which line intersects the surface of the earth in two points, for one of which ζ is positive and for the other negative. But, taking ζ only with the positive sign, we must also have $\sin \varepsilon$ positive. For the northern limit, therefore, when $\lambda = \varepsilon$, $\sin \lambda$ must be positive, which, according to the equation $\cos \beta \sin \lambda = \sin \delta'$, can be the case only when δ' is positive. Hence the formula $\varphi = \beta + \gamma$ gives the most northern limit of visibility only when the star is in north declination. For similar reasons, the formula $\varphi = \beta - \gamma$ gives the southern limit only when the star is in south declination. The second limit of visibility in each case must evidently be one of the points in which the general northern or southern limiting curve meets the rising and setting limits,—namely, the points where $\zeta = 0$, and consequently, also, $\sin \varepsilon = 0$, $\cos \varepsilon = \pm 1$, which conditions reduce the general formula for $\sin \varphi$ to the following:

$$\sin \varphi = (\sin N \cos \gamma \pm \cos N \sin \gamma) \cos \delta' = \sin (N \pm \gamma) \cos \delta'$$

If $\cos N$ is taken with the positive sign only, the upper sign in this equation will give the most northern limit to be used when the southern limit has been found by the formula $\varphi = \beta - \gamma$; and the lower sign will give the southern limit to be used when the northern limit has been found by the formula $\varphi = \beta + \gamma$.

Finally, since the epoch T_0 is arbitrary, we may assume for it the time of true conjunction in right ascension when $x_0 = 0$, and we shall then have

$$a = \cos \gamma = y_0 \sin N \pm k$$

The above discussion leads to the following simple arrangement of the formulæ

$$\left.\begin{array}{lr} \cos \gamma_1 = y_0 \sin N \pm 0.2723 & (\gamma < 180°) \\ \sin \beta = \sin N \cos \delta' & (\beta < \ \ 90°) \\ \quad \varphi_1 = \beta \pm \gamma_1 & \\ \cos \gamma_2 = y_0 \sin N \mp 0.2723 & \\ \sin \varphi_2 = \sin (N \mp \gamma_2) \cos \delta' & (N < \ \ 90°) \end{array}\right\} \quad (604)$$

in which the upper or the lower signs are to be used, according as the declination of the star is north or south. When the declination is north, φ_1 will be the northern limit and φ_2 the southern; and the reverse when the declination is south. The angle N is here supposed to be less than 90°, and is found by the formula

$$\tan N = \frac{x'}{y'}$$

always considering y' as well as x' to be positive.

When the cylindrical shadow extends beyond the earth, north or south, we shall obtain imaginary values for γ_1 or γ_2. The following obvious precepts must then be observed:

1st. When $\cos \gamma_1$ is imaginary, the occultation is visible beyond the pole which is elevated above the principal plane of reference, and, therefore, we must put for the extreme limit $\varphi_1 = +\ 90°$, or $\varphi_1 = -\ 90°$, according to the sign of δ'.

2d. When $\cos \gamma_2$ is imaginary, the value of φ_2 will be the latitude of that point of the (great circle) intersection of the principal plane and the earth's surface which lies nearest the depressed pole; that is, we must take $\varphi_2 = \delta' - 90°$, or $\varphi_2 = \delta' + 90°$, according as δ' is positive or negative.

It is also to be observed that the numerical value of φ_1 obtained by the formula $\varphi_1 = \beta \pm \gamma_1$ may exceed 90°, in which case the true value is either $\varphi_1 = 180° - (\beta \pm \gamma_1)$, or $\varphi_1 = -\ 180° - (\beta \pm \gamma_1)$, since these values have the same sine.

Example.—Find the limiting parallels of latitude for the occultation of *Aldebaran*, April 15, 1850.

We have found, page 559, for this occultation,

$$y_0 = +\ 0.6360 \qquad x' = 0.5886 \qquad y' = 0.1086$$

Hence, with $\delta' = 16°\ 12'$, we find

$N =$	$79°\ 33'$	$\log \sin \beta =$	9.9751
$y_0 \sin N =$	$+\ 0.6255$	$\beta =$	$70°\ 47'$
$k =$	0.2723	$\gamma_1 =$	$26 \quad 8$
$\cos \gamma_1 =$	$+\ 0.8978$	$\beta + \gamma_1 =$	$96 \quad 55$
$\cos \gamma_2 =$	$+\ 0.3532$	$\varphi_1 =$	$83 \quad 5$
$\gamma_2 =$	$69°\ 19'$		
$N - \gamma_2 =$	$10 \quad 14$	$\varphi_2 =$	$9 \quad 49$

It is hardly necessary to observe that the occultation is not visible at all the places included between the extreme latitudes thus found, since the true limiting curves do not coincide with the parallels of latitude, but cut the meridians at various angles, as is illustrated by the southern limit in our diagram of a solar eclipse, p. 504. Unless a place is considerably within the assigned limits, it may, therefore, be necessary in many cases to make a special computation, by the method of Art. 345, to determine whether the occultation can be observed.

OCCULTATIONS OF PLANETS BY THE MOON.

347. If the disc of a planet were always a circle, and fully illuminated, its occultation by the moon might be computed by the general method used for solar eclipses by merely substituting the parallax and semidiameter of the planet for those of the sun; and this is the method which has generally been prescribed by writers on this subject. But with the telescopes now in use, and especially with the aid of the electro-chronograph, it is possible to observe the instants of contact with the planet's limb to such a degree of accuracy that it appears to be worth while to take into account the true figure of the visible illuminated portion of the planet. Moreover, the investigation of this true figure possesses an intrinsic interest which justifies entering upon it here somewhat at length.

In order to embrace at once all cases, I shall consider the planet as a spheroidal body which even when fully illuminated presents an elliptical outline, and when partially illuminated presents an outline composed of two ellipses, of which one is the boundary of the spheroid and the other is the limit of illumination on the side of the planet towards the observer. I begin with the determination of the first of these ellipses.

348. *To find the apparent form of the disc of a spheroidal planet.**—Let us first express the apparent place of any point of the surface of the planet, by referring it to three planes perpendicular to each other, of which the plane of xy coincides with the plane of the planet's equator, while the axis of z coincides with the axis of rotation. In this system, let

$x, y, z =$ the co-ordinates of the point on the surface of the planet,
$\xi, \eta, \zeta =$ those of the observer.

Straight lines drawn from the observer to the centre of the planet and to the point on its surface determine their apparent places on the celestial sphere. If these places are referred to the great circle which corresponds to the planet's equator, and if we put

$\lambda, \lambda' =$ the geocentric longitudes of the apparent places of the planet's centre and the point on its surface, reckoned from the axis of x, in the great circle of the planet's equator,
$\beta, \beta' =$ the latitudes of these places referred to the great circle of the planet's equator,
$\rho, \rho' =$ the distances of the centre of the planet and the point on its surface from the observer,

we shall have (Arts. 32 and 33)†

$$\left.\begin{aligned} \rho \cos\beta \cos\lambda &= -\xi \\ \rho \cos\beta \sin\lambda &= -\eta \\ \rho \sin\beta &= -\zeta \end{aligned}\right\} \qquad (605)$$

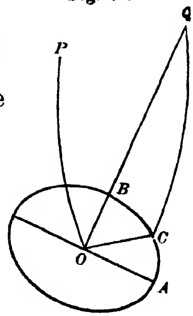
$$\left.\begin{aligned} \rho' \cos\beta' \cos\lambda' &= x - \xi \\ \rho' \cos\beta' \sin\lambda' &= y - \eta \\ \rho' \sin\beta' &= z - \zeta \end{aligned}\right\} \qquad (606)$$

* The method of investigation here adopted, so far as relates to the apparent form of the disc, is chiefly derived from Bessel, *Astronomische Untersuchungen*, Vol. I. Art. VI.

† The group (606) may be deduced by supposing for a moment that the position of the observer is referred to a system of planes parallel to the first, but having its origin at the point on the surface of the planet. The co-ordinates in this system are equal to those in the first increased respectively by x, y, and z. The negative sign in the second members of both groups results from the consideration that the longitude of the observer as seen from the planet is $180° + \lambda$, or $180° + \lambda'$; and his latitude, $-\beta$, or $-\beta'$. Compare Art. 98.

Now, let O and C, Fig. 47, be the apparent places of the planet's centre and the point on its surface, projected upon the celestial sphere; Q the pole of the planet's equator; P the pole of the earth's equator; and let

Fig. 47.



$s' =$ the apparent distance of C from $O =$ the arc OC,

$p' =$ the position angle of C reckoned at O, from the declination circle OP towards the east, $= POC$,

$p =$ the position angle of the pole of the planet $= POQ$;

then, in the triangle QOC, we have

$$\begin{aligned}\sin s' \sin (p' - p) &= \cos \beta' \sin (\lambda' - \lambda)\\ \sin s' \cos (p' - p) &= \cos \beta \sin \beta' - \sin \beta \cos \beta' \cos (\lambda' - \lambda)\end{aligned}$$

Multiplying these by ρ', and substituting the expressions (605) and (606), we obtain

$$\begin{aligned}\rho' \sin s' \sin (p' - p) &= - x \sin \lambda + y \cos \lambda\\ \rho' \sin s' \cos (p' - p) &= - x \sin \beta \cos \lambda - y \sin \beta \sin \lambda + z \cos \beta\end{aligned}$$

or, since s' is very small and $\rho' \sin s'$ or $\rho' s'$ differs insensibly from $\rho \sin s'$ or $\rho s'$,

$$\left.\begin{aligned}\rho s' \sin (p' - p) &= - x \sin \lambda + y \cos \lambda\\ \rho s' \cos (p' - p) &= - x \sin \beta \cos \lambda - y \sin \beta \sin \lambda + z \cos \beta\end{aligned}\right\} \quad (607)$$

These equations apply to any point on the surface of the planet. If we apply them to those points in which the visual line of the observer is *tangent* to that surface, they will determine the curve which forms the apparent disc. The equation of an ellipsoid of revolution whose axes are a and b, of which b is the axis of revolution, is

$$1 = \frac{xx}{aa} + \frac{yy}{aa} + \frac{zz}{bb} \quad (608)$$

and the equation of a tangent line passing through the point whose co-ordinates are ξ, η, and ζ is

$$1 = \frac{x\xi}{aa} + \frac{y\eta}{aa} + \frac{z\zeta}{bb} \quad (609)$$

The distances ξ, η, and ζ are very great in comparison with x,

y, and z. If we divide (609) by ρ, the quotients $\frac{\xi}{\rho}, \frac{\eta}{\rho}, \frac{\zeta}{\rho}$ will be of the same order as $\frac{x}{a}, \frac{y}{a}, \frac{z}{b}$, but the quotient $\frac{1}{\rho}$ will be inappreciable in relation to the quotients $\frac{x}{aa}, \frac{y}{aa}, \frac{z}{bb}$. Performing this division, therefore, and substituting the values of ξ, η, and ζ from (605), we may write for the equation of the tangent line

$$0 = \frac{x \cos\beta \cos\lambda}{aa} + \frac{y \cos\beta \sin\lambda}{aa} + \frac{z \sin\beta}{bb} \tag{610}$$

If the curve ACB, Fig. 47, is referred to rectangular axes passing through the apparent centre O of the planet, one of which is in the direction of the pole of the planet, and if u and v denote the co-ordinates of any point of the curve, so that

$$\begin{aligned} u &= s' \sin(p' - p) \\ v &= s' \cos(p' - p) \end{aligned}$$

the equations (607) and (610) will enable us to determine x, y, and z in terms of u and v. Putting

$$\frac{bb}{aa} = 1 - ee$$

the three equations become

$$\begin{aligned} \rho u &= -\ x \sin\lambda + y \cos\lambda \\ \rho v &= -(x \cos\lambda + y \sin\lambda) \sin\beta + z \cos\beta \\ 0 &= (x \cos\lambda + y \sin\lambda)(1 - ee) \cos\beta + z \sin\beta \end{aligned}$$

from which we derive

$$\begin{aligned} -x \sin\lambda + y \cos\lambda &= \rho u \\ -x \cos\lambda - y \sin\lambda &= \rho v \frac{\sin\beta}{1 - ee\cos^2\beta} \\ z &= \rho v \frac{(1 - ee)\cos\beta}{1 - ee\cos^2\beta} \end{aligned}$$

Substituting these values in (608) and putting

$s = \frac{a}{\rho}$ = the greatest apparent semidiameter of the planet,

$$c = \sqrt{(1 - ee\cos^2\beta)}$$

we find

$$ss = uu + \frac{vv}{cc} \tag{611}$$

which is the equation of the outline of the planet as projected upon the celestial sphere, or upon a plane passed through the centre of the planet at right angles to the line of vision. It represents an ellipse whose axes are $2s$ and $2s\sqrt{(1 - ee\cos^2\beta)}$, e being the eccentricity of the planet's meridians. The minor axis (OB, Fig. 47) lies in the direction of the great circle drawn to the pole of the planet's equator.

We next proceed to determine what portion of this ellipse is illuminated and visible from the earth.

349. *To find the apparent curve of illumination of a planet's surface.*—If the sun be regarded as a point (which will produce no sensible error in this problem), the curve of illumination of the planet, *as seen from the sun*, can be determined by conditions quite similar to those employed in the preceding problem; for we have only to substitute the co-ordinates expressing the sun's position with reference to the planet, instead of those of the observer. If, therefore, we put

A, B = the heliocentric longitude and latitude of the centre of the planet referred to the great circle of the planet's equator,

the equation of the tangent line from the sun to the planet, being of the same form as (610), will be

$$0 = \frac{x\cos B\cos A}{aa} + \frac{y\cos B\sin A}{aa} + \frac{z\sin B}{bb} \tag{612}$$

If each point which satisfies this condition be projected upon the celestial sphere by a line from the observer on the earth, and u and v again denote the co-ordinates of the projected curve, we have here, also, to satisfy the equations

$$\left.\begin{aligned} \rho u &= -x\sin\lambda + y\cos\lambda \\ \rho v &= -(x\cos\lambda + y\sin\lambda)\sin\beta + z\cos\beta \end{aligned}\right\} \tag{613}$$

in which λ and β have the same signification as in the preceding article. The values of x, y, and z, determined by the three equations (612), (613), being substituted in the equation of the ellipsoid, we obtain the relation between u and v, or the equation

of the required curve of illumination as seen from the earth. In order to facilitate the substitution, let us put

$$\begin{aligned} x_1 &= -x \sin\lambda + y\cos\lambda \\ y_1 &= x\cos\lambda + y\sin\lambda \end{aligned}$$

from which follow

$$\begin{aligned} x &= -x_1 \sin\lambda + y_1\cos\lambda \\ y &= x_1\cos\lambda + y_1\sin\lambda \end{aligned}$$

At the same time, let us introduce the auxiliaries β_1 and B_1 dependent upon β and B by the assumed relations

$$\left.\begin{aligned} \frac{1}{g}\cos\beta_1 &= \cos\beta & \qquad \frac{1}{G}\cos B_1 &= \cos B \\ \frac{1}{g}\sin\beta_1 &= \frac{a}{b}\sin\beta & \qquad \frac{1}{G}\sin B_1 &= \frac{a}{b}\sin B \end{aligned}\right\} \quad (614)$$

Then the three equations become

$$\begin{aligned} 0 &= x_1\cos B_1\sin(A-\lambda) + y_1\cos B_1\cos(A-\lambda) + \frac{a}{b}z\sin B_1 \\ \rho u &= x_1 \\ \frac{a}{b}g\rho v &= -y_1\sin\beta_1 + \frac{a}{b}z\cos\beta_1 \end{aligned}$$

from which we derive

$$\begin{aligned} x_1 &= \rho u \\ N y_1 &= -\rho u\cos\beta_1\cos B_1\sin(A-\lambda) - \frac{a}{b}g\rho v\sin B_1 \\ N\frac{a}{b}z &= -\rho u\sin\beta_1\cos B_1\sin(A-\lambda) + \frac{a}{b}g\rho v\cos B_1\cos(A-\lambda) \end{aligned}$$

where, for brevity, N is put for $\sin\beta_1\sin B_1 + \cos\beta_1\cos B_1\cos(A-\lambda)$.

Before substituting these expressions in the equation of the ellipsoid, it will be well to consider the geometrical signification of the quantities β_1 and B_1. If we draw straight lines from the centre of the planet to the earth and to the sun, the latitudes of the points in which these lines intersect the surface of the planet will be β and B. If these points be projected upon the surface of a sphere circumscribed about the ellipsoid, by perpendiculars to its equator, the latitudes of the projected points will be β_1 and B_1; and g and G will be the corresponding radii of the ellipsoid. If now these projected points are referred to the celestial sphere, by lines from the planet's centre, they will form with the pole Q of the planet's equator a spherical triangle QOS, in which the

angle Q will be $A - \lambda$; and the sides including this angle will be $90° - \beta_1 = QO$, $90° - B_1 = QS$. Denoting the angle at O by w, and the side OS by V, we shall have

$$\left.\begin{aligned} \cos V &= \sin \beta_1 \sin B_1 + \cos \beta_1 \cos B_1 \cos (A - \lambda) \\ \sin V \cos w &= \cos \beta_1 \sin B_1 - \sin \beta_1 \cos B_1 \cos (A - \lambda) \\ \sin V \sin w &= \cos B_1 \sin (A - \lambda) \end{aligned}\right\} \quad (615)$$

in which V is very nearly the angular distance between the sun and the earth as seen from the planet.

This triangle also gives

$$\begin{aligned} \sin B_1 &= \cos V \sin \beta_1 + \sin V \cos \beta_1 \cos w \\ \cos B_1 \cos (A - \lambda) &= \cos V \cos \beta_1 - \sin V \sin \beta_1 \cos w \\ \cos B_1 \sin (A - \lambda) &= \sin V \sin w \end{aligned}$$

By these equations the above expressions for x_1, y_1, and z are reduced to

$$\begin{aligned} \cos V . x_1 &= \quad \rho u \cos V \\ \cos V . y_1 &= -\rho u \sin V \sin w \cos \beta_1 \\ &\quad - \frac{a}{b} g \rho v (\cos V \sin \beta_1 + \sin V \cos \beta_1 \cos w) \\ \cos V \cdot \frac{a}{b} z &= -\rho u \sin V \sin w \sin \beta_1 \\ &\quad + \frac{a}{b} g \rho v (\cos V \cos \beta_1 - \sin V \sin \beta_1 \cos w) \end{aligned}$$

Substituting these in (608), observing that $xx + yy = x_1x_1 + y_1y_1$, we have

$$\begin{aligned} \cos^2 V \cdot \frac{aa}{\rho\rho} &= uu \cos^2 V \\ &\quad + \left[(u \sin w + \frac{a}{b} gv \cos w) \sin V \cos \beta_1 + \frac{a}{b} gv \cos V \sin \beta_1\right]^2 \\ &\quad + \left[(u \sin w + \frac{a}{b} gv \cos w) \sin V \sin \beta_1 - \frac{a}{b} gv \cos V \cos \beta_1\right]^2 \end{aligned}$$

Developing the squares in the second member, and putting s for $\frac{a}{\rho}$, and also

$$c = \sqrt{(1 - ee \cos^2 \beta)} = \frac{b}{ag}$$

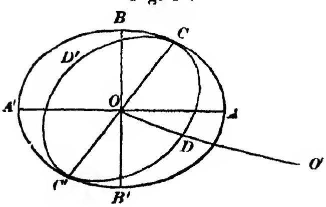
we shall find

$$ss = \left(u \cos w - v \cdot \frac{\sin w}{c}\right)^2 + \left(u \sin w + v \cdot \frac{\cos w}{c}\right)^2 \sec^2 V \quad (616)$$

which is the required equation of the curve of illumination, as seen from the earth, projected upon the celestial sphere. It represents an ellipse whose centre is at the origin but whose axes are, in general, inclined to the axes of co-ordinates, and, consequently, to the axes of the ellipse of equation (611). The equation (611) is only the particular case of (616) which corresponds to $V = 0$, or the case of full illumination.

350. We have yet to determine what portions of the apparent disc are bounded by the two curves respectively. If $ABA'B'$, Fig. 48, is the ellipse of (611), which I shall call the *first ellipse*, and $CDC'D'$ that of (616), which I shall distinguish as the *second ellipse*, the visible outline of the planet is composed of one-half the first and one-half the second curve, and these halves either begin or end at the points C and C', which are the common points of tangency of the two curves. These points satisfy both equations; and, therefore, putting u_1 and v_1 for the co-ordinates of either point, and subtracting (611) from (616), we find

Fig. 48.



$$0 = \left(u_1 \sin w + v_1 \frac{\cos w}{c} \right)^2 \tan^2 V$$

which is satisfied, in general, by taking

$$u_1 \sin w + v_1 \frac{\cos w}{c} = 0$$

Denoting the position angle corresponding to u_1 v_1, by p_1, we have $u_1 = s_1 \sin (p_1 - p)$, $v_1 = s_1 \cos (p_1 - p)$. Substituting these values, and also putting

$$c_1 \sin w_1 = \sin w \qquad\qquad c_1 \cos w_1 = \frac{\cos w}{c} \tag{617}$$

the preceding condition becomes

$$c_1 s_1 \cos (p_1 - p - w_1) = 0$$

whence

$$p_1 = p + w_1 \mp 90° \tag{618}$$

which expresses the position angles of both C and C'. If we draw the arc ODO', Fig. 48, making the angle $BOO' = w$, and

take $OO' = V$, the point O' will be nearly the position of the planet as seen from the sun, and the arc V will be the measure of the angular distance between the sun and the earth as viewed from the planet. If we assume $\sin w$ to be positive in equations (615), as we are at liberty to do, the arc V will be reckoned from the planet *eastward* from 0° to 360°. Now, so long as V is less than 180°, the west limb will evidently be the full limb, and when V is greater than 180°, the east limb will be the full limb. Hence we infer that a point whose given position angle is p' is on the east limb when

$$p' > p + w_1 - 90° \quad \text{and} \quad < p + w_1 + 90°$$

but on the west limb when

$$p' < p + w_1 - 90° \quad \text{and} \quad > p + w_1 + 90°$$

When $V > 90°$ and $< 270°$, the planet is crescent; but when $V > 270°$ and $< 90°$, it is gibbous. In the case of a crescent planet there are two points, one on the full and the other on the crescent limb, corresponding to the same position angle: hence in observations of a crescent planet the point of observation on the limb will not be sufficiently determined by the position angle alone; it will be necessary for the observer to distinguish the crescent from the full limb in his record.

351. In order to apply the preceding theory, it is necessary to find the quantities p, β, λ, B, Λ. The direction of the axis of x in Art. 348 was left indeterminate, and may be assumed at pleasure, but it is most convenient to let it pass through the ascending node of the planet's equator on the equinoctial, so that λ and Λ will be reckoned from this node. The position of the node must, therefore, be known, and this we derive from the researches of physical astronomers. If we put

n = the longitude of the ascending node of the planet's equator on the equinoctial,
i = the inclination of the planet's equator to the equinoctial,

we have at any given time t, for the planets Jupiter and Saturn, the only ones whose figures are sensibly spheroidal,

$$\text{For Jupiter.}\quad \begin{cases} n = 357^\circ\, 56'\, 25'' + 3''.59\,(t - 1850) \\ i = 25^\circ\, 25'\, 49'' + 0''.66\,(t - 1850) \end{cases}$$

$$\text{For Saturn.}\quad \begin{cases} n = 125^\circ 13'\, 54'' + 128''.76\,(t - 1850) + 0''.0605\,(t - 1850)^2 \\ i = 7^\circ 10'\, 10'' - 15''.08\,(t - 1850) + 0''.0035\,(t - 1850)^2 \end{cases}$$

in which t is expressed in years.*

The values for Saturn apply either to its equator or the rings, which are sensibly in the same plane.

If now we put

α', δ' = the right ascension and declination of the planet,

we can convert α' and δ' into λ and β by Art. 23; we shall merely have to substitute in (29) or (31) $\alpha' - n$ for α, δ' for δ, and i for ε. The angle p is here the position angle of the pole of the planet reckoned from the declination circle of the planet towards the *east;* but in Art. 25 the angle η is the position angle reckoned towards the west, and, therefore, we shall have to put $\eta = 360^\circ - p$ in (33). Hence we obtain the following formulæ for β, λ, and p:

$$\left.\begin{aligned} f \sin F &= \tan \delta' & f' \sin \lambda &= \cos (F - i) \\ f \cos F &= \sin (\alpha' - n) & f' \cos \lambda &= \cos F \cot (\alpha' - n) \\ \tan \beta &= \sin \lambda \tan (F - i) & & \\ \tan F' &= \tan i \sin (\alpha' - n) & \tan p &= -\frac{\sin F' \cot (\alpha' - n)}{\cos (F' - \delta')} \end{aligned}\right\} \quad (619)$$

To find A and B, we avail ourselves of the heliocentric longi tude and latitude of the planets given in the British Almanac, and as these quantities are referred to the ecliptic, while A and B are referred to the planet's equator, we must know the relative position of these circles. Putting

N' = the longitude of the node of the planet's equator on the ecliptic,
I' = the inclination of the planet's equator to the ecliptic,
N = the arc of the planet's equator between the equinoctial and the ecliptic,

* These values I have deduced from the data given in DAMOISEAU'S *Tables Écliptiques des Satellites de Jupiter*, Paris, 1836; and BESSEL'S *Bestimmung der Lage und Grösse des Saturns-Ringes und der Figur und Grösse des Saturns, Astronom. Nach.*, Vol. XII. p. 167.

we deduce from the data of BESSEL and DAMOISEAU, for a given year t,

$$\text{For Jupiter.}\left\{\begin{array}{l} N'=335^\circ\,40'\,46''+49''.80\,(t-1850) \\ I'=\quad 2^\circ\ \ 8'\,51''+\ \ 0''.43\,(t-1850) \\ N\ =336^\circ\,33'\,18''+46''.55\,(t-1850) \end{array}\right.$$

$$\text{For Saturn.}\left\{\begin{array}{l} N'=167^\circ\,31'\,52''+46''.62\,(t-1850) \\ I'=\ \ 28^\circ\,10'\,27''-\ \ 0''.35\,(t-1850) \\ N=\ \ 43^\circ\,31'\,34''-86''.75\,(t-1850)-0''.0625(t-1850)^2 \end{array}\right.$$

and these values for Saturn also apply to the rings.

Finally, if we put

A', B' = the heliocentric longitude and latitude of the planet, referred to the ecliptic,

the formulæ (29) or (31) will serve to convert $A'-N'$ and B' into $A-N$ and B; and they become

$$\left.\begin{array}{ll} K\sin M=\tan B' & K'\sin(A-N)=\cos(M-I') \\ K\cos M=\sin(A'-N') & K'\cos(A-N)=\cos M\cot(A'-N') \\ \multicolumn{2}{c}{\tan B=\sin(A-N)\tan(M-I')} \end{array}\right\}\quad(620)$$

352. The preceding complete theory admits of several abridgments in its application to the different planets, varying according to the features peculiar to each.

Jupiter.—The inclination of Jupiter's equator to the ecliptic is so small that the quantity $c=\sqrt{(1-ee\cos^2\beta)}$ never differs sensibly from $\sqrt{(1-ee)}$, which, according to STRUVE's measures, is 0.92723. I shall, therefore, use as a constant the value $\log c=9.9672$. Again, on account of the small inclinations both of Jupiter's equator and of his orbit to the ecliptic, the angle w never differs much from 90°, and, since this angle is required only in computing the gibbosity of the planet (which never exceeds 0″.5), it is plain that we may take $w=90^\circ$, and that V may be found with sufficient accuracy by the formula

$$V=A-\lambda$$

or, indeed, by the formula

$$V=A'-\lambda' \qquad (621)$$

in which A' and λ' are, respectively, the heliocentric and geocentric longitudes of the planet, the former being taken directly

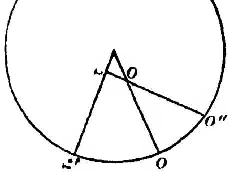
from the British Almanac, and the latter computed from the geocentric right ascension and declination by Art. 23: so that for this planet the equations (615), (619), (620) will be dispensed with, except only the last two equations of (619), which will be required in finding p.

Saturn.—The inclination of Saturn's equator to the ecliptic is over 28°, and therefore the quantity $c = \sqrt{(1 - ee \cos^2 \beta)}$ will have sensibly different values at different times. The value of β is, however, given in the table for Saturn's Ring in our Ephemerides (where it is usually denoted by l). The value of ee is 0.1865, or $\log ee = 9.2706$. The gibbosity of Saturn is altogether insensible; so that we shall have occasion to use only the equation (611), or in any formula that may be derived from the more general equation (616) we shall have to put $V = 0$. The angle p is also given in the table for the ring.

Saturn's Ring.—The ring may be here regarded as an ellipsoid of revolution whose minor axis $= 0$. Hence we have only to make $e = 1$ in our formulæ to obtain the equation of its elliptical outline. This gives $c = \sqrt{(1 - \cos^2 \beta)} = \sin \beta$, which value being substituted in (611), we have at once the required equation, while the position of the ellipse is given at once by the angle p from the table above referred to.

Mars, Venus, and Mercury.—These planets may be regarded as spherical in the computation of their occultations, and we shall, therefore, have to consider only their crescent and gibbous phases. To adapt our formulæ to the case of a spherical body, we have only to put $e = 0$, or $c = 1$. Since in this case we are concerned only with the *apparent figure* of a partially illuminated spherical body, we may, for the convenience of computation, assume any point as the pole of the planet; and it will be most natural to assume the point which is the pole of the great circle whose plane passes through the sun, the earth, and the planet. The direction of this pole is evidently the same as that of the line joining the cusps of the partially illuminated disc. This makes $\beta = 0$, $B = 0$, in (615), and, consequently, $V = A - \lambda$. But, as the adopted equator of the planet is here a variable plane, we can no longer use the form (620) for finding A. A very simple and direct process for finding V offers itself. Let E, S, O, Fig. 49, repre-

Fig. 49.



sent the centres of the earth, the sun, and the planet; $S'O'O''$, the great circle of the celestial sphere whose plane passes through the three bodies; S' and O', the geocentric places of the sun and the planet; O'', the heliocentric place of the planet. Then $O'O''$ is the arc heretofore denoted by V, and, in the infinite sphere, is the measure of the angle $O'OO'' = SOE$. Putting then $V = O'O''$, $\gamma = S'O'$, and also

$R' = SO =$ the heliocentric distance of the planet,
$R = SE =$ " " " earth,

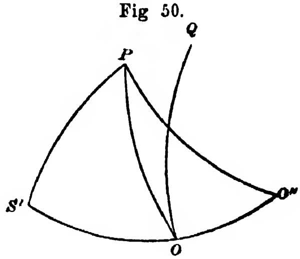
we have

$$\sin V = \frac{R}{R'} \sin \gamma$$

We might find V directly from the three known sides of the triangle SOE; but, as we have yet to find p, and γ comes out at the same time with p in a very simple manner, it will be preferable to employ the above form.

To find p and γ, let S', O', O'', Fig. 50, be the three places above referred to, and P the pole of the equinoctial. Draw $O'Q$ perpendicular to the great circle $S'O'O''$. This perpendicular passes through the adopted pole of the planet, and we have $PO'Q = p$, or $PO'S' = 90° - p$, and $S'O' = \gamma$. Hence, denoting by δ' and D the declination of the planet and the sun, and by α' and A their right ascensions respectively, the spherical triangle $PS'O'$ gives

Fig 50.



$$\left.\begin{aligned} \cos\gamma &= \sin\delta' \sin D + \cos\delta' \cos D \cos(\alpha' - A) \\ \sin\gamma \sin p &= \cos\delta' \sin D - \sin\delta' \cos D \cos(\alpha' - A) \\ \sin\gamma \cos p &= \cos D \sin(\alpha' - A) \end{aligned}\right\} \quad (622)$$

Hence, introducing an auxiliary to facilitate the computation, both p and V will be found by the following formulæ:

$$\left.\begin{aligned} \tan F &= \tan D \sec(\alpha' - A) \\ \tan p &= \cot(\alpha' - A) \sin(F - \delta') \sec F \\ \sin V &= \frac{R}{R'} \cdot \frac{\sin(\alpha' - A)\cos D}{\cos p} \end{aligned}\right\} \quad (623)$$

In this method of finding V we do not determine whether it is

greater or less than 90°. This is of no importance in computing an actual observation, but only in *predicting* the phase of the planet, whether crescent or gibbous. For the latter purpose we must have recourse to the triangle *SEO* of Fig. 49, the three sides of which are given in the Ephemeris.

The value of V being found, the equation (616) will be used to determine the apparent outline after substituting $c = 1$ and $w = 90°$, whereby it becomes

$$s^2 = v^2 + u^2 \sec^2 V$$

The value of s in our equations is supposed to be given. It will be most convenient to deduce it from the apparent semidiameter of the planet when at a distance from the earth equal to the earth's mean distance from the sun, which is the unit employed in expressing their geocentric distances in the Ephemeris. Thus, denoting the mean semidiameter by s_0, and the geocentric distance by r', we have (Art. 128)

$$s = \frac{s_0}{r'} \tag{624}$$

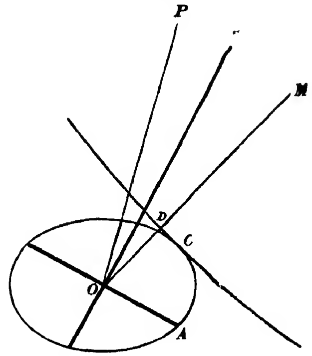
and s_0 may be taken from the following table:

	s_0	Authority.
MERCURY	3″.34	LE VERRIER, *Theory of Mercury.*
VENUS	8 .55	PEIRCE, *Am. Ephemeris.*
MARS	5 .05	" " "
JUPITER	99 .70	STRUVE, *Astr. Nach.*, No. 139.
SATURN	81 .36	BESSEL, *Astr. Nach.*, No. 275.
SATURN'S RINGS.		
Outer semi-major axis of outer ring	187 .56	" " "
Inner " " " "	165 .07	STRUVE, *Astr. Nach.*, No. 139, reduced to agree with BESSEL's measures of the outer diameter of the outer ring.
Outer " " inner "	161 .27	
Inner " " " "	124 75	

353. *To find the longitude of a place from the observed contact of the moon's limb with the limb of a planet.*—In the following investigation, it is assumed that the quantities p, w, V, c, are known for the time of the occultation. They may be computed by the above methods for the time of conjunction of the moon and planet, and regarded as constant for the same occultation over the earth in general.

Let O, Fig. 51, be the apparent centre of the planet, and C the point of contact of its limb with that of the moon. Let OM be drawn from O towards the moon's centre, intersecting the moon's limb in D. Since the apparent semidiameter of any of the planets is never greater than 31″, it is evident that no appreciable error can result from our assuming that the small portion CD of the moon's limb coincides sensibly with the common tangent to the two bodies drawn at C. If, then, the planet were a spherical body with the radius OD, the observed time of contact would not be changed. We may, therefore, reduce the occultation of a planet to the general case of eclipse of one spherical body by another, by substituting the perpendicular OD for the radius of the disc of the eclipsed body. Let s'' denote this perpendicular; let OA and OQ be the axes of u and v respectively, to which the curve of illumination is referred by the equation (616); and let ϑ be the angle QOD which the perpendicular s'' makes with the axis of v. The equation of the tangent line CD referred to these axes is

Fig. 51.



$$u \sin \vartheta + v \cos \vartheta = s'' \tag{625}$$

We have also in the curve

$$\frac{dv}{du} = -\tan \vartheta$$

Differentiating the equation (616), therefore, we have

$$\left(u \cos w - \frac{v \sin w}{c}\right)\left(\cos w + \frac{\tan \vartheta \sin w}{c}\right)$$
$$+\left(u \sin w + \frac{v \cos w}{c}\right)\left(\sin w - \frac{\tan \vartheta \cos w}{c}\right)\sec^2 V = 0$$

By means of this equation, together with (616) and (625), we can eliminate u and v, and thus obtain the relation between s and s''. To abbreviate, put

$$x = u \cos w - \frac{v \sin w}{c}$$

$$y = u \sin w + \frac{v \cos w}{c}$$

and also

$$c' \sin \vartheta' = \frac{\sin \vartheta}{c} \qquad c' \cos \vartheta' = \cos \vartheta \qquad (626)$$

then the three equations become

$$x \cos (\vartheta' - w) - y \sin (\vartheta' - w) \sec^2 V = 0$$
$$x^2 + y^2 \sec^2 V = s^2$$
$$x \sin (\vartheta' - w) + y \cos (\vartheta' - w) = \frac{s''}{cc'}$$

From the first and second of these we find

$$x = \frac{s \sin (\vartheta' - w)}{\sqrt{[1 - \cos^2 (\vartheta' - w) \sin^2 V]}}$$
$$y = \frac{s \cos (\vartheta' - w) \cos^2 V}{\sqrt{[1 - \cos^2 (\vartheta' - w) \sin^2 V]}}$$

which substituted in the third give

$$s'' = s cc' \sqrt{[1 - \cos^2 (\vartheta' - w) \sin^2 V]}$$

Hence, if we put

$$\left.\begin{aligned} \sin \chi &= \cos (\vartheta' - w) \sin V \\ s'' &= s \, . \, cc' \cos \chi \end{aligned}\right\} \qquad (627)$$

we have

We have seen (Art. 352) that in all practical cases we may take $w = 90°$, and, therefore, instead of (626) and (627) we may employ the following:

$$\left.\begin{aligned} \tan \vartheta' &= \frac{\tan \vartheta}{c} \\ \sin \chi &= \sin \vartheta' \sin V \\ s'' &= \frac{s \sin \vartheta \cos \chi}{\sin \vartheta'} \end{aligned}\right\} \qquad (628)$$

If the occultation of a *cusp* of Venus or Mercury is observed, we have at once $s'' = s \cos \vartheta$ (for the axis of v coincides with the line joining the cusps), and we do not require V.

The value of s'' is to be substituted in (486) for the apparent semidiameter of the eclipsed body. In that formula, H denotes the apparent semidiameter at the distance unity: therefore, we must now substitute the value

$$\sin H = r' \sin s''$$

or, by (624) and (628),

$$\sin H = \frac{\sin s_0 \sin \vartheta \cos \chi}{\sin \vartheta'} \tag{629}$$

Since f is here very small, we may put $\tan f = \sin f$, and the formula for L (488) becomes

$$\begin{aligned} L &= (z - \zeta) \sin f \pm k \\ &= (z - \zeta) \frac{\sin H}{r'g} \pm k \pm (z - \zeta) \frac{k \sin \pi_0}{r'g} \end{aligned}$$

Hence, putting

$$k' = k + (z - \zeta) \frac{k \sin \pi_0}{r'g} \tag{630}$$

we have

$$L = (z - \zeta) \frac{\sin H}{r'g} \pm k' \tag{631}$$

When the angle ϑ is known, therefore, the preceding formulæ will determine L, with which the computation will be carried out in precisely the same form as in the case of a solar eclipse, Art. 329. To find ϑ, let OP, Fig. 51, be drawn in the direction of the pole of the equinoctial; then we have $POQ = p$, and, denoting POM by Q,

$$\vartheta = Q - p$$

and Q has here the same signification as in the general equations (567), as shown in Art. 295: so that when N and ψ have been found by (568) and (569), we have $Q = N + \psi$, or

$$\vartheta = N + \psi - p \tag{632}$$

But to compute ψ by (569) we must know L, and this involves H, which depends upon ϑ. The problem can, therefore, be solved only by successive approximations; but this is a very slight objection in the present case, since the only formulæ to be repeated are those for L and ψ, and the second approximation will mostly be final. It can only be in a case such as the occultation of Saturn's ring, where the outline of the eclipsed body is very elliptical, and especially when the contact occurs near the northern or southern limb of the moon, that it may be necessary (for extreme accuracy) to compute H a second time and, consequently, ψ a third time.

The formula (629) is adapted to the general case of an ellip-

soidal body partially illuminated, the point of contact being on the defective limb. When the point of contact is on the full limb, we have only to put $V = 0$, and the formula becomes

$$\sin H = \frac{\sin s_0 \sin \vartheta}{\sin \vartheta'} \tag{633}$$

and for the full limb of a spherical planet (Venus, Mercury, and Mars) we have $H = s_0$.

In the first approximation we may take $L = \pm k$.

354. Sometimes it may not be known from the record of the observation whether the point of contact is on the full or the defective limb of the planet. This might be determined by the method of Art. 350; but, since that method supposes the position angle p' to be given, which we do not here employ, the following more direct and simple process may be used. In that article the common point of tangency of the two curves of the full and defective limbs was determined by the condition

$$u_1 \sin w + v_1 \frac{\cos w}{c} = 0$$

in which u_1 and v_1 denotes the co-ordinates of the point of tangency. In the notation of Art. 353 this is simply $y_1 = 0$; and since we have

$$y_1 = \frac{s \cos(\vartheta_1 - w) \cos^2 V}{\sqrt{[1 - \cos^2(\vartheta_1 - w) \sin^2 V]}}$$

it follows that we must have

$$\cos(\vartheta_1 - w) = 0 \qquad \text{or} \qquad \vartheta_1 = w \mp 90°$$

Hence, when, as in our present application, we take $w = 90°$, we have

$$\vartheta_1 = 0 \qquad \text{or} \qquad \vartheta_1 = 180°$$

Hence *a point is to be regarded as on the east limb for values of* ϑ *between* 0° *and* 180°, *and on the west limb for values of* ϑ *between* 180° *and* 360°; and (Art. 350) *the east or the west limb is defective according as* V *is between* 0° *and* 180° *or between* 180° and 360°.

But, since $\sin \vartheta'$ and $\sin \vartheta$ have the same sign, we deduce from this a still more simple rule; for we have $\sin \chi = \sin \vartheta' \sin V$, whence it follows that *the observed point is on the defective limb when* $\sin \chi$ *is positive, and on the full limb when* $\sin \chi$ *is negative.*

355. In the cases of the planets Neptune, Uranus, and the asteroids, the occultation of their centres will be observed, and it will be most convenient to compute by the method for a fixed star, only substituting for π the difference of the moon's and planet's horizontal parallaxes—that is, the *relative parallax*—in the formulæ for x and y, Art. 341.

This artifice of using the relative parallax may also be used with advantage for Jupiter and Saturn.

Having thus found x and y as for a fixed star, we shall have, in the preceding method,

$$L = (z - \zeta)\frac{\sin H}{r'} \pm k \tag{634}$$

the other formulæ remaining unchanged.

EXAMPLE 1.—Several occultations of Saturn's Ring were observed by Dr. KANE at Van Rensselaer Harbor on the northwest coast of Greenland during the second Grinnell Expedition in search of Sir JOHN FRANKLIN.* The first of these was as follows:

1853 December 12th, Van Rensselaer Mean Time	
Immersion, contact of last point of ring, . . .	$14^h\ 20^m\ 48^s.8$
Emersion, " " " " . .	14 54 18 .3

The assumed longitude of the place of observation was $\omega = 4^h\ 43^m\ 32^s$ west of Greenwich. The latitude was $\varphi = 78°\ 37'\ 4''$, whence

$$\log \rho \sin \varphi' = 9.989862 \qquad \log \rho \cos \varphi' = 9.296642$$

I. From the Nautical Almanac we take for 1853 Dec. 12, 19^h,

$$p = -2°\ 37'.3 \qquad l = 24°\ 0'.4 \qquad \text{whence } \log c = \log \sin l = 9.6094$$

and from page 578, the outer ring only being observed,

$$s_0 = 187''.56 \qquad \log \sin s_0 = 6.9587$$

* "Astronomical Observations in the Arctic Seas by ELISHA KENT KANE, M.D., U.S.N. Reduced and discussed by CHARLES A. SCHOTT, Assistant U.S. Coast Survey." Published by the Smithsonian Institution, May, 1860.

II. We shall compute the elements of the occultation for the centre of the planet for the Greenwich hours 18^h, 19^h, and 20^h. For these times we take the following quantities from the Nautical Almanac, applying to them the corrections determined by Mr. Schott from the Greenwich observations of this date:

Moon.

Gr. T.	α	δ	π
18^h	$3^h\ 36^m\ 55^s.23$	$+18°\ 2'\ 47''.5$	$54'\ 7''.68$
19	38 53 .92	12 13 .9	7 .22
20	40 52 .81	21 35 .7	6 .76

Saturn.

	α'	δ'	π'	$\log r'$
18^h	$3^h\ 39^m\ 9^s.88$	$+17°\ 14'\ 28''.4$	$1''.05$	0.9126
19	9 .16	26 .5		
20	8 .44	24 .5		

The corrections applied to the Nautical Almanac values to obtain the above are $\Delta\alpha = -0^s.22$, $\Delta\delta = -5''.0$, $\Delta\alpha' = +0^s.15$, $\Delta\delta' = -8''.9$, $\Delta\pi = +0''.3$, this last correction being derived from Mr. Adams's Table in the Nautical Almanac for 1856.

We shall use the relative parallax, and compute as for a fixed star, taking $\pi - \pi'$ for π, namely

	π
18^h	$54'\ 6''.73$
19	6 .17
20	5 .71

whence we find for the moon's co-ordinates,

Gr. T.	x	x'	y	y'
18^h	-0.59152	$+0.52457$	$+0.89382$	$+0.17436$
19	-0.06690	$+0.52466$	$+1.06817$	$+0.17434$
20	$+0.45781$	$+0.52475$	$+1.24250$	$+0.17432$

and, taking $z = r = \dfrac{1}{\sin \pi}$ for 19^h, as sufficiently accurate,

$$z = 63.54$$

III. For the co-ordinates of the place of observation:

	Immersion.	Emersion.
Local mean time t	$14^h\ 20^m\ 48^s.8$	$14^h\ 54^m\ 18^s.3$
$t + \omega$	19 4 20 .8	19 37 50 .3
Local sid. time μ	117° 4′ 59″.7	125° 28′ 44″.7

and hence, by the formulæ on p. 550,

ξ	+ 0.17529	+ 0.18685
η	+ 0.90575	+ 0.91363
ζ	+ 0.38	+ 0.35
$z - \zeta$	63.16	63.19

IV. Assuming now two epochs corresponding nearly to the times of observation, the remainder of the computation *in extenso* is as follows:

	Immersion.	Emersion.
Assumed T_0	$19^h.07 =$ $19^h\ 4^m\ 12^s$	$19^h.63 =$ $19^h\ 37^m\ 48^s$.
x_0	− 0.03017	+ 0.26365
y_0	+ 1.08037	+ 1.17800
$x_0 - \xi = m \sin M$	− 0.20546	+ 0.07680
$y_0 - \eta = m \cos M$	+ 0.17462	+ 0.26437
M	310° 21′ 38″	16° 11′ 56″
$\log m$	9.43079	9.43980
$x' = n \sin N$	+ 0.52467	+ 0.52472
$y' = n \cos N$	+ 0.17434	+ 0.17433
N	71° 37′ 10″	71° 37′ 20″
$\log n$	9.74263	9.74266

Then, for a first appproximation, by the formula

$$\sin \psi = \frac{m \sin (M - N)}{\pm k}$$

and observing that the immersion is here an interior contact and the emersion an exterior contact, we have

		Immersion.	Emersion.
	$\log \sin (M - N)$	n9.93188	n9.91559
	$\log m$	9.43079	9.43980
$(L = \mp k)$	ar. co. $\log L$	n0.56441	0.56441
	$\log \sin \psi$	9.92708	n9.91980
	*ψ	57° 43′.2	303° 45′.5
	$N - p$	74 14.5	74 16.6
	$N + \psi - p = \vartheta$	131 57.7	18 0.1
	$\log \tan \vartheta$	n0.0462	9.5118
	$\log c$	9.6094	9.6094
	$\log \tan \vartheta'$	n0.4368	9.9024
	$\log \sin \vartheta$	n9.8713	9.4900
	ar. co. $\log \sin \vartheta'$	n0.0273	0.2047
	$\log \left(\frac{z - \zeta}{r'}\right) \sin s_0$	7.8465	7.8467
	†$\log a$	7.7451	7.5414
	a	0.00556	0.00348
	$\mp k$	$-$ 0.27264	+ 0.27264
	$a \mp k = L$	$-$ 0.26708	+ 0.27612
	$\log L$	n9.42664	9.44110

Applying the difference between $\log L$ and $\log k$ to $\log \sin \psi$, we find, for our second approximation,

	Immersion.	Emersion.
Corrected $\log \sin \psi$	9.93603	9.91429
" ψ	59° 39′.6	304° 49′.5
" ϑ	133 54.1	19 4.1
$\log \tan \vartheta$	n0.0167	9.5387
$\log \tan \vartheta'$	n0.4073	9.9293
$\log \sin \vartheta$	n9.8577	9.5141
ar. co. $\log \sin \vartheta'$	n0.0310	0.1887
	7.8465	7.8467
Corrected $\log a$	7.7352	7.5495
" a	0.00543	0.00354
" L	$-$ 0.26721	+ 0.27618
" $\log L$	n9.42685	9.44119
Final value of $\log \sin \psi$	9.93582	n9.91420
$\log \cos \psi$	9.70403	9.75688

* The angle ψ is to be taken so that $L \cos \psi$ shall be negative for immersion and positive for emersion, Art. 329.

† Putting $a = (z - \zeta) \frac{\sin H}{r'} = \frac{\sin \vartheta}{\sin \vartheta'} \cdot \frac{z - \zeta}{r'} \cdot s_0$

	Immersion.	Emersion.
$h = 3600, \quad \log b = \log \dfrac{h L \cos \psi}{n}$	$n2.94455$	3.01171
$\log c = \log \dfrac{hm \cos (M - N)}{n}$	$n2.95956$	3.00741
b	$- 880^s.1$	$+ 1027^s.3$
c	$- 911\ .1$	$+ 1017\ .2$
$b - c = \tau$	$+ \ \ 31\ .0$	$+ \ \ \ 10\ .1$
Gr. Time of obs. $= T_0 + \tau = T$	$19^h\ 4^m\ 43^s.0$	$19^h\ 37^m\ 58^s.1$
$T - t = \omega$	$4\ \ 43\ \ 54\ .2$	$4\ \ 43\ \ 39\ .8$

If now we wish to form the equations of condition for determining the effect of errors in the data, we proceed precisely as in the case of a solar eclipse, page 533, and find

	Immersion.	Emersion.
$\log \nu \tan \psi$	0.5341	$n0.4596$
$\log \nu \sec \psi$	0.5983	0.5454

where $\log \nu = \log \dfrac{3600}{n\pi} = 0.3023$. Hence, neglecting the terms depending on the correction of the parallax and of the eccentricity of the meridian, the equations of condition are

$$\text{(Im.)} \quad \omega_1 = 4^h\ 43^m\ 54^s.2 - 2.001\ \gamma + 3.421\ \vartheta - 3.965\ \pi\Delta k$$
$$\text{(Em.)} \quad \omega_1 = 4\ \ 43\ \ 39\ .8 - 2.001\ \gamma - 2.881\ \vartheta + 3.511\ \pi\Delta k$$

Eliminating ϑ from these equations, we have

$$\omega_1 = 4^h\ 43^m\ 46^s.4 - 2.001\ \gamma + 0.092\ \pi\Delta k$$

An error of $1''$ in the moon's semidiameter (represented by $\pi\Delta k$) would, therefore, have no sensible effect upon this combined result; and since γ must also be very small, as we have corrected the places of the moon and planet by the Greenwich observations, we can adopt, as the definite result from this observation,

$$\omega_1 = 4^h\ 43^m\ 46^s.4$$

It will be observed that in this example OUDEMANS's value, $k = 0.27264$, has been employed; but our final equation shows that the result would have been sensibly the same if we had taken the usual value 0.27227; for the reduction of the result to that which the latter value of k would have given is only $0.092 \times 3247 \times (-\ 0.00037) = -\ 0^s.11$.

EXAMPLE 2.—The occultation of *Venus*, April 24, 1860, was observed at the U. S. Military Academy, West Point ($\omega = 4^h\ 55^m\ 51^s$, $\varphi = 41°\ 23'\ 31''.2$), and at Albany ($\omega = 4^h\ 54^m\ 59^s.4$, $\varphi = 42°\ 39'\ 49''.5$), as follows:

	West Point. Sid. time	Albany. Mean time.
Immersion.		
First contact, planet's full limb	$10^h\ 46^m\ 53^s.35$	$8^h\ 31^m\ 1^s.9$
Disappearance of cusp	10 47 47 .80	8 31 54 .2

The observations were made with the large refractors of the West Point and Dudley observatories.

I. To find p for the cusp observations, we have for the Greenwich time $13^h.478$, which is the mean of the times of the observations at the two places, and will serve for both,

$$\text{Planet,}\ \alpha' = 78°\ 38'.6 \qquad \delta' = 25°\ 59'.1$$
$$\text{Sun,}\ A = 32\ \ 45.5 \qquad D = 13\ \ 12.9$$

whence, by (623),

$$p = -\ 7°\ 27'.3$$

and, from p. 578,

$$s_0 = 8''.55 \qquad \log \sin s_0 = 5.6175$$

II. We shall compute the moon's co-ordinates only for the Greenwich times $13^h.4$ and $13^h.5$. For these times the *American Ephemeris* furnishes the following data:

Moon.

Gr. T.	α	δ	π
$13^h.4$	79° 12′ 16″.8	+ 26° 43′ 1″.6	57′ 6″.6
13 .5	79 15 58 .5	26 43 4 .3	57 6 .7

Venus.

	α'	δ'	$\log r'$
$13^h.4$	78° 38′ 23″.3	+ 25° 59′ 2″.5	9.9193
13 .5	78 38 40 .7	25 59 4 .3	9.9193

Hence, by the formulæ of I. and II., p. 452, we find

	a	d	$\log g$
$13^h.4$	78° 38′ 17″.2	+ 25° 58′ 54″.5	9.9987
13 .5	78 38 34 .0	25 58 56 .3	9.9987

	x	x'	y	y'
13.4	+ 0.531695	+ 0.53390	+ 0.773681	+ 0.00480
13.5	+ 0.585085		+ 0.774161	
	$z = 60.19$			

III. For the co-ordinates of the places of observation:

	West Point.		Albany.	
	Full limb	Cusp.	Full limb.	Cusp.
Local mean time t	$8^h\ 33^m\ 43^s.72$	$8^h\ 34^m\ 38^s.02$	$8^h\ 31^m\ 1^s.90$	$8^h\ 31^m\ 54^s.20$
$t + \omega$	13 29 34 .72	13 30 29 .02	13 26 1 .30	13 26 53 .60
μ	161° 43′ 20″.3	161° 56′ 57″.0	161° 2′ 44″.3	161° 15′ 50″.9
$\log \rho \sin \phi'$	9.818064		9.828792	
$\log \rho \cos \phi'$	9.875814		9.867157	
ξ	+ 0.745828	+ 0.746178	+ 0.730013	+ 0.730378
η	+ 0.551616	+ 0.552909	+ 0.563428	+ 0.564641
ζ	+ 0.37	+ 0.37	+ 0.38	+ 0.38
$z - \zeta$	59.82	59.82	59 81	59.81

IV. Assuming $T_0 = 13^h.45$, we find, for this time,

x_0	+ 0.558390			
y_0	+ 0.773921			
$x_0 - \xi = m \sin M$	− 0.187438	− 0.187788	− 0.171623	− 0.171988
$y_0 - \eta = m \cos M$	+ 0.222305	+ 0.221012	+ 0.210493	+ 0.209280
M	319° 51′ 50″	319° 38′ 47″	320° 48′ 30″	320° 35′ 11″
$\log m$	9.463563	9.462425	9.433915	9.432783
N	89° 29′ 6″			
$\log n$	9.727480			

Then, for the observations of the full limb, we have for both places, by (631), putting $H = s_0$,

	$\log (z - \zeta)$	1.7768	. . . . . .	1.7768
	ar. co. $\log r'g$	0.0820	. . . . . .	0.0820
$k = 0.27264$	constant	5.0542	$\log \sin s_0$	5.6175
0.00082	log	6.9130		
$k' = 0.27346$				
0.00299	. . . .		log (1)	7.4763
$L = 0.27645$				

	West Point.	Albany.
$M - N$	230° 22′ 44″	231° 19′ 24″
ψ	234 6 57	230 4 55
τ	+ 2^m 37^s.7	— 52^s.7
T_0	13^h 27^m 0^s.	13^h 27^m 0^s.
T	13 29 37 .7	13 26 7 .3
$T - t = \omega$	4 55 54 .0	4 55 5 .4

For the observations of the cusps we can employ the preceding values of ψ as a first approximation; and hence we proceed as follows:

	West Point.	Albany.
$N + \psi - p = \vartheta$	331° 3′.4	327° 1′.3
log cos ϑ	9.9421	9 9237
log (1)	7.4763	7.4763
	7.4184	7.4000
	0.00262	0 00251
k'	0.27346	0.27346
L	0.27084	0.27095
$M - N$	230° 9′ 41″	231° 6′ 5″
log sin $(M - N)$	n9.885278	n9.891124
log m	9.462425	9.432783
ar. co. log L	0.567287	0.567111
log sin ψ	n9.914990	n9.891018
ψ	235° 18′.5	231° 5′.0
Corrected ϑ	332 14 .9	328 1 .4
log cos ϑ	9.9469	9.9285
log (1)	7.4763	7.4763
	7.4232	7.4048
	0.00265	0.00254
Corrected L	0.27081	0.27092
ar. co. log L	0.567335	0.567159
Corrected log sin ψ	n9.915038	9.891066
τ	+ 3^m 33^s 7	— 0^s.4
$T_0 + \tau = T$	13^h 30^m 33^s.7	13^h 26^m 59^s.6
$T - t = \omega$	4 55 55 .7	4 55 5 .4

Finally, if we wish to form the equations of condition for correcting these results for errors in the data, including an error in the planet's semidiameter, we proceed as for an eclipse of the

sun, p. 533. For the full limb we have only to substitute Δs_0 for ΔH; but for the cusp we must evidently substitute $\Delta s_0 \cos\vartheta$ for ΔH. It will be more accurate to restore $r'g$ in the place of r', since g here differs sensibly from unity. We shall thus find

$$\begin{aligned}
\omega' &= 4^h\,55^m\,54^s.0 - 1.967\,\gamma + 2.720\,\vartheta - 3.358\,\pi\Delta k - 4.061\,\Delta s_0 \\
\omega' &= 4\;\;55\;\;55\,.7 - 1.967\,\gamma + 2.844\,\vartheta - 3.459\,\pi\Delta k + 3.697\,\Delta s_0 \\
\omega'' &= 4\;\;55\;\;\;\;5\,.4 - 1.967\,\gamma + 2.352\,\vartheta - 3.067\,\pi\Delta k - 3.704\,\Delta s_0 \\
\omega'' &= 4\;\;55\;\;\;\;5\,.4 - 1.967\,\gamma + 2.438\,\vartheta - 3.134\,\pi\Delta k + 3.349\,\Delta s_0
\end{aligned}$$

where ω' and ω'' denote the true longitudes. Hence, also,

$$\begin{aligned}
\omega' - \omega'' &= +\,48^s.6 + 0.368\,\vartheta - 0.291\,\pi\Delta k - 0.357\,\Delta s_0 \\
\omega' - \omega'' &= +\,50\,.3 + 0.406\,\vartheta - 0.325\,\pi\Delta k + 0.348\,\Delta s_0
\end{aligned}$$

and the mean is

$$\omega' - \omega'' = +\,49^s.5 + 0.387\,\vartheta - 0.308\,\pi\Delta k - 0.005\,\Delta s_0$$

The effect of an error in s_0 upon the difference of longitude of the two places is, therefore, insensible; but, to eliminate ϑ and $\pi\Delta k$, observations of the emersion should also be used. The effect of γ and ϑ upon ω' and ω'' can only be eliminated by means of observations of the moon's place at a standard observatory on the day of the observation, as we have already shown in other examples.

TRANSITS OF VENUS AND MERCURY.

356. The transits of Venus and Mercury may be computed by the method for solar eclipses, substituting the planet for the moon. In the formulæ (486), (487), &c., we must employ

for Venus, $k = 0.9975$
for Mercury, $k = 0.3897$

which are the values which result from the apparent semidiameters of these planets adopted on p. 578.

Since b is no longer a small quantity, it will be necessary to employ the exact formulæ (479) instead of (481).

The longitude of a place at which the transit is observed may be computed from each of the four contacts of the limb of the sun and planet, by the formulæ of Art. 329. These observations, however, are of little use in determining an unknown longitude, on account of the great effect of small errors in the assumed

parallax upon the computed time; but, on the other hand, when the longitude is previously known, each observation furnishes an equation of condition of the form (584) for determining the correction of the parallax. In developing this equation, however, we supposed $g = 1$, in the formula (486), and we must, therefore, here restore the true value. We may take

$$g = 1 - b = 1 - \frac{\pi'}{\pi} = \frac{\pi - \pi'}{\pi}$$

in which π and π' are the assumed horizontal parallaxes of the planet and sun respectively at the time of the observation. Instead of the form for l employed on p. 449, we shall now take the more correct form

$$l = \frac{H}{r'g\pi} \pm \frac{k}{g}$$

If we denote the sun's semidiameter at the time of the observation by s', that of the planet by s, we have $s' = \frac{H}{r'}$, $s = \pi k$, and hence

$$l = \frac{s' \pm s}{g\pi}$$

and instead of (581) we shall have

$$\Delta L = \Delta l = \frac{\Delta(s' \pm s)}{g\pi} - \frac{s' \pm s}{g\pi} \cdot \frac{\Delta\pi}{\pi}$$

Omitting the term depending upon Δee, which can never be appreciable in the transits of the planets, the equation (582) will now become

$$\omega' - \omega = -\nu\gamma + \nu \tan \psi \,.\, \vartheta + \frac{\nu \sec \psi}{g} \Delta(s' \pm s)$$
$$+ \nu \left[n\,(t + \omega - T_1) - \varkappa \tan \psi - \frac{s' \pm s}{g\pi} \sec \psi \right] \Delta\pi \qquad (635)$$

where γ and ϑ have the signification (583); ω' is the true longitude, and ω that which is computed from the observation.

Since, by KEPLER'S laws, the *ratio* of the mean distances of any two planets is accurately known from their periods, the ratio $\frac{\pi}{\pi_0}$ is also known, and will not be changed by substituting the corrected values $\pi + \Delta\pi$ and $\pi_0 + \Delta\pi_0$: in other words we shall have

$$\frac{\Delta\pi}{\Delta\pi_0} = \frac{\pi}{\pi_0} \qquad \text{or} \qquad \Delta\pi_0 = \frac{\pi_0}{\pi} \Delta\pi$$

The discussion of all the equations of condition of the form (635) will, therefore, give not only the correction $\Delta\pi$ of the planet's parallax, but also, by the last-mentioned relation, that of the solar parallax.*

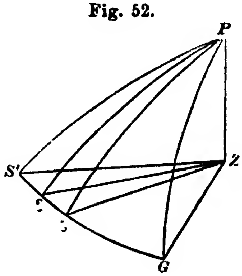
The transits of Venus will afford a far more accurate determination of this parallax than those of Mercury; for, on account of its greater proximity to the earth, the difference in the duration of the transit at different places will be much greater, and the coefficient of $\Delta\pi$ in the final equations proportionally great.

Although the general method for eclipses may also be extended to the prediction of the transits of the planets (by Art. 322), yet it is more convenient in practice to follow a special method in which advantage is taken of the circumstance that the parallaxes of both bodies are so small that their squares and higher powers may be neglected. LAGRANGE's method for this purpose is the most simple, and, in the improved form which I shall give to it in the following article, most accurate.

357. *To predict the times of ingress and egress for a given place.*—We first find the times of ingress and egress *for the centre of the earth*, from which the times for any place on the surface are readily deduced.

Let α, δ, α', δ' be the right ascensions and declinations of the planet and the sun for an assumed time T_0, at the first meridian, near the time of conjunction. Let m denote the apparent distance of the centres at this time. Let S' and S, Fig. 52, be the geocentric places of the centres of the sun and planet, P the pole; then, denoting the angles $PS'S$ and PSS' by P' and $180° - P$, the triangle PSS' gives

Fig. 52.



$$\sin \tfrac{1}{2} m \sin \tfrac{1}{2}(P + P') = \sin \tfrac{1}{2}(\alpha - \alpha') \cos \tfrac{1}{2}(\delta + \delta')$$
$$\sin \tfrac{1}{2} m \cos \tfrac{1}{2}(P + P') = \cos \tfrac{1}{2}(\alpha - \alpha') \sin \tfrac{1}{2}(\delta - \delta')$$

But, since $\tfrac{1}{2} m$ is at the time of a contact only about 8′, we may without appreciable error substitute it for its sine, and,

* Another method of forming the equations, apparently shorter, but in reality, where many observations are to be reduced, not more convenient than the rigorous method, will be found in ENCKE's *Die Entfernung der Sonne von der Erde. aus dem Venusdurchgange von* 1761 *hergeleitet;* and *Der Venusdurchgang von* 1769

writing M for $\frac{1}{2}(P + P')$, we may regard the following equations as practically exact:

$$\left.\begin{aligned} m \sin M &= (\alpha - \alpha') \cos \delta_0 \\ m \cos M &= \delta - \delta' \end{aligned}\right\} \quad (636)$$

in which $\delta_0 = \frac{1}{2}(\delta + \delta')$.

Now, let the required time of contact be $T = T_0 + \tau$, and put

a = the relative hourly motion of the two bodies in right ascension,
= the planet's hourly motion — the sun's,
d = the relative hourly motion in declination,

then at the time T the differences of right ascension and declination are $\alpha - \alpha' + a\tau$ and $\delta - \delta' + d\tau$. If further we put

s, s' = the apparent semidiameters of the planet and sun, respectively,

the apparent distance of the centres at the time T is $s' \pm s$, the lower sign being employed for inner contacts; and if the value of M at this time is Q, we have

$$\begin{aligned} (s' \pm s) \sin Q &= (\alpha - \alpha') \cos \delta_0 + a \cos \delta_0 . \tau \\ (s' \pm s) \cos Q &= \delta - \delta' + d\tau \end{aligned}$$

Putting, therefore,

$$\left.\begin{aligned} n \sin N &= a \cos \delta_0 \\ n \cos N &= d \end{aligned}\right\} \quad (637)$$

we have

$$\begin{aligned} (s' \pm s) \sin Q &= m \sin M + n \sin N . \tau \\ (s' \pm s) \cos Q &= m \cos M + n \cos N . \tau \end{aligned}$$

which, solved in the usual manner, give

$$\left.\begin{aligned} &\sin \psi = \frac{m \sin (M - N)}{s' \pm s} \\ &\tau = \frac{s' \pm s}{n} \cos \psi - \frac{m}{n} \cos (M - N) \\ &Q = N + \psi \qquad\qquad T = T_0 + \tau \end{aligned}\right\} \quad (638)$$

where $\cos \psi$ is to be taken with the negative sign for ingress and with the positive sign for egress. The angle Q is (as in eclipses) the position angle of the point of contact.

The formulæ (636), (637), and (638) serve for the complete prediction for the centre of the earth.

To find the time of a contact for any point of the surface of the earth, let m be the geocentric apparent distance of the centres of the two bodies at any given time; m' the apparent distance, at the same time, as seen from a point on the earth's surface in latitude φ and longitude ω; π and π' the equatorial horizontal parallaxes of the planet and sun respectively; ζ and ζ' their geocentric zenith distances; ρ the radius of the earth for the latitude φ. The apparent zenith distances are $\zeta + \rho\pi \sin \zeta$ and $\zeta' + \rho\pi' \sin \zeta'$: these approximations being quite exact where the parallaxes are so small. Let Z, Fig. 52, be the geocentric zenith of the place, S and S' the true places of the bodies. The distance $SS' = m$ will become the apparent distance m' if we increase the sides ZS and ZS' by $\rho\pi \sin \zeta$ and $\rho\pi' \sin \zeta'$; and, if we regard these small increments as differentials, we shall have, by the first equation of (46),

$$m' - m = -\rho\pi \sin \zeta \cos S + \rho\pi' \sin \zeta' \cos S'$$

where $S = 180° - ZSS'$, and $S' = ZS'S$.

Let S_0 be the middle point of the arc SS', and denote the angle ZS_0S by S_0, the arc ZS_0 by ζ_0; then we have

$$\begin{aligned} -\sin \zeta \cos S &= \sin \tfrac{1}{2} m \cos \zeta_0 - \cos \tfrac{1}{2} m \sin \zeta_0 \cos S_0 \\ \sin \zeta' \cos S' &= \sin \tfrac{1}{2} m \cos \zeta_0 + \cos \tfrac{1}{2} m \sin \zeta_0 \cos S_0 \end{aligned}$$

which give

$$m' - m = \rho \left[(\pi + \pi') \sin \tfrac{1}{2} m \cos \zeta_0 - (\pi - \pi') \cos \tfrac{1}{2} m \sin \zeta_0 \cos S_0\right]$$

If then g and γ are determined by the conditions

$$\left.\begin{aligned} g \sin \gamma &= (\pi + \pi') \sin \tfrac{1}{2} m \\ g \cos \gamma &= (\pi - \pi') \cos \tfrac{1}{2} m \end{aligned}\right\} \quad (639)$$

we have

$$m' - m = g\rho (\sin \gamma \cos \zeta_0 - \cos \gamma \sin \zeta_0 \cos S_0)$$

Produce the arc $S'S$, and take $S_0G = 90° + \gamma$. Then, denoting the arc ZG by λ, the triangle ZGS_0 gives

$$\cos \lambda = -\sin \gamma \cos \zeta_0 + \cos \gamma \sin \zeta_0 \cos S_0$$

and the expression for m' becomes

$$m' = m - g\rho \cos \lambda \quad (640)$$

This remarkably simple form was first given by LAGRANGE,* with the difference only that he regarded the earth as a sphere, which amounts to supposing ρ to be constant. Under this supposition, it follows from the equation that, *at any given time, the apparent distance of the bodies is the same for all places on the surface of the earth which have the same value of λ*; that is, *for all places whose zeniths are in a small circle described from the point G as a pole with the polar distance $ZG = \lambda$.*

The computation of m' will, therefore, be extremely simple after the position of the point G is determined. The quantity γ is determined by (639), for which, however, we can take

$$\left.\begin{aligned} \tan\gamma &= \frac{\pi + \pi'}{\pi - \pi'} \tan \tfrac{1}{2} m \\ g &= \pi - \pi' \end{aligned}\right\} \quad (641)$$

Let A and D denote the right ascension and declination of the point G. Those of the point S_0 are very nearly $\alpha_0 = \frac{1}{2}(\alpha + \alpha')$ and $\delta_0 = \frac{1}{2}(\delta + \delta')$: so that in the triangle PS_0G we have the angle $S_0PG = A - \alpha_0$, the side $PS_0 = 90° - \delta_0$, and for the angle PS_0G we can take $M = \frac{1}{2}(PSG + PS'G)$ as in (636). Hence we have

$$\left.\begin{aligned} \cos D \sin (A - \alpha_0) &= \cos\gamma \sin M \\ \cos D \cos (A - \alpha_0) &= -\cos\delta_0 \sin\gamma - \sin\delta_0 \cos\gamma \cos M \\ \sin D &= -\sin\delta_0 \sin\gamma + \cos\delta_0 \cos\gamma \cos M \end{aligned}\right\} \quad (642)$$

or, adapted for logarithms,

$$\left.\begin{aligned} f \sin F &= \sin\gamma & \cos D \sin (A - \alpha_0) &= \cos\gamma \sin M \\ f \cos F &= \cos\gamma \cos M & \cos D \cos (A - \alpha_0) &= -f \sin (\delta_0 + F) \\ & & \sin D &= f \cos (\delta_0 + F) \end{aligned}\right\} \quad (642^*)$$

For any given time T, therefore, we can find m and M by (636), then γ and g by (641), and hence the values of A and D by (642). Now, let μ be the sidereal time (at the first meridian) corresponding to T, and put

$$\Theta = \mu - A$$

then, in the triangle PGZ, we have the angle $GPZ = \Theta - \omega$, and hence, φ' being the geocentric latitude of Z,

$$\cos\lambda = \sin\varphi' \sin D + \cos\varphi' \cos D \cos (\Theta - \omega) \quad (643)$$

with which the value of m' will be found by (640).

---

* *Memoirs of the Berlin Academy for* 1766. The above extremely simple demonstration I suppose to be new.

In order to apply these formulæ in predicting the time of a contact at a given place, we observe, first, that this time differs but a few minutes from the time of the same contact for the centre of the earth, and during these few minutes we may assume the distance m to vary uniformly.

Let T be the time of the geocentric contact, and T' the required time of the contact at the place, both times being reckoned at the first meridian. At the time T the geocentric distance $= s' \pm s$, and at the time T' the apparent distance $m' = s' \pm s$ (neglecting here the augmentation of the semidiameters, which are too minute to be considered in merely predicting the phenomenon); but at this time T' the geocentric distance has become

$$m = s' \pm s + (T' - T)\frac{dm}{dt}$$

where $\frac{dm}{dt}$ denotes the change of m in the unit of time. These values substituted in (640) give

$$(T' - T)\frac{dm}{dt} = g\rho \cos \lambda$$

Differentiating (636), we find

$$\begin{aligned} \frac{dm}{dt} \sin M + \frac{dM}{dt} m \cos M &= a \cos \delta_0 = n \sin N \\ \frac{dm}{dt} \cos M - \frac{dM}{dt} m \sin M &= d \qquad\quad\; = n \cos N \end{aligned}$$

whence

$$\frac{dm}{dt} = n \cos (M - N)$$

But, since at the time T we have $m = s' \pm s$, we also have for this time, by (638), $M - N = \psi$, and, therefore,

$$\frac{dm}{dt} = n \cos \psi$$

which gives

$$T' = T + \frac{g\rho \cos \lambda}{n \cos \psi} \tag{644}$$

in which the values of n and ψ found in the computation for the centre of the earth are to be employed. The value of λ to be employed must be that which results from the preceding formulæ at the time T. Now, at this time the value of the angle

M is Q, which is found by (638), and this value is to be employed in (642), while in (641) we take $m = s' \pm s$.

The formula for T' will be

$$T' = T + \frac{\pi - \pi'}{n \cos \psi} \left[\rho \sin \varphi' \sin D + \rho \cos \varphi' \cos D \cos (\Theta - \omega)\right] \quad (645)$$

in which T, n, ψ, D, Θ, $\pi - \pi'$ are all constants, found in the computation for the centre: so that the computation for a particular place requires only this single formula in which the latitude and longitude of the place are to be substituted.

358. The necessary formulæ for the complete prediction are recapitulated as follows:

I.—FOR THE CENTRE OF THE EARTH.

Assume a convenient time T_0 near the time of true conjunction of the sun and the planet, or this time itself, reckoned at the first meridian, and find for this time the values of α, δ for the planet; α', δ' for the sun; the semidiameters s and s'; and the relative changes in right ascension and declination, a and d, in the unit of time. Then, putting $\delta_0 = \frac{1}{2}(\delta + \delta')$, compute

$$m \sin M = (\alpha - \alpha') \cos \delta_0 \qquad n \sin N = a \cos \delta_0$$
$$m \cos M = \delta - \delta' \qquad n \cos N = d$$

$$\sin \psi = \frac{m \sin (M - N)}{s' \pm s}$$

where $s' + s$ is to be employed for exterior contact, and $s' - s$ for interior contact. Putting $h = 3600$, to reduce the terms to seconds, we then find

$$T = T_0 + h\left(\frac{s' \pm s}{n}\right) \cos \psi - \frac{hm}{n} \cos (M - N)$$

in which $\cos \psi$ is to be taken with the negative sign for ingress and with the positive sign for egress.

For the greatest precision, the computation may be repeated separately for ingress and egress, taking for T_0 the value of T first computed.

As in solar eclipses, if T_1 denotes the time of nearest approach of the centres of the bodies, and Δ_1 the distance at this time, we have

$$\Delta_1 = m \sin (M - N) \qquad T_1 = T_0 - \frac{hm}{n} \cos (M - N)$$

II.—CONSTANTS.

For each of the computed values of T take the corresponding values of N and ψ from the preceding computation. Then

$$Q = N + \psi$$

Take the horizontal parallaxes π and π' of the planet and the sun, and compute A and D by the formulæ

$$\tan\gamma = \frac{\pi + \pi'}{\pi - \pi} \cdot \tan\tfrac{1}{2}(s' \pm s) \qquad \begin{aligned} f \sin F &= \sin\gamma \\ f \cos F &= \cos\gamma \cos Q \end{aligned}$$

$$\begin{aligned} \cos D \sin (A - \alpha_0) &= \cos\gamma \sin Q \\ \cos D \cos (A - \alpha_0) &= -f \sin (\delta_0 + F) \\ \sin D &= f \cos (\delta_0 + F) \end{aligned}$$

in which α_0 is the mean of the right ascensions of the planet and sun, and δ_0 the mean of their declinations, at the time T.

Find the sidereal time μ at the first meridian corresponding to T. Then form the three constants

$$\Theta = \mu - A \qquad B = \frac{\pi - \pi'}{n \cos\psi} h \sin D \qquad C = \frac{\pi - \pi'}{n \cos\psi} h \cos D$$

III.—FOR A GIVEN PLACE WHOSE LATITUDE IS φ AND WEST LONGITUDE ω.

Find the values of $\rho \sin\varphi'$ and $\rho \cos\varphi'$ by the geodetic table. The required time of the phenomenon at the place is

$$T' = T + B \cdot \rho \sin\varphi' + C \cdot \rho \cos\varphi' \cos (\Theta - \omega)$$

The local time will be $T' - \omega$. The angle Q will express the angular distance of the point of contact reckoned on the sun's limb from its north point towards the east, and will be very nearly the same for all places on the earth.

EXAMPLE.—Compute the times of ingress and egress for the transit of *Mercury*, November 11, 1861.

I. *For the centre of the earth.*—Let us take as the first meridian that of Washington, and employ the elements given in the American Ephemeris.

The Washington mean time of conjunction in right ascension is November 11, $14^h\ 59^m\ 43^s.6$, which we shall adopt as the value of T_0. For this time we have

$\alpha = \alpha' =$ 227° 31′ 8″.5	☿ Hourly motion in R. A.	$= -$ 3′ 9″.0	
	☉ " " "	$= +$ 2 32 .7	
	a	$= -$ 5 41 .7	$= -341''.7$
☿ $\delta = -$ 17 32 45 .1	" " in Dec.	$= +$ 1 43 .8	
☉ $\delta' = -$ 17 44 44 .6	" " "	$= -$ 0 40 .6	
$\delta_0 = -$ 17 38 44 .9	d	$= +$ 2 24 .4	$= +144''.4$
$\delta - \delta' = +$ 0 11 59 .5			
	☉ Semidiameter s' =	16′ 12″.55	
☿ $\pi =$ 12″.68	☿ " s =	4 .94	
☉ $\pi' =$ 8 .67	For external contacts, $s' + s$ =	16 17 .49	$= 977''.49$

Since for T_0 we have $\alpha = \alpha'$, we also have $M = 0°$, $m = \delta - \delta' = 719''.5$. We then find, by the preceding formulæ,

$\log n =$	2.55170	$T_0 =$	14ʰ 59ᵐ 43ˢ.6
$N = -$	66° 5′.1	$-\dfrac{hm}{n}\cos(M - N) =$	$-$ 49 7 .8
$M - N = +$	66 5 .1		
$\log \sin \psi =$	9.82793	Middle of Transit, $T_1 =$	14 10 35 .8
For Ingress, $\psi =$	137° 42′.7	$h\left(\dfrac{s' + s}{n}\right)\cos\psi = \mp$	2 1 48 .0
For Egress, $\psi =$	42 17 .3		
		Ingress, $T =$	12 8 47 .8
		Egress, $T =$	16 12 23 .8

The least distance of the centres $= m \sin(M - N) = 10' 57''.7$

II. *Constants.*—We find, for both ingress and egress, $\log \tan \gamma = 8.10094$, and then the following quantities:

	Ingress.	Egress.
Q	71° 37′.6	$-$ 23° 47′.8
$\log f$	9.49891	9.96252
F	2° 17′.5	0° 47′.3
δ_0	$-$ 17 40 .3	$-$ 17 38 .1
$\delta_0 + F$	$-$ 15 22 .8	$-$ 16 50 .8
$A - \alpha_0$	84 57 .7	$-$ 56 16 .0
α_0	227 32 .0	227 30 .8
A	312 29 .7	171 14 .8
$\log \sin D$	9.48307	9.94348
$\log \cos D$	9.97892	9.68009
T	12ʰ 8ᵐ 47ˢ.8	16ʰ 12ᵐ 23ˢ.8
Sid. T. Wash. mean noon	15 23 17 .8	15 23 17 .8
	1 59 .7	2 39 .7
μ	3 34 5 .3	7 38 21 .3
μ (in arc)	53° 31′.3	114° 35′.3
$\mu - A = \Theta$	101 1 .6	303 20 .5
$\log B$	n1.2217	1.6821
$\log C$	n1.7176	1.4187

III. For any place on the surface of the earth we have, therefore, in mean Washington time,

$$\begin{array}{llllll} \text{Ingress,} & T' = 12^h\ 8^m\ 47^s.8 & - 16^s.66\,\rho \sin\phi' & - 52^s.19\,\rho\cos\phi' \cos(101^\circ\ 1'.6 - \omega) \\ \text{Egress,} & T' = 16\ \ 12\ \ 23\ .8 & + 48\ .10\,\rho\sin\phi' & + 26\ .23\,\rho\cos\phi'\cos(303\ \ 20.5 - \omega) \end{array}$$

or, in a more convenient form, giving the logarithms of the constant factors,

$$\begin{array}{llll} \text{Ingress,} & T' = 12^h\ 8^m\ 47^s.8 & - [1.2217]\,\rho\sin\phi' & + [1.7176]\,\rho\cos\phi'\cos(\omega + 78^\circ\ 58'.4) \\ \text{Egress,} & T' = 16\ \ 12\ \ 23\ .8 & + [1.6821]\,\rho\sin\phi' & + [1.4187]\,\rho\cos\phi'\cos(\omega + 56\ \ 39\ .5) \end{array}$$

To determine whether the phenomenon is visible at the given place, we have only to determine whether the sun is above the horizon at the computed time. All the places at which it will be visible will be readily found by the aid of an artificial terrestrial globe, by taking that point where the sun is in the zenith at the time T, and describing a great circle from this point as a pole. All places within the hemisphere containing this pole evidently have the sun above the horizon. In the present example this point at ingress is in latitude $-17^\circ\ 43'$ and longitude $186^\circ\ 2'$ west from Washington; and at egress it is in latitude $-17^\circ\ 46'$ and longitude $247^\circ\ 4'$. The whole transit is invisible in the United States, and in Europe only the egress is visible.

For the egress at *Altona*, $\varphi = 53^\circ\ 32'.8$, $\omega = 350^\circ\ 3'.5$, we find

$$\begin{array}{rr} T' = & 16^h\ 13^m\ 13^s.0 \\ \omega = - & 5\ \ 47\ \ 57\ .4 \\ \hline \text{Altona mean time of egress} = & 22\ \ \ 1\ \ 10\ .4 \end{array}$$

The time actually observed by PETERSEN and PAPE was $22^h\ 1^m\ 8^s.5$.* The error of the prediction is very small, and proves the excellence of LE VERRIER's Theory of Mercury, from which the places in the American Ephemeris were derived.

OCCULTATION OF A FIXED STAR BY A PLANET.

359. Very small stars disappear to the eye when near the bright limb of a planet, before they are actually occulted by it; and the occultations of stars of sufficient brightness to be observed at the limb of the planet are so rare that it has not been thought worth while to incur the labor of predicting their oc-

* *Astron. Nach.*, Vol. LVI. p. 239.

currence. But in case such an occultation has been observed at different points on the earth, it may be reduced by Art. 341, substituting the planet for the moon. Such observations would be especially valuable for determining the planet's parallax by a discussion of the equations of condition of the form given on p. 552. If the occultation occurred near the stationary points of the planet, there would be a long interval between the immersion and the emersion; the coefficient of $\Delta\pi$ in the final equations would be proportionally large, and therefore a very accurate determination of this quantity might be expected. If, therefore, means can be found to make the occultation of the smaller stars by a planet a distinctly observable phenomenon, this mode of finding a planet's parallax (and, consequently, also the solar parallax) may become of real practical value.*

It may be added that some advantage might be derived from the occultations of small stars by the dark limb of Venus.

CHAPTER XI.

PRECESSION, NUTATION, ABERRATION, AND ANNUAL PARALLAX OF THE FIXED STARS.

360. I HAVE hitherto treated of those problems only in which the apparent geocentric places of the celestial bodies are supposed to be known; and these have been chiefly problems which may be regarded as arising from the earth's diurnal motion, or in some way modified by it. According to the definition of our subject (Art. 1), Spherical Astronomy embraces also those problems which arise from the earth's annual motion "so far as this affects the apparent positions of the heavenly bodies upon the celestial sphere." I shall therefore proceed now to consider those uranographical corrections, affecting the apparent geocentric places of the stars, which result from the motion of the earth in its orbit, and, consequently, also those which result

* See a paper by PROF. A. C. TWINING, *Enquiries concerning stellar occultations by the moon and planets, &c.*, Am. Journal of Science for July, 1858.

from the changes in the position of the plane of the orbit and the plane of the equator.

361. The variations of astrononomical elements are usually divided into *secular* and *periodic*.

Secular variations are very slow changes, which proceed through ages (*secula*), so that for a number of years, or even centuries in some cases, they are nearly proportional to the time.

Periodic variations are relatively quick changes, which oscillate between their extreme values in so short a period that they cannot be regarded as proportional to the time except for very small intervals.*

The *true position* of a celestial body, or of a celestial plane, at a given time, is that which it actually has at that time; its *mean position* is that which it would have at that time if it were freed from its periodic variations.

362. The plane of the ecliptic, or of the earth's orbit, is a slowly moving plane. Its position at any epoch, as the beginning of the year 1800, can be adopted as a *fixed plane*, to which its position at any other time may be referred.

The plane of the equator is also a moving plane. Its inclination to the fixed plane and the direction of the line in which it intersects that plane are constantly changing, thus causing variations in the obliquity of the ecliptic and in the position of the equinoctial points.

363. The latitudes and declinations of stars are therefore subject to variations which do not arise from the motions of the stars, but from the shifting of the planes of reference; and the longitudes and right ascensions are in like manner subject to variations from the shifting of the vernal equinox, which is their common point of reference, or origin, from which both are reckoned.

Under the head of *precession* are considered those parts of these variations which are *secular;* namely, those which arise from the motions of the *mean ecliptic* and the *mean equator*.

---

* Most of the secular variations also have periods, though of great length, and therefore not yet in all cases well defined: so that, strictly speaking, the distinction between secular and periodic variations is only an arbitrary one, established for practical convenience between variations of *long* and *short* periods.

Under the head of *nutation* are embraced those parts of these variations which are *periodic*, and result from the difference between the motions of the true ecliptic and equator and those of the mean ecliptic and equator.

PRECESSION.

364. *Luni-solar precession.*—It is shown in physical astronomy that the attraction of the sun and moon upon that portion of the matter of our globe which is accumulated about the equator, and by which its figure is rendered spheroidal, combined with the rotation of the earth on its axis, continually shifts the position of the plane of the equator (without, however, changing its inclination to the plane of the fixed ecliptic). The line of the equinoxes, or the intersection of the two planes, is thus caused to revolve slowly in the plane of the ecliptic in a direction opposite to that in which longitudes are reckoned; the result of which is a common annual increase in the longitudes of all the stars, reckoned on the fixed ecliptic by a quantity which is called the *luni-solar precession.*

The luni-solar precession is, then, the effect of a motion of *the equator upon the ecliptic.*

365. *Planetary precession.*—The mutual attraction between the planets and the earth tends continually to draw the earth out of the plane in which it is revolving; that is, to change the position of the plane of the orbit, but without changing the position of the earth's equator. The equator here being regarded as fixed, and the ecliptic as moving, the effect is a revolution of the line of intersection, or of the equinoxes, *in the plane of the equator*, in a direction which is the same as that in which right ascensions are reckoned. There is thus caused a common annual decrease in the right ascensions of all the stars, which is called the *planetary precession.*

The *planetary precession* is, then, the effect of a motion of *the ecliptic upon the equator.*

366. The luni-solar precession does not affect the latitudes of stars; but since it changes their longitudes it must also change both their right ascensions and declinations (Art. 26). The planetary precession does not affect the declination of stars, but

changes their right ascensions, their longitudes, and their latitudes (Art. 23).

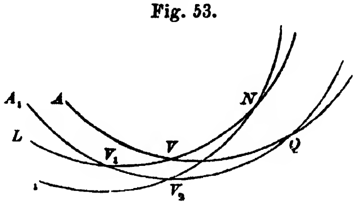
367. *Obliquity of the ecliptic.*—Since by the mutual action of the planets the position of the plane of the (mean) ecliptic is changed while that of the equator remains fixed, the mutual inclination of these planes, or the obliquity of the ecliptic, is changed.

The action of the sun and moon in causing luni-solar precession does not directly produce any change in the obliquity of the ecliptic; but, in consequence of the change produced by the planets, the attraction of the sun and moon is modified: so that there results an additional very minute change of the inclination of the mean equator to the fixed plane of reference.

These changes produce small changes in the co-ordinates of the stars, which, being secular in their character, are combined with the preceding in deducing the general precession.

368. *To find the general precession in longitude, and the position of the mean ecliptic, at a given time.*—Let NL, Fig. 53, be the fixed ecliptic, or the mean ecliptic at the beginning of the year 1800; AQ, the mean equator, and V the mean vernal equinox, or, as it is briefly called, the mean equinox, of 1800. In the figure, let the longitudes be reckoned from V towards N. Let VV_1 be the luni-solar precession in longitude in the time t, and A_1Q the mean equator at the time $1800 + t$. By the action of the planets, the ecliptic in the same time is moved into the position NL_1: so that V_1V_2 is the planetary precession in the time t, and V_2 is the mean equinox at the time $1800 + t$.

Fig. 53.



The point N may be called the *ascending node* of the mean ecliptic on the fixed ecliptic.

The difference between NV and NV_2 is called the *general precession in longitude*, being that part of the change of the longitudes of the stars which is common to all of them.

Now, let us put

$\varepsilon_0 =$ the mean obliquity of the ecliptic for 1800,
$\quad = NVQ,$

$\varepsilon_1 =$ the obliquity of the fixed ecliptic at the time $1800 + t$,
$\quad = NV_1Q,$

ε = the mean obliquity of the ecliptic at the time 1800 + t,
= NV_2Q,

ϑ = the planetary precession in the interval t,
= V_1V_2,

ψ = the luni-solar precession in the interval t,
= VV_1,

ψ_1 = the general precession in the interval t,
= $NV_2 - NV$,

Π = the longitude of the ascending node of the mean ecliptic at the time 1800 + t, reckoned on the fixed ecliptic from the mean equinox of 1800,
= VN,

π = the inclination of the mean ecliptic to the fixed ecliptic at the time 1800 + t,
= V_1NV_2.

The first five of these quantities will be here assumed as known from the investigations of physical astronomers. The following are their values, according to STRUVE and PETERS,* for the epoch 1800:

$$\left.\begin{aligned}
\varepsilon_0 &= 23°\ 27'\ 54''.22 \\
\varepsilon_1 &= \varepsilon_0 + 0''.00000735\,t^2 \\
\varepsilon &= \varepsilon_0 - 0''.4738\,t - 0''.0000014\,t^2 \\
\vartheta &= 0''.15119\,t - 0''.00024186\,t^2 \\
\psi &= 50''.3798\,t - 0''.0001084\,t^2
\end{aligned}\right\} \quad (646)$$

from which we can find ψ_1, Π, and π, as follows. In the triangle NV_1V_2 we have

$$\begin{aligned}
\pi &= V_1NV_2 & \vartheta &= V_1V_2 \\
180° - \varepsilon &= NV_2V_1 & \Pi + \psi &= NV_1 \\
\varepsilon_1 &= NV_1V_2 & \Pi + \psi_1 &= NV_2
\end{aligned}$$

and hence, by the GAUSSIAN equations [Sph. Trig. (44)]

---

* DR. C. A. F. PETERS, *Numerus Constans Nutationis*, pp. 66 et 71. The observations at Dorpat give 0''.4645 for the annual diminution of the obliquity, and this is adopted in the American Ephemeris instead of 0''.4738 which results from theory and is subject to an error in the estimated mass of Venus. The difference, however, is so small that either number will serve to represent the actually observed obliquity for half a century within 0''.5.

I have here adopted the precession constant (50''.3798) given by PETERS, rather for the convenience of the reader (this being employed in the English and American Almanacs) than on account of its superior accuracy. Recent researches rather confirm BESSEL'S constant (50''.36354). See MADLER'S *Die Eigenbewegungen der Fixsterne*, Dorpat, 1856, p. 11.

$$\left.\begin{aligned}\cos\tfrac{1}{2}\pi\sin\tfrac{1}{2}(\psi-\psi_1)&=\sin\tfrac{1}{2}\vartheta\cos\tfrac{1}{2}(\varepsilon+\varepsilon_1)\\ \cos\tfrac{1}{2}\pi\cos\tfrac{1}{2}(\psi-\psi_1)&=\cos\tfrac{1}{2}\vartheta\cos\tfrac{1}{2}(\varepsilon-\varepsilon_1)\\ \sin\tfrac{1}{2}\pi\sin(\Pi+\tfrac{1}{2}\psi+\tfrac{1}{2}\psi_1)&=\sin\tfrac{1}{2}\vartheta\sin\tfrac{1}{2}(\varepsilon+\varepsilon_1)\\ \sin\tfrac{1}{2}\pi\cos(\Pi+\tfrac{1}{2}\psi+\tfrac{1}{2}\psi_1)&=\cos\tfrac{1}{2}\vartheta\sin\tfrac{1}{2}(\varepsilon-\varepsilon_1)\end{aligned}\right\}\quad(647)$$

The angles $\frac{1}{2}\vartheta$ and $\frac{1}{2}(\varepsilon-\varepsilon_1)$ are so small that their cosines may always be put equal to unity, and, consequently, also those of $\frac{1}{2}\pi$ and $\frac{1}{2}(\psi-\psi_1)$; while for their sines we may substitute the arcs. We thus obtain at once, from the first two equations,

$$\psi-\psi_1=\vartheta\cos\tfrac{1}{2}(\varepsilon+\varepsilon_1)$$

where we can take, with sufficient accuracy,

$$\begin{aligned}\cos\tfrac{1}{2}(\varepsilon+\varepsilon_1)&=\cos(\varepsilon_0-0''.2369\,t)\\ &=\cos\varepsilon_0+0''.2369\,t\sin 1''\sin\varepsilon_0\end{aligned}$$

and hence, by substituting the values of ϑ and ε_0 from (646),

$$\begin{aligned}\psi-\psi_1&=0''.1387\,t-0''.0002218\,t^2\\ \psi_1&=50''.2411\,t+0''.0001134\,t^2\end{aligned}\qquad(648)$$

The sum of the squares of the last two equations of (647) gives

$$\pi^2=\vartheta^2\sin^2\tfrac{1}{2}(\varepsilon+\varepsilon_1)+(\varepsilon-\varepsilon_1)^2$$

in which we may take

$$\sin^2\tfrac{1}{2}(\varepsilon+\varepsilon_1)=\sin^2\varepsilon_0-0''.2369\,t\sin 1''\sin 2\varepsilon_0$$

and then, substituting the values of ϑ, ε_0, and $\varepsilon-\varepsilon_1$, we obtain

$$\pi^2=0''.228111\,t^2-0''.0000033234\,t^3$$

and, by extracting the root,

$$\pi=0''.4776\,t-0''.0000035\,t^2\qquad(649)$$

The quotient of the third equation of (647) divided by the fourth gives

$$\tan(\Pi+\tfrac{1}{2}\psi+\tfrac{1}{2}\psi_1)=\frac{\vartheta}{\varepsilon-\varepsilon_1}\sin\tfrac{1}{2}(\varepsilon+\varepsilon_1)$$

in which we have

$$\begin{aligned}\frac{\vartheta}{\varepsilon-\varepsilon_1}&=\frac{0.15119\,t-0.00024186\,t^2}{-0.4738\,t-0.00000875\,t^2}\\ &=-0.3191+0.00051636\,t\end{aligned}$$

and

$$\sin \tfrac{1}{2}(\varepsilon + \varepsilon_1) = \sin \varepsilon_0 - 0''.2369t \sin 1'' \cos \varepsilon_0$$

whence

$$\tan(\Pi + \tfrac{1}{2}\psi + \tfrac{1}{2}\psi_1) = -0.127062 + 0.00020595t$$

If, then, we put

$$\tan \Pi_0 = -0.127062$$

or

$$\Pi_0 = 172^\circ\ 45'\ 31''$$

and also

$$\Pi_1 = \Pi + \tfrac{1}{2}\psi + \tfrac{1}{2}\psi_1$$

we have

$$\tan \Pi_1 - \tan \Pi_0 = (\Pi_1 - \Pi_0) \sin 1'' \sec^2 \Pi_0 = 0.00020595\,t$$

$$\Pi_1 - \Pi_0 = \frac{0.00020595\,t \cos^2 \Pi_0}{\sin 1''} = 41''.805t$$

whence

$$\Pi_1 = \Pi + \tfrac{1}{2}\psi + \tfrac{1}{2}\psi_1 = 172^\circ\ 45'\ 31'' + 41''.805t$$

and, subtracting from this the quantity

$$\tfrac{1}{2}\psi + \tfrac{1}{2}\psi_1 = 50''.310t$$

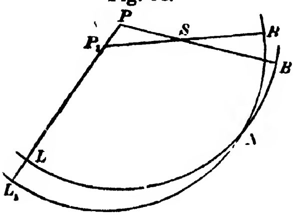
we have, finally,

$$\Pi = 172^\circ\ 45'\ 31'' - 8''.505t \tag{650}$$

The equation (648) determines the general precession, and (649) and (650) the position of the mean ecliptic.

369. *To find the precession in longitude and latitude of a given star, from the epoch* 1800.—Let LNB (Fig. 54) be the fixed ecliptic of 1800; L_1NB_1 the mean ecliptic at the given time $1800 + t$; P and P_1 the poles of these circles respectively. The node N is the pole of the great circle PP_1L_1 joining P and P_1. Let S be the star, and put

Fig. 54.



$L =$ the star's given mean longitude for 1800, reckoned from the mean equinox of that year,

$B =$ the star's given mean latitude for 1800,

$\lambda, \beta =$ the mean longitude and latitude for $1800 + t$.

We have in the figure (as in Fig. 53)

$$NB = L - \Pi \qquad SB = B$$

$$NB_1 = \lambda - \Pi - \psi_1 \qquad SB_1 = \beta$$

and in the triangle PSP_1 we have

$$\begin{aligned}
PP_1 &= LL_1 = LNL_1 = \pi \\
PS &= 90^\circ - B \\
P_1S &= 90^\circ - \beta \\
SPP_1 &= BL = 90^\circ + L - \Pi \\
SP_1P &= 180^\circ - L_1B_1 = 180^\circ - (90^\circ + \lambda - \Pi - \psi_1) \\
&= 90^\circ - (\lambda - \Pi - \psi_1)
\end{aligned}$$

so that, by the fundamental equations of Sph. Trig.,

$$\left.\begin{aligned}
\cos\beta\cos(\lambda - \Pi - \psi_1) &= \cos B\cos(L - \Pi) \\
\cos\beta\sin(\lambda - \Pi - \psi_1) &= \cos B\sin(L - \Pi)\cos\pi + \sin B\sin\pi \\
\sin\beta &= -\cos B\sin(L - \Pi)\sin\pi + \sin B\cos\pi
\end{aligned}\right\} \quad (651)$$

Instead of these rigorous formulæ, we may deduce approximate ones, which will be sufficient in all practical cases, as follows. Neglecting the square of π (that is, putting $\cos\pi = 1$), let the first equation be multiplied by $\sin(L - \Pi)$, the second by $\cos(L - \Pi)$; the difference of the products is

$$\cos\beta\sin(\lambda - L - \psi_1) = \sin\pi\sin B\cos(L - \Pi)$$

The sum of the products obtained by multiplying the same equations by $\cos(L - \Pi)$ and $\sin(L - \Pi)$, respectively, is

$$\cos\beta\cos(\lambda - L - \psi_1) = \cos B + \sin\pi\sin B\sin(L - \Pi)$$

and the quotient of these last equations is

$$\tan(\lambda - L - \psi_1) = \frac{\sin\pi\tan B\cos(L - \Pi)}{1 + \sin\pi\tan B\sin(L - \Pi)}$$

which developed in series (Pl. Trig., Art. 257) gives

$$\lambda - L - \psi_1 = \pi\tan B\cos(L - \Pi) - \tfrac{1}{2}\pi^2\tan^2 B\sin 2(L - \Pi) - \text{\&c.}$$

where, however, since we here neglect the square of π, the first term of the series suffices: so that we have

$$\lambda - L = \psi_1 + \pi\tan B\cos(L - \Pi) \qquad (652)$$

Here ψ_1 appears as the precession in longitude common to all the stars, and the term $\pi\tan B\cos(L - \Pi)$ as that which varies with the star.

The last equation of (651) gives

$$\sin\beta - \sin B = -\sin\pi\cos B\sin(L - \Pi)$$

whence, neglecting π^2 as before,

$$\beta - B = -\pi \sin(L - \Pi) \tag{653}$$

The values of ψ_1, π, and Π being found for the time $1800 + t$, by means of (648), (649), and (650), the formulæ (652) and (653) determine the required precession in the longitude and latitude, and, consequently, also the mean place of the star for the given date.

370. *To find the precession in longitude and latitude between any two given dates.*—Suppose λ and β are given for $1800 + t$, and λ' and β' are required for $1800 + t'$. Denoting by L and B the longitude and latitude for 1800, we shall have, by (652),

$$\begin{aligned} \lambda - L &= \psi_1 + \pi \tan B \cos(L - \Pi) \\ \lambda' - L &= \psi_1' + \pi' \tan B \cos(L - \Pi') \end{aligned}$$

where ψ_1', π', Π' are the quantities given by (648), (649), and (650) when t' is substituted for t. If we subtract the first of these equations from the second, and at the same time introduce the auxiliaries a and A, determined by the conditions

$$\begin{aligned} a \sin A &= (\pi' + \pi) \sin \tfrac{1}{2}(\Pi' - \Pi) \\ a \cos A &= (\pi' - \pi) \cos \tfrac{1}{2}(\Pi' - \Pi) \end{aligned}$$

we find

$$\lambda' - \lambda = \psi_1' - \psi_1 + a \cos\left(L - \frac{\Pi' + \Pi}{2} - A\right) \tan B$$

and in the same manner, from (653),

$$\beta' - \beta = -a \sin\left(L - \frac{\Pi' + \Pi}{2} - A\right)$$

For the values of A and a we have

$$\tan A = \frac{\pi' + \pi}{\pi' - \pi} \tan \tfrac{1}{2}(\Pi' - \Pi) = \frac{t' + t}{t' - t} \tan \tfrac{1}{2}(\Pi' - \Pi)$$

or, by (650),

$$A = \left(\frac{t' + t}{2}\right) \frac{\Pi' - \Pi}{t' - t} = -8''.505 \left(\frac{t' + t}{2}\right)$$

so that $\cos A$ may be put equal to unity, and therefore we have

$$a = \pi' - \pi$$

We may also put $\tan\beta$ instead of $\tan B$ in the above formulæ, since the error in $\lambda' - \lambda$ thus produced will be only a term in π^2; and for L we may take $\lambda - \psi_1$: so that if we put

$$L - \frac{\Pi' + \Pi}{2} - A = \lambda - M$$

and then substitute the numerical values of our constants, we shall have the following formulæ for computing the precession from $1800 + t$ to $1800 + t'$:

$$\left.\begin{aligned}
M &= \quad 172^\circ\,45'\,31'' + t\,.\,50''.241 - (t' + t)\,8''.505 \\
\lambda' - \lambda &= \quad (t' - t)\,[50''.2411 + (t' + t)\,0''.0001134] \\
&\quad + (t' - t)\,[0''.4776 - (t' + t)\,0''.0000035]\cos(\lambda - M)\tan\beta \\
\beta' - \beta &= -(t' - t)\,[0''.4776 - (t' + t)\,0''.0000035]\sin(\lambda - M)
\end{aligned}\right\} \quad (654)$$

These are the same as Bessel's formulæ in the *Tabulæ Regiomontanæ*, except that we have here employed the constants given by Peters, and the epoch to which t and t' are referred is 1800.

To find the annual precession in longitude for a given date.—If we divide the equations (654) by $t' - t$, the quotients

$$\frac{\lambda' - \lambda}{t' - t}, \qquad \frac{\beta' - \beta}{t' - t}$$

will express the mean annual precession between the two dates; and if we then suppose t' and t to differ by an infinitesimal quantity, or put $t' = t$, these quotients will become the differential coefficients which express the annual precession for the instant $1800 + t$; namely,

$$\left.\begin{aligned}
\frac{d\lambda}{dt} &= \quad 50''.2411 + 0''.0002268t \\
&\quad + [0''.4776 - 0.0000070t]\cos(\lambda - M)\tan\beta \\
\frac{d\beta}{dt} &= -[0''.4776 - 0.0000070t]\sin(\lambda - M) \\
&\text{in which} \\
M &= \quad 172^\circ\,45'\,31'' + 33''.23t
\end{aligned}\right\} \quad (655)$$

Example.—For the star *Spica*, we have, for the beginning of the year 1800,

the mean longitude,	$L =$	$201^\circ\ 3'\ 5''.97$
the mean latitude,	$B = -$	$2^\circ\ 2'\ 22''.64$

Find its mean longitude and latitude for the beginning of the year 1860.

First. By the direct formulæ (652) and (653).—We find, by (648), (649), and (650), for $t = 60$,

$$\psi_1 = 50' \, 14''.874$$
$$\pi = 28''.6434$$
$$\Pi = 172^\circ \, 37' \, 1''$$

whence

$$L - \Pi = 28^\circ \, 26' \, 5''$$
$$\pi \tan B \cos (L - \Pi) = - \quad 0''.897$$
$$\pi \sin (L - \Pi) = + 13''.639$$

and hence, by (652) and (653), the precession is

$$\lambda - L = 50' \, 14''.874 - 0''.897 = 50' \, 13''.977$$
$$\beta - B = - 13''.639$$

and the mean longitude and latitude for 1860.0 are

$$\lambda = \quad 201^\circ \, 53' \, 20''.95$$
$$\beta = - \quad 2^\circ \, 2' \, 36''.28$$

Second. By the use of the annual precession.—The mean annual precession for the sixty years from 1800 to 1860 is the annual precession for 1830. Hence, by taking $t = 30$ in (655), and denoting by λ_0 and β_0 the longitude and latitude for 1830,

$$\frac{d\lambda}{dt} = \quad 50''.2479 + 0''.4774 \cos (\lambda_0 - M) \tan \beta_0$$
$$\frac{d\beta}{dt} = - 0''.4774 \sin (\lambda_0 - M)$$
$$M = \quad 173^\circ \, 2' \, 8''.$$

To compute these, we can employ approximate values of λ_0 and β_0, found by adding the general precession for thirty years to L, and neglecting the terms in π; namely,

$$\lambda_0 = 201^\circ \, 28'.2 \qquad \beta_0 = - 2^\circ \, 2'.6$$

and hence $\lambda_0 - M = 28^\circ \, 26'.1$,

$$\frac{d\lambda}{dt} = 50''.2329 \qquad \frac{d\beta}{dt} = - 0''.2274$$

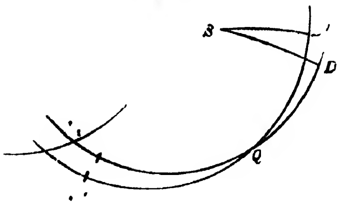
These multiplied by 60 give the whole precession from 1800 to 1860,

$$\lambda - L = 50' \, 13''.97 \qquad \beta - B = - 13''.64$$

agreeing with the values found above.

371. *Given the mean right ascension and declination of a star for any date* 1800 + t, *to find the mean right ascension and declination for any other date* 1800 + t'.—Let $V_1 V_1'$ (Fig. 55) be the fixed ecliptic of 1800, $V_1 Q$ the mean equator of 1800 + t, $V_1' Q$ the mean equator of 1800 + t', Q the intersection of these circles (or the ascending node of the second upon the first). The position of the point Q is found as follows. The arc $V_1 V_1'$ is the luni-solar precession for the interval $t' - t$: so that, distinguishing by accents the quantities obtained by (646) when t' is put for t, we have, in the triangle $Q V_1 V_1'$,

Fig. 55.



$$V_1 V_1' = \psi' - \psi, \qquad Q V_1 V_1' = 180^\circ - \varepsilon_1, \qquad Q V_1' V_1 = \varepsilon_1',$$

and putting

$$Q V_1 = 90^\circ - z, \qquad Q V_1' = 90^\circ + z', \qquad V_1 Q V_1' = \Theta,$$

we find, by GAUSS's equations of Sph. Trig.,

$$\left.\begin{aligned}
\cos \tfrac{1}{2}\Theta \sin \tfrac{1}{2}(z' + z) &= \sin \tfrac{1}{2}(\psi' - \psi) \cos \tfrac{1}{2}(\varepsilon_1' + \varepsilon_1) \\
\cos \tfrac{1}{2}\Theta \cos \tfrac{1}{2}(z' + z) &= \cos \tfrac{1}{2}(\psi' - \psi) \cos \tfrac{1}{2}(\varepsilon_1' - \varepsilon_1) \\
\sin \tfrac{1}{2}\Theta \sin \tfrac{1}{2}(z' - z) &= \cos \tfrac{1}{2}(\psi' - \psi) \sin \tfrac{1}{2}(\varepsilon_1' - \varepsilon_1) \\
\sin \tfrac{1}{2}\Theta \cos \tfrac{1}{2}(z' - z) &= \sin \tfrac{1}{2}(\psi' - \psi) \sin \tfrac{1}{2}(\varepsilon_1' + \varepsilon_1)
\end{aligned}\right\} \quad (656)$$

which determine Θ, z, and z' in a rigorous manner. But, since $\frac{1}{2}(\varepsilon_1' - \varepsilon_1)$ is exceedingly small, we can always put unity for its cosine, and the arc for the sine, and, consequently, the same may be done in the case of the arc $\frac{1}{2}(z' - z)$; we thus obtain the following simple but accurate formulæ:

$$\left.\begin{aligned}
\tan \tfrac{1}{2}(z' + z) &= \tan \tfrac{1}{2}(\psi' - \psi) \cos \tfrac{1}{2}(\varepsilon_1' + \varepsilon_1) \\
\tfrac{1}{2}(z' - z) &= \frac{\tfrac{1}{2}(\varepsilon_1' - \varepsilon_1)}{\tan \tfrac{1}{2}(\psi' - \psi) \sin \tfrac{1}{2}(\varepsilon_1' + \varepsilon_1)} \\
\sin \tfrac{1}{2}\Theta &= \sin \tfrac{1}{2}(\psi' - \psi) \sin \tfrac{1}{2}(\varepsilon_1' + \varepsilon_1)
\end{aligned}\right\} \quad (657)$$

If V_2 and V_2' are the positions of the mean equinox in 1800 + t and 1800 + t', $V_1 V_2$ is the planetary precession for the first and $V_1' V_2'$ that for the second of these times, which being denoted by ϑ and ϑ' we have

$$V_2 Q = 90^\circ - z - \vartheta$$
$$V_2' Q = 90^\circ + z' - \vartheta'$$

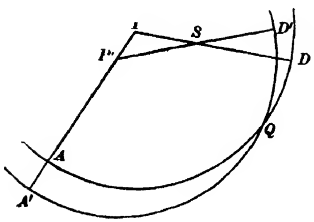
If then we put

α, δ = the given mean right ascension and declination of a star S, for $1800 + t$,
α', δ' = those required for $1800 + t'$,

we have $\alpha = V_2 D$, and $\alpha' = V_2' D'$, and consequently,

$$QD = V_2 D - V_2 Q = \alpha + z + \vartheta - 90°,$$
$$QD' = V_2' D' - V_2' Q = \alpha' - z' + \vartheta' - 90°,$$

Fig. 56.



Now, let P and P' (Fig. 56) be the poles of the equator at the times $1800 + t$, $1800 + t'$, AQD, $A'QD'$, the two positions of the equator at these times, as in Fig. 55; S the star. Q is the pole of the great circle $PP'A'$ joining the poles P and P', and, therefore, $PP' = AA' = AQA' = \Theta$, and in the triangle $PP'S$ we have

$$PS = 90° - \delta, \qquad P'S = 90° - \delta', \qquad PP' = \Theta$$
$$SPP' = \qquad AD = 90° + QD = \qquad \alpha + z + \vartheta$$
$$SP'P = 180° - A'D' = 90° - QD' = 180° - (\alpha' - z' + \vartheta')$$

Hence, by the fundamental equations of Spherical Trigonometry,

$$\left.\begin{aligned} \cos\delta' \sin(\alpha' - z' + \vartheta') &= \cos\delta \sin(\alpha + z + \vartheta) \\ \cos\delta' \cos(\alpha' - z' + \vartheta') &= \cos\delta \cos(\alpha + z + \vartheta) \cos\Theta - \sin\delta \sin\Theta \\ \sin\delta' &= \cos\delta \cos(\alpha + z + \vartheta) \sin\Theta + \sin\delta \cos\Theta \end{aligned}\right\} \quad (658)$$

We have thus a rigorous and direct solution of our problem by finding, first, Θ, z, and z' from (656), and hence α' and δ' by (658), employing the values of ε, ψ, ϑ for the time $1800 + t$, and of ε', ψ', ϑ' for the time $1800 + t'$, as given by (646) for the two dates.

372. The formulæ (658) may be adapted for logarithmic computation by the introduction of an auxiliary angle in the usual manner; or we may employ the GAUSSIAN equations, which, if we denote the angle at the star by C, and for the sake of brevity put

$$A = \alpha + z + \vartheta \qquad A' = \alpha' - z' + \vartheta' \qquad (659)$$

give

$$\cos \tfrac{1}{2}(90° + \delta') \sin \tfrac{1}{2}(A'+C) = \cos \tfrac{1}{2}(90° + \delta - \Theta) \sin \tfrac{1}{2} A$$
$$\cos \tfrac{1}{2}(90° + \delta') \cos \tfrac{1}{2}(A'+C) = \cos \tfrac{1}{2}(90° + \delta + \Theta) \cos \tfrac{1}{2} A$$
$$\sin \tfrac{1}{2}(90° + \delta') \sin \tfrac{1}{2}(A'-C) = \sin \tfrac{1}{2}(90° + \delta - \Theta) \sin \tfrac{1}{2} A$$
$$\sin \tfrac{1}{2}(90° + \delta') \cos \tfrac{1}{2}(A'-C) = \sin \tfrac{1}{2}(90° + \delta + \Theta) \cos \tfrac{1}{2} A$$

373. We may, however, obtain greater precision by computing the differences between A and A' and between δ and δ'. From the first two equations of (658) we deduce

$$\cos \delta' \sin (A'-A) = \cos \delta \sin A \sin \Theta \left[\tan \delta + \tan \tfrac{1}{2}\Theta \cos A\right]$$
$$\cos \delta' \cos (A'-A) = \cos \delta - \cos \delta \cos A \sin \Theta \left[\tan \delta + \tan \tfrac{1}{2}\Theta \cos A\right]$$

so that, if we put

$$p = \sin \Theta (\tan \delta + \tan \tfrac{1}{2} \Theta \cos A)$$

we have

$$\tan (A' - A) = \frac{p \sin A}{1 - p \cos A} \qquad (660)$$

and, by NAPIER's Analogy,*

$$\tan \tfrac{1}{2}(\delta' - \delta) = \tan \tfrac{1}{2}\Theta \, \frac{\cos \tfrac{1}{2}(A' + A)}{\cos \tfrac{1}{2}(A' - A)}$$

EXAMPLE.—The mean place of *Polaris* for 1755, according to the *Tabulæ Regiomontanæ*, is

$$\alpha = 10° \; 55' \; 44''.955 \qquad \delta = 87° \; 59' \; 41''.12$$

it is required to reduce this place to the mean equator and equinox of 1820.

For 1755 we take $t = -45$; and for 1820, $t' = +20$; and, by (646), we find—

For 1755.	For 1820.
$\psi = -37' \; 47''.31$	$\psi' = +16' \; 47''.55$
$\vartheta = \quad - 7''.29$	$\vartheta' = \quad + 2''.93$
$\varepsilon_1 = 23° \; 27' \; 54''.23488$	$\varepsilon_1' = 23° \; 27' \; 54''.22294$

and hence

$$\tfrac{1}{2}(\psi' - \psi) = 27' \; 17''.43$$
$$\tfrac{1}{2}(\varepsilon_1' - \varepsilon_1) = - \; 0''.00597$$
$$\tfrac{1}{2}(\varepsilon_1' + \varepsilon_1) = 23° \; 27' \; 54''.23$$

* The formulæ (657), (658), (659), (660) are those given by BESSEL in the *Tabulæ Regiomontanæ*.

with which the formulæ (657) give

$$\begin{aligned} \tfrac{1}{2}(z'+z) &= +\ 25'\ 2''.02 \\ \tfrac{1}{2}(z'-z) &= \quad -\ 1''.89 \\ z &= +\ 25'\ 3''.91 \\ z' &= +\ 25'\ 0''.13 \\ \log\sin\tfrac{1}{2}\Theta &= 7.499823 \end{aligned}$$

Then, by the formulæ (660), we find

$$A = \alpha + z + \vartheta = 11^\circ\ 20'\ 41''.57$$

$\log p$	9.256676	$\log\tan\frac{1}{2}\Theta$	7.499825
$\log\sin A$	9.293836	$\log\cos\frac{1}{2}(A'+A)$	9.989446
$\log\cos A$	9.991430	$\log\sec\frac{1}{2}(A'-A)$	0.000101
$\log p\cos A$	9.248106	$\log\tan\frac{1}{2}(\delta'-\delta)$	9.489372
ar. co. $\log(1-p\cos A)$	0.084629		
$\log\tan(A'-A)$	8.635141		

$$A'-A = 2^\circ\ 28'\ 18''.08 \qquad \delta'-\delta = 21'\ 12''.99$$
$$A' = 13^\circ\ 48'\ 59''.65$$
$$\alpha' = A' + z' - \vartheta' = 14^\circ\ 13'\ 56''.85 \qquad \delta' = 88^\circ\ 20'\ 54''.11$$

374. *To find the annual precession in right ascension and declination.*—In computing the precession for a single year, the square of Θ becomes insensible, and we may take, instead of (660), the approximate formula

$$A'-A = \alpha'-\alpha-(z'+z)+\vartheta'-\vartheta = \Theta\sin\alpha\tan\delta$$

and from (657) we then have, with sufficient accuracy,

$$\begin{aligned} z'+z &= (\psi'-\psi)\cos\varepsilon_1 \\ \Theta &= (\psi'-\psi)\sin\varepsilon_1 \end{aligned}$$

Substituting these values in the above formula, and then dividing by $t'-t$, we have

$$\frac{\alpha'-\alpha}{t'-t} = \frac{\psi'-\psi}{t'-t}\cos\varepsilon_1 - \frac{\vartheta'-\vartheta}{t'-t} + \frac{\psi'-\psi}{t'-t}\sin\varepsilon_1\sin\alpha\tan\delta$$

which gives the *annual* precession between the times $1800+t$ and $1800+t'$, the unit of time being one year. But, in order that the formula may express the rate of change at the instant $1800+t$, we must suppose the interval $t'-t$ to become infinitely small; that is, we must write the formula thus:

$$\frac{d\alpha}{dt} = \frac{d\psi}{dt} \cos \epsilon_1 - \frac{d\vartheta}{dt} + \frac{d\psi}{dt} \sin \epsilon_1 \sin \alpha \tan \delta$$

and similarly, from the last equation of (660),

$$\frac{d\delta}{dt} = \frac{d\psi}{dt} \sin \epsilon_1 \cos \alpha$$

Putting then

$$\left.\begin{aligned} m &= \frac{d\psi}{dt} \cos \epsilon_1 - \frac{d\vartheta}{dt} \\ n &= \frac{d\psi}{dt} \sin \epsilon_1 \end{aligned}\right\} \quad (661)$$

we find, by (646),

$$\begin{aligned} \frac{d\psi}{dt} \cos \epsilon_1 &= (50''.3798 - 0''.0002168\, t) \cos \epsilon_1 \\ &= 46''.2135 - 0''.00019887\, t \\ \frac{d\vartheta}{dt} &= 0''.1512 - 0''.00048372\, t \end{aligned}$$

and hence

$$\left.\begin{aligned} m &= 46''.0623 + 0''.0002849\, t \\ n &= 20''.0607 - 0''.0000863\, t \end{aligned}\right\} \quad (662)$$

and the annual precession in right ascension and declination for the time $1800 + t$ is found by the formulæ

$$\left.\begin{aligned} \frac{d\alpha}{dt} &= m + n \sin \alpha \tan \delta \\ \frac{d\delta}{dt} &= n \cos \alpha \end{aligned}\right\} \quad (663)$$

These formulæ may be used for computing the whole precession between any two dates, if we multiply the annual precession at the *middle time* between the two dates by the number of years in the interval.

Example.—The mean right ascension and declination of *Spica* for 1800 are, by the *Tabulæ Regiomontanæ*,

$$\begin{aligned} \alpha &= 13^h\, 14^m\, 40^s.5057 \\ \delta &= -10^\circ\ 6'\ 46''.843 \end{aligned}$$

Find the mean right ascension (α') and declination (δ') for 1860

We have, for 1830, by making $t = 30$ in (662),

$$m = 46''.0708 \qquad n = 20''.0581$$

and, for a first approximation, taking $\alpha' = \alpha$, $\delta' = \delta$, we have, by (663),

$$\frac{d\alpha}{dt} = +\ 47''.22 \qquad \frac{d\delta}{dt} = -\ 19''.00$$

The approximate precession for sixty years is, therefore,

$$\text{in R. A.,} + 2833'' = +\ 188^s.9 \qquad \text{in dec.,} - 1140''$$

which, applied to α and δ, give the approximate values for 1860,

$$\alpha' = 13^h\ 17^m\ 49^s.4 \qquad \delta' = -\ 10°\ 25'\ 47''$$

The means between these values and those of α and δ are

$$\alpha_0 = 13^h\ 16^m\ 15^s. \qquad \delta_0 = -\ 10°\ 16'\ 17''$$

which being employed in (663) give the more correct annual precession for 1830,

$$\frac{d\alpha}{dt} = +\ 47''.2579 \qquad \frac{d\delta}{dt} = -\ 18''.9582$$

The true precession for sixty years is then

$$\text{in R. A.,} + 2835''.474 = 3^m\ 9^s.0316, \qquad \text{in dec.,} - 18'\ 57''.492,$$

which applied to α and δ give the mean place for 1860,

$$\alpha' = 13^h\ 17^m\ 49^s.5373 \qquad \delta' = -\ 10°\ 25'\ 44''.335$$

and these values agree almost precisely with those found by the rigorous method of Art. 371.

375. *To find the position of the pole of the equator at a given time.*— The precession causes the pole of the equator to revolve about the pole of the ecliptic (nearly) in a small circle whose polar distance is equal to the obliquity of the ecliptic. The time in which the pole will make a complete revolution and return to the same position (small changes in the obliquity of the ecliptic not considered) is the value of t given by the equation

$$50''.2411\,t + 0''.0001134\,t^2 = 360° \times 60 \times 60 = 1296000''$$

which gives

$$t = 24447 \text{ years;}$$

or, in round numbers, since the precession is not known with

sufficient precision to determine so great a period exactly, $t = 24500$ years.

To find the position of the pole for any indeterminate time $1800 + t'$, we have only to observe that if P, in Fig. 56, is the pole for a fixed time $1800 + t$, P' that for the time $1800 + t'$, the right ascension of P', reckoned from the equinox of $1800 + t$, is equal to that of the point Q diminished by 90°. The right ascension of Q is V_2Q in Fig. 55, and, in Art. 371, we have found

$$V_2Q = 90^\circ - z - \vartheta$$

Hence, if we put

$A, D =$ the right ascension and declination of the pole at the time $1800 + t'$, referred to the equator and equinox of $1800 + t$,

we have

$$A = -z - \vartheta$$
$$D = 90^\circ - \Theta$$

which will become known by computing ψ, ψ', ε, ε', ϑ for the times $1800 + t$, $1800 + t'$, and then z and Θ by (657)

An approximate solution is obtained by neglecting the variation of ε, and, consequently, taking $z' = z$, and also neglecting ϑ: so that

$$\left.\begin{aligned} \tan A &= -\tan \tfrac{1}{2}(\psi' - \psi)\cos \varepsilon_0 \\ \sin(45^\circ - \tfrac{1}{2}D) &= \sin \tfrac{1}{2}(\psi' - \psi)\sin \varepsilon_0 \end{aligned}\right\} \quad (664)$$

The ambiguity in determining A by its tangent is removed by observing that $\cos A$ and $\cos \frac{1}{2}(\psi' - \psi)$ must have the same sign so long as $\psi' - \psi$ does not exceed 360°, as we readily infer from the equations (656).

For example, if we wish to find the position of the pole for the year 14000, referred to the equinox of 1850, we take $t = 50$, $t' = 12200$; whence $\psi' - \psi = 165^\circ\ 33'$, and

$$A = 277^\circ\ 52' \qquad\qquad D = 43^\circ\ 28'$$

The position of α *Lyræ* for 1850 is

$$\alpha = 277^\circ\ 58' \qquad\qquad \delta = 38^\circ\ 39'$$

consequently, this star, in the year 14000, will be within five degrees of the pole, and will become the pole star of that period.

PROPER MOTION OF THE FIXED STARS.

376. When from direct observations of the apparent positions of the stars we deduce their mean places, we find that the changes in these mean places between distant dates do not agree with those which arise solely from the precession, but that each star appears to have a small motion of its own, which is, therefore, designated as its *proper motion*.*

This proper motion is partly real—arising from the absolute motion of the star in space; and partly apparent—arising from the motion of our own sun, with the planets, whereby our point of view is changed. It will be shown hereafter how these two motions are to be distinguished from each other; but we here consider only the resultant of both.

The path of a star upon the celestial sphere is assumed to coincide with the arc of a great circle, and the proper motion in this circle to be uniform or proportional to the time. It is not probable that either hypothesis is strictly true; but that portion of its whole orbit which a star appears to describe even in several centuries is so small that, in the observations thus far practicable, no sensible departure from uniform motion or from motion in a great circle could become sensible.

377. In order to distinguish the proper motion from the precession, the star's observed mean place at two different dates must be referred to the same mean equinox. Suppose, therefore, that α and δ are the observed mean right ascension and declination for the time $1800 + t$, and α' and δ' those for $1800 + t'$. If we start from the first place, and, computing the precession for the interval $t' - t$, find the values (α') and (δ') for $1800 + t'$, the whole proper motion in the interval, *referred to the equinox of* $1800 + t'$, is

$$\Delta\alpha' = \alpha' - (\alpha') \qquad \Delta\delta' = \delta' - (\delta')$$

But if we start from the second place, and, reducing it to the first time, find (α) and (δ), the proper motion in the interval, *referred to the equinox of* $1800 + t$, is

$$\Delta\alpha = (\alpha) - \alpha \qquad \Delta\delta = (\delta) - \delta$$

* The student must remember that precession does not affect the relative positions of the stars, but only shifts the circles of reference. The proper motion changes the relative positions or the apparent *configuration* of the stars.

378. *To reduce a star's mean place from one epoch to another, when the proper motion is given.*—Let α, δ, be the given place for $1800 + t$, and let the given *annual* proper motion in right ascension and declination, referred to the equinox of this date, be denoted by $d\alpha$ and $d\delta$. To reduce to the date $1800 + t'$, we first find the whole proper motion in the interval, by the formulæ

$$\Delta\alpha = d\alpha\,(t' - t) \qquad\qquad \Delta\delta = d\delta\,(t' - t)$$

Then, putting

$$(\alpha) = \alpha + \Delta\alpha \qquad\qquad (\delta) = \delta + \Delta\delta$$

we compute the precession by the formulæ of Arts. 371 to 374, employing in these formulæ (α) and (δ) for α and δ.

If the proper motion $(\Delta\alpha', \Delta\delta')$ had been given for the epoch $1800 + t'$, we should first have computed the precession with the given values α and δ, and, having applied it, if (α') and (δ') were the resulting values, we should have finally $\alpha' = (\alpha') + \Delta\alpha'$, $\delta' = (\delta') + \Delta\delta'$.

379. *To reduce the proper motion in right ascension and declination from one epoch to another.*—If, in Fig. 56, P and P' are the poles of the equator for the epochs $1800 + t$ and $1800 + t'$ respectively, and we suppose the star S to vary its position, the present problem requires us to deduce the relations between the variations of the parts of the triangle SPP', the side PP' being the only constant part. Observing the notation of Art. 371, we have (since z, ϑ, z', ϑ' do not depend upon the star's place)

$$\begin{aligned}
d(SPP') &= d(\alpha + z + \vartheta) = d\alpha \\
d(SP'P) &= d(180^\circ - \alpha' + z' - \vartheta') = -\,d\alpha' \\
d(SP) &= -\,d\delta \\
d(SP') &= -\,d\delta'
\end{aligned}$$

and hence, by the formulæ (47) and (46), putting γ for the angle at the star,

$$\left.\begin{aligned}
\cos\delta'.\,d\alpha' &= d\alpha\cos\delta\cos\gamma + d\delta\sin\gamma \\
d\delta' &= -\,d\alpha\cos\delta\sin\gamma + d\delta\cos\gamma
\end{aligned}\right\} \quad (665)$$

in which

$$\sin\gamma = \frac{\sin\Theta\sin(\alpha + z + \vartheta)}{\cos\delta'} = \frac{\sin\Theta\sin(\alpha' - z' + \vartheta')}{\cos\delta}$$

$$\cos\gamma = \frac{\cos\Theta - \sin\delta\sin\delta'}{\cos\delta\cos\delta'}$$

In computing these, it will usually suffice to find γ by its sine alone, since $\cos\gamma$ will always be positive except in the rare case where the star is so near the pole that $\cos\Theta < \sin\delta \sin\delta'$.

The formulæ (665) are equally applicable whether $d\alpha$, $d\delta$, $d\alpha'$, $d\delta'$ denote the annual proper motion or the whole proper motion in the given interval.

Example.—The mean place of *Polaris* for 1755 was

$$\alpha = 10° 55' 44''.955 \qquad \delta = 87° 59' 41''.12$$

and, by the application of the precession, this place reduced to 1820 was found, on page 616, to be

$$(\alpha') = 14° 13' 56''.85 \qquad (\delta') = 88° 20' 54''.11$$

But the mean place for 1820, derived from observation, was, according to Bessel in the *Tabulæ Regiomontanæ*,

$$\alpha' = 14° 15' 22''.575 \qquad \delta' = 88° 20' 54''.27$$

Hence, the proper motion from 1755 to 1820, referred to the mean equinox of 1820, was

$$\Delta\alpha' = +\ 85''.725 \qquad \Delta\delta' = +\ 0''.16$$

or the annual motion

$$d\alpha' = +\ 1''.31885 \qquad d\delta' = +\ 0''.00246$$

Now, to reduce this proper motion to the year 1755, we may employ the formulæ (665), by exchanging $d\alpha$ with $d\alpha'$ and $d\delta$ with $d\delta'$, and taking γ with the negative sign, since Θ is negative for the interval from 1820 to 1755; or we may avoid the change of notation and of sign by deducing from (665) the following:

$$\begin{aligned} \cos\delta \,.\, d\alpha &= d\alpha' \cos\delta' \cos\gamma - d\delta' \sin\gamma \\ d\delta &= d\alpha' \cos\delta' \sin\gamma + d\delta' \cos\gamma \end{aligned}$$

From the example on page 616, we find

$$\log\sin\Theta = 7.800851 \qquad \alpha + z + \vartheta = 11° 20' 41''.57$$

with which and $\delta' = 88° 20' 54''.27$ we find

$$\log\sin\gamma = 8.634966 \qquad \log\cos\gamma = 9.999596$$

and hence for 1755 we find

$$d\alpha = +\ 1''.08228 \qquad\qquad d\delta = +\ 0''.004098$$

If, now, there had been given both the mean place and the proper motion for 1755, namely,

$$\alpha = 10^\circ\ 55'\ 44''.955 \qquad\qquad \delta = 87^\circ\ 59'\ 41''.12$$
$$d\alpha = +\ 1''.08228 \qquad\qquad d\delta = +\ 0''.004098$$

to find the mean place for 1820, we should first take

$$(\alpha) = 10^\circ\ 55'\ 44''.955 + 1''.08228 \times 65 = 10^\circ\ 56'\ 55''.30$$
$$(\delta) = 87\ \ 59\ \ 41\ .12 + 0\ .004098 \times 65 = 87\ \ 59\ \ 41\ .39$$

and employing these values, instead of α and δ, in (659) and (660), we should find

$$\alpha + z + \vartheta = A = 11^\circ\ 21'\ 51''.92$$
$$\log p = 9.256691$$
$$A' - A = 2^\circ\ 28'\ 33''.45$$
$$\tfrac{1}{2}(\delta' - \delta) = \quad 10'\ 36''.44$$

whence

$$\alpha' = 14^\circ\ 15'\ 22''.57 \qquad\qquad \delta' = 88^\circ\ 20'\ 54''.27$$

as given above.

380. *The proper motion on a great circle.*—If we denote this by ρ, and the angle which the great circle makes with the circle of declination of the star by χ, reckoning the angle from the north towards the east, we have

$$\rho \sin \chi = \Delta\alpha \cos \delta \qquad\qquad \rho \cos \chi = \Delta\delta$$

Thus, we find, in the preceding example, for *Polaris* in 1755,

$$\rho = 0''.03809 \qquad\qquad \chi = 83^\circ\ 49'.4$$

and in 1820,

$$\rho = 0''.03809 \qquad\qquad \chi' = 86^\circ\ 17'.8$$

where the accent of χ' is used for the second epoch, but ρ is necessarily the same for both epochs.

It is evident, moreover, that we have $\chi' = \chi + \gamma$, and hence, if ρ and χ have been found for one epoch, it is only necessary to compute γ to obtain the reduction to another epoch, for we then have, by (665),

$$\cos \delta'\, d\alpha' = \rho \sin (\chi + \gamma) = \rho \sin \chi'$$
$$d\delta' = \rho \cos (\chi + \gamma) = \rho \cos \chi'$$

NUTATION.

381. By the luni-solar precession, combined with the diminution of the obliquity of the ecliptic, the mean pole of the equator is carried around the pole of the ecliptic, but gradually approaching it. But the true pole of the equator has at the same time a small subordinate motion around the mean pole, which is called *nutation.* This motion, if it existed alone, would be nearly in an ellipse whose major axis would be $18''.5$ and minor axis $13''.7$, the major axis being directed towards the pole of the ecliptic; and a revolution of the true around the mean pole would be completed in a period of about nineteen years. This period is the time of a complete revolution of the moon's ascending node on the ecliptic, upon the position of which the principal terms of the nutation depend.

This periodic nutation of the pole involves a corresponding *nutation of the obliquity of the ecliptic* $= \Delta\varepsilon$, and a *nutation of the equinox in longitude*, or, briefly, a *nutation in longitude* $= \Delta\lambda$, which are expressed by the following formulæ* for the year 1800:

$$\begin{aligned}\Delta\varepsilon = {} & 9''.2231 \cos \Omega - 0''.0897 \cos 2\Omega + 0''.0886 \cos 2\text{☾} \\ & + 0''.5510 \cos 2\odot + 0''.0093 \cos(\odot + \Gamma) \\ \Delta\lambda = {} & -17''.2405 \sin \Omega + 0''.2073 \sin 2\Omega - 0''.2041 \sin 2\text{☾} + 0''.0677 \sin(\text{☾} - \Gamma') \\ & - 1''.2694 \sin 2\odot + 0''.1279 \sin(\odot - \Gamma) - 0''.0213 \sin(\odot + \Gamma')\end{aligned} \tag{666}$$

in which

Ω = the longitude of the ascending node of the moon's orbit, referred to the mean equinox,
☾ = the moon's true longitude,
$\odot$ = the sun's true longitude,
Γ = the true longitude of the sun's perigee,
Γ' = the true longitude of the moon's perigee.

The quantity $\Delta\lambda$ is also called the *equation of the equinoxes.*

The coefficient of $\cos \Omega$ in the formulæ for $\Delta\varepsilon$ is called the *constant of nutation.* The coefficient of $\sin \Omega$ in the formula for $\Delta\lambda$ is equal to this constant multiplied by $-2 \cot 2\varepsilon_0$, in which

---

* PETERS, *Numerus Constans Nutationis*, p. 46. Some exceedingly small terms, and others of short period, are here omitted, as, even if they are not altogether insensible in a single observation, their effect disappears in the mean of a number of observations.

$\varepsilon_0 = 23° \, 27' \, 54''.2$. These coefficients, however, are not absolutely constant: so that, according to PETERS, the formulæ for 1900 will be

$$\begin{aligned}\Delta\varepsilon = \;& 9''.2240 \cos \Omega - 0''.0896 \cos 2\Omega + 0''.0885 \cos 2☾ \\ & + 0''.5507 \cos 2\odot + 0''.0092 \cos (\odot + \Gamma) \\ \Delta\lambda = & - 17''.2577 \sin \Omega + 0''.2073 \sin 2\Omega - 0''.2041 \sin 2☾ + 0''.0677 \sin (☾ - \Gamma') \\ & - 1''.2695 \sin 2\odot + 0''.1275 \sin (\odot - \Gamma) - 0''.0213 \sin (\odot + \Gamma)\end{aligned} \tag{667}$$

Since the attractions of the sun and moon upon the earth do not disturb the position of the ecliptic, but only that of the equator and its intersection with the ecliptic, the nutation does not affect the latitudes of stars, and its effect upon their longitudes is simply to increase them all by the same quantity $\Delta\lambda$.

382. *To find the nutation in right ascension and declination for a given star at a given time.*—Let α and δ denote the mean right ascension and declination of the star at the given time; α' and δ' the *true* right ascension and declination at this time, or the mean place corrected for the nutation. Let the coefficients of the formulæ for $\Delta\varepsilon$ and $\Delta\lambda$ be found for the given year by interpolation between the values for 1800 and 1900, and then, taking Ω, ☾, $\odot$, Γ, and Γ' from the Ephemeris for the given date (the day of the year, and, for the greatest precision, the hour of the day), we can compute the values of $\Delta\varepsilon$ and $\Delta\lambda$. We can then have either a rigorous or an approximate solution of our problem.

A rigorous solution may be obtained by employing the formulæ (656), (658), and (659), substituting $\varepsilon + \frac{1}{2}\Delta\varepsilon$, $\Delta\varepsilon$, $\Delta\lambda$, $\alpha + z$, and $\alpha' - z'$ for $\frac{1}{2}(\varepsilon_1' + \varepsilon_1)$, $\varepsilon_1' - \varepsilon_1$, $\psi' - \psi$, A and A', respectively.

Another rigorous solution is obtained by first computing the mean longitude λ and latitude β, from the given α and δ, and the mean obliquity ε, by Art. 23. Then, with the true longitude $\lambda + \Delta\lambda$, the true latitude β, and the true obliquity $\varepsilon + \Delta\varepsilon$, we can compute the true right ascension α' and declination δ' by Art. 26.

But, since $\Delta\varepsilon$ and $\Delta\lambda$ are very small, an approximate solution by means of differential formulæ will be sufficiently accurate, and practically more convenient. The effect of varying λ and ε by $\Delta\lambda$ and $\Delta\varepsilon$, while β is constant, is, by the equations (53),

$$\alpha' - \alpha = \Delta\lambda \cdot \frac{\cos\eta \cos\beta}{\cos\delta} - \Delta\varepsilon \cos\alpha \tan\delta$$

$$\delta' - \delta = \Delta\lambda \cdot \sin\eta \cos\beta + \Delta\varepsilon \sin\alpha$$

in which η is the position angle at the star, as defined in Art. 25

Substituting the values of $\cos\eta\cos\beta$ and $\sin\eta\cos\beta$ there given, we have

$$\begin{aligned} \alpha' - \alpha &= \Delta\lambda\,(\cos\varepsilon + \sin\varepsilon\sin\alpha\tan\delta) - \Delta\varepsilon\cos\alpha\tan\delta \\ \delta' - \delta &= \Delta\lambda\sin\varepsilon\cos\alpha + \Delta\varepsilon\sin\alpha \end{aligned}$$

Hence, substituting the values of $\Delta\lambda$ and $\Delta\varepsilon$ for 1800, with $\varepsilon = 23°\ 27'\ 54''$, and also the values for 1900 with $\varepsilon = 23°\ 27'\ 7''$, we find

$$\begin{aligned} \alpha' - \alpha = \\ &- (\underset{15''.8321}{15''.8148} + \underset{6''.8682}{6''.8650}\sin\alpha\tan\delta)\sin\text{☊} - \underset{9''.2240}{9''.2231}\cos\alpha\tan\delta\cos\text{☊} \\ &+ (\ 0''.1902 + 0''.0825\sin\alpha\tan\delta)\sin 2\text{☊} + 0''.0897\cos\alpha\tan\delta\cos 2\text{☊} \\ &- (\ 0''.1872 + 0''.0813\sin\alpha\tan\delta)\sin 2\text{☾} - 0''.0886\cos\alpha\tan\delta\cos 2\text{☾} \\ &+ (\ 0''.0621 + 0''.0270\sin\alpha\tan\delta)\sin(\text{☾} - \Gamma') \\ &- (\ 1''.1644 + 0''.5055\sin\alpha\tan\delta)\sin 2\odot - 0''.5510\cos\alpha\tan\delta\cos 2\odot \\ &+ (\ 0''.1173 + 0''.0509\sin\alpha\tan\delta)\sin(\odot - \Gamma) \\ &- (\ 0''.0195 + 0''.0085\sin\alpha\tan\delta)\sin(\odot + \Gamma) - 0''.0093\cos\alpha\tan\delta\cos(\odot + \Gamma) \end{aligned} \tag{668}$$

$$\begin{aligned} \delta' - \delta = &- \underset{6''.8682}{6''.8650}\cos\alpha\sin\text{☊} + \underset{9''.2240}{9''.2231}\sin\alpha\cos\text{☊} \\ &+ 0''.0825\cos\alpha\sin 2\text{☊} - 0'.0897\sin\alpha\cos 2\text{☊} \\ &- 0''.0813\cos\alpha\sin 2\text{☾} + 0''.0886\sin\alpha\cos 2\text{☾} \\ &+ 0''.0270\cos\alpha\sin(\text{☾} - \Gamma') \\ &- 0''.5055\cos\alpha\sin 2\odot + 0''.5510\sin\alpha\cos 2\odot \\ &+ 0''.0509\cos\alpha\sin(\odot - \Gamma) \\ &- 0''.0085\cos\alpha\sin(\odot + \Gamma) + 0''.0093\sin\alpha\cos(\odot + \Gamma) \end{aligned}$$

The values of the coefficients which sensibly change during the century are given for 1900 in small figures below the values for 1800.*

Previous to the investigations of PETERS, the only terms retained in the nutation formula were those depending on ☊, 2☊, 2☾, and 2⊙. Of the additional terms added by him, I have retained only those which can have any sensible effect in the actual state of the art of astronomical observation.

383. *General tables for the nutation in right ascension and declination.*—Of the various tables proposed for facilitating the compu-

* If we take into account the squares of $\Delta\lambda$ and $\Delta\varepsilon$ and their product in the development of $\alpha' - \alpha$ and $\delta' - \delta$ in series, some of the coefficients are changed, but only by two or three units in the last decimal place. Compare the formulæ of the text with those given by PETERS in the *Numerus Constans*, and by STRUVE in the *Astronom. Nach.*, No. 486.

tation of the nutation formulæ, the most compendious are those computed by NICOLAI, according to the form suggested by GAUSS, and included in WARNSTORFF'S edition of SCHUMACHER'S *Hülfstafeln.* In these tables the new constants are adopted from PETERS, as in the preceding formulæ, and the epoch is 1850.

For the *lunar* nutation in right ascension, the first table gives, with the argument ☊, the quantity

$$- 15''.8235 \sin ☊ = c$$

The two remaining terms in the first line of our formula are reduced to a single term by assuming auxiliaries b and B, also given in the tables with the argument ☊, determined by the conditions

$$b \sin (☊ + B) = 6''.8666 \sin ☊$$
$$b \cos (☊ + B) = 9''.2235 \cos ☊$$

Thus, the first line of the formula, containing the principal terms of the lunar nutation in right ascension, becomes

$$c - b \cos (☊ + B - \alpha) \tan \delta$$

By the use of the same auxiliaries, the first two terms of the lunar nutation in declination are reduced to the following:

$$- b \sin (☊ + B - \alpha)$$

For the solar nutation, the second table gives, with the argument $2\odot$, the quantity

$$- 1''.1644 \sin 2\odot = g$$

and the two remaining terms involving $2\odot$ are reduced to a single one by the auxiliaries f and F, given in the table, which are determined by the conditions

$$f \sin (2\odot + F) = 0''.5055 \sin 2\odot$$
$$f \cos (2\odot + F) = 0''.5510 \cos 2\odot$$

so that the solar nutation in right ascension is

$$g - f \cos (2\odot + F - \alpha) \tan \delta$$

and the solar nutation in declination is

$$- f \sin (2\odot + F - \alpha)$$

Almost all the remaining terms of the formulæ may also be found by means of the table for the solar nutation. The coefficients of the terms in $2☊$ and $2☾$ are about one-sixth part of

those of the terms in $2\odot$, while the signs of the terms in 2Ω are the opposite to those in $2\odot$: hence, to find the value of these terms, we can enter the table first with the argument $2\Omega + 180^\circ$ ($= 2\Omega + \mathrm{VI}^s$), and then with $2☾$; and, computing the nutation in each case by the above forms for the solar nutation, take $\frac{1}{6}$, or more exactly $\frac{6}{37}$, of the sum of the results. The terms in $\odot + \Gamma$ are obtained by entering the table with the argument $\odot + \Gamma$ and taking $\frac{1}{60}$ of the results. The terms in $\odot - \Gamma$ will be found in the most simple manner by multiplying the annual precession [given by (663), and usually computed in connection with the nutation] by $\frac{1}{393} \sin(\odot - \Gamma)$; and the terms in $☾ - \Gamma'$ by multiplying the annual precession by $\frac{1}{742} \sin(☾ - \Gamma')$.

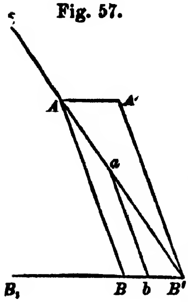
The computation even with the aid of these tables is sufficiently tedious. Their chief recommendation is their brevity; but where the nutation is to be computed very frequently, more extended tables are required, such, for example, as are given in the 3d vol. of the *Washington Observations*, Appendix C, by Professors HUBBARD, COFFIN, and KEITH.

ABERRATION.

384. The *apparent* direction of a star from the earth is determined by the direction of the telescope through which it is observed. But this apparent direction differs from the true one in consequence of the motion of the earth combined with the progressive motion of light; for the telescope, partaking of the movement of the earth, is changing its position while the light is descending through its axis.

Let us distinguish between the two instants t and t' when the ray of light from the star arrives respectively at the object-end and at the eye-end of the axis of the telescope.

Fig. 57.



Let A, B (Fig. 57) be the position of the object and eye end of the telescope at the instant t; A', B', their positions at the instant t'; BB', the motion of the earth in the interval $t' - t$, in which the ray SAB' from the star is describing the line AB'. Then it is evident that, while $B'A$ is the true direction of the star, $B'A'$ is the apparent direction as given by the telescope.* Moreover, supposing the motion of the earth for

* GAUSS: *Theoria Motus Corporum Coelestium*, p. 68.

so small an interval to be rectilinear and uniform, and the motion of light to be uniform, the lines BA and $B'A'$ are parallel, and the ray of light during its progress from A to B', is constantly in the axis of the telescope; for instance, when the telescope is in the position ba, the ray will have reached the point a, and we have

$$Aa : Bb = AB' : BB'$$

The difference of apparent direction thus caused by the motion of the earth combined with that of light is called the *aberration of the fixed stars.* When we also take into account the motion of the luminous body, as in the case of the planets, another species of aberration occurs, which will be considered hereafter, under the name of the *planetary* aberration.

The whole displacement of the star produced by aberration is in the plane passed through the star and the line of the observer's motion, and the star appears to be thrown *forward* in this plane in the direction of that motion. Thus, in the figure the whole aberration is the angle $SB'A'$; and, if we conceive the plane of the lines SB' and BB' to be produced to the celestial sphere, this plane will be that of a great circle drawn through the place of the star and the points of the sphere in which the line BB' meets it. The displacement of the star will be the arc of this circle subtending the angle $SB'A'$ and measured from the star towards that point of the sphere towards which the observer is moving.

385. *To find the aberration of a star in the direction of the observer's motion.*—Let

$\vartheta = AB'B_1 =$ the true direction of the star referred to the line $B'B_1$,
$=$ the arc of a great circle of the sphere joining the star's true place and the point *from* which the observer is moving,
$\vartheta' =$ the apparent direction of the star referred to the same line, $= ABB_1$,
$V =$ the velocity of light,
$v =$ the velocity of the observer;

then the aberration in the plane of motion is the angle $A'B'A = B'AB = \vartheta' - \vartheta$, and the triangle ABB' gives

$$\frac{\sin(\vartheta' - \vartheta)}{\sin \vartheta'} = \frac{BB'}{AB'} = \frac{v}{V}$$

As $\vartheta' - \vartheta$ is very small, we may put the arc for the sine; and if we then also put

$$k = \frac{v}{V \sin 1''} \tag{669}$$

we shall have

$$\vartheta' - \vartheta = k \sin \vartheta' \tag{670}$$

where the constant k may be regarded as known from the velocities of light and of the observer.

386. The motion of the observer on the surface of the earth is the resultant of the motion of the earth in its orbit and its rotation on its axis; that is, of its *annual* and *diurnal* motions. These may be separately considered.

The *annual aberration* is that part of the total aberration which results from the earth's annual motion. It may be called the aberration for the earth's centre.

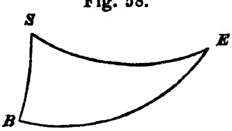
The *diurnal aberration* is that part of the total aberration which results from the earth's diurnal motion. It obviously varies with the position of the observer on the earth's surface, and vanishes for an observer at the poles.

387. *To find the annual aberration of a star in longitude and latitude.*—Let

λ, β = the true longitude and latitude of the star,
λ', β' = the apparent longitude and latitude (affected by aberration),
$\odot$ = the true longitude of the sun.

The point of the sphere from which the earth appears to be moving is a point in the ecliptic whose longitude is $90° + \odot$ (the eccentricity of the earth's orbit being here neglected), and the mean velocity of the earth in its orbit may be supposed to be substituted in (669): so that k is known.

Fig. 58.



If, then, BE (Fig. 58) is an arc of the ecliptic, E the point from which the earth is moving, S the true place of the star, and if SB is drawn perpendicular to BE, we have, in the right triangle SBE,

$$SB = \beta, \qquad BE = 90° + \odot - \lambda, \qquad SE = \vartheta,$$

and hence, if we denote the angle E by γ, we have

$$\left.\begin{aligned} \sin\vartheta\sin\gamma &= \sin\beta \\ \sin\vartheta\cos\gamma &= \cos\beta\cos(\odot-\lambda) \\ \cos\vartheta &= -\cos\beta\sin(\odot-\lambda) \end{aligned}\right\} \quad (671)$$

The apparent place of the star is on the great circle ES at the distance ϑ' from S: so that, if we now suppose S to be the apparent place, the angle γ is not changed, and we have

$$\left.\begin{aligned} \sin\vartheta'\sin\gamma &= \sin\beta' \\ \sin\vartheta'\cos\gamma &= \cos\beta'\cos(\odot-\lambda') \\ \cos\vartheta' &= -\cos\beta'\sin(\odot-\lambda') \end{aligned}\right\} \quad (672)$$

If, then, the true place of the star is given, the equations (671) may be used to determine γ and ϑ; then ϑ' will be found from (670), and, finally, λ' and β' will be found from (672). This is the direct and rigorous solution of the problem; but a more convenient solution is obtained by eliminating ϑ and γ as follows. We find, from the equations (671) and (672),

$$\sin\vartheta'\cos\vartheta\cos\gamma = -\cos\beta\cos\beta'\sin(\odot-\lambda)\cos(\odot-\lambda')$$
$$\sin\vartheta\cos\vartheta'\cos\gamma = -\cos\beta\cos\beta'\cos(\odot-\lambda)\sin(\odot-\lambda')$$

the difference of which is

$$\sin(\vartheta'-\vartheta)\cos\gamma = -\cos\beta\cos\beta'\sin(\lambda'-\lambda)$$

whence

$$\lambda'-\lambda = -\frac{(\vartheta'-\vartheta)\cos\gamma}{\cos\beta\cos\beta'} = -\frac{k\sin\vartheta'\cos\gamma}{\cos\beta\cos\beta'}$$

or

$$\lambda'-\lambda = -k\frac{\cos(\odot-\lambda')}{\cos\beta} \quad (673)$$

Again, we find, from our equations,

$$\cot\gamma = \cot\beta'\cos(\odot-\lambda') = \cot\beta\cos(\odot-\lambda)$$

by which β' can be found from β after λ' has been found by (673), or we may find the difference between β' and β thus:

$$\tan\beta'-\tan\beta = \tan\beta'\left[\frac{\cos(\odot-\lambda')-\cos(\odot-\lambda)}{\cos(\odot-\lambda')}\right]$$

$$\sin(\beta'-\beta) = \frac{2\sin\frac{1}{2}(\lambda'-\lambda)\sin[\odot-\frac{1}{2}(\lambda'+\lambda)]\sin\beta'\cos\beta}{\cos(\odot-\lambda')}$$

whence, taking $2\sin\frac{1}{2}(\lambda'-\lambda) = \sin(\lambda'-\lambda)$, we obtain, by means of (673),

$$\beta'-\beta = -k\sin[\odot-\tfrac{1}{2}(\lambda'+\lambda)]\sin\beta' \quad (674)$$

The equations (673) and (674) are almost rigorously exact; but, since the value of k is only about $20''$, a sufficient degree of accuracy will be obtained if in the second members we put λ and β for λ' and β'. The formulæ for the annual aberration in longitude and latitude thus become

$$\left.\begin{array}{l} \lambda' - \lambda = -k \cos(\odot - \lambda) \sec \beta \\ \beta' - \beta = -k \sin(\odot - \lambda) \sin \beta \end{array}\right\} \quad (675)$$

in which the value of the constant, according to STRUVE,* is

$$k = 20''.4451$$

These last formulæ may be directly deduced by differentiating the equations (671).

If we retain terms of the second order in developing (673) and (674), we shall find that the following quantities will be added to the second members of (675):

$$-\tfrac{1}{2} k^2 \sin 1'' \sin 2(\odot - \lambda) \sec^2 \beta$$

and $$-\tfrac{1}{4} k^2 \sin 1'' \tan \beta - \tfrac{1}{4} k^2 \sin 1'' \cos 2(\odot - \lambda) \tan \beta$$

But the term $-\frac{1}{4} k^2 \sin 1'' \tan \beta$ being constant may be omitted, since it will be included in the expression of the star's *mean* place, which (Art. 361) involves the non-periodic elements of the star's position. Retaining, therefore, only the periodic terms —namely, those involving $\odot$—the more complete formulæ will be

$$\left.\begin{array}{l} \lambda' - \lambda = -20''.4451 \cos(\odot - \lambda) \sec \beta - 0''.0010133 \sin 2(\odot - \lambda) \sec^2 \beta \\ \beta' - \beta = -20''.4451 \sin(\odot - \lambda) \sin \beta - 0''.0005067 \cos 2(\odot - \lambda) \tan \beta \end{array}\right\} (675^*)$$

The last terms will be sensible only for stars very near the pole.

Terms of the second order not multiplied by $\tan \beta$ or $\sec \beta$ are wholly insensible, and have been disregarded in the deduction of the above formulæ.

388. It is easy to prove, from the equations (675), that the effect of the aberration is the same as if the star actually moved in a circle parallel to the plane of the ecliptic; the diameter of the circle being equal to the distance of the star multiplied by $\sin k$. This circle will be seen projected upon a plane tangent to the sphere at the mean place of the star, as an ellipse whose major axis is $\sin k$ and minor axis $\sin k \sin \beta$, the radius of the

* *Astron. Nach.*, No. 484.

sphere being unity. The period in which a star appears to describe this ellipse is a sidereal year.

389. *To find the annual aberration in right ascension and declination.*—Let

$A, D =$ the right ascension and declination of the point E (*from* which the earth is moving);

then, in the triangle formed by the point E, the star, and the pole of the equator, the sides are $90° - D$, $90° - \delta$, and ϑ; and the angle opposite to ϑ is $A - \alpha$. If then we suppose the side ϑ to vary, the corresponding variations of the angle $A - \alpha$ and the side $90° - \delta$ may be directly deduced by the differential formulæ of Art. 34. The angle at E and the side $90° - D$ being constant, we find

$$\begin{aligned} \cos\delta \,.\, d\alpha &= - d\vartheta \sin C \\ d\delta &= - d\vartheta \cos C \end{aligned}$$

where C denotes the angle at the star. For determining C, our triangle gives

$$\begin{aligned} \sin\vartheta \sin C &= \cos D \sin (A - \alpha) \\ \sin\vartheta \cos C &= \cos\delta \sin D - \sin\delta \cos D \cos (A - \alpha) \end{aligned}$$

In (670) we may employ $\sin\vartheta$ for $\sin\vartheta'$: so that, putting $\alpha' - \alpha$ and $\vartheta' - \vartheta$ for $d\alpha$ and $d\vartheta$, we find

$$\left.\begin{aligned} \alpha' - \alpha &= -k \sec\delta \cos D \sin (A - \alpha) \\ \delta' - \delta &= -k\,[\cos\delta \sin D - \sin\delta \cos D \cos (A - \alpha)] \end{aligned}\right\} \quad (676)$$

The quantities A and D are found from the right triangle formed by the equator, the ecliptic, and the declination circle drawn through E, by the formulæ,

$$\left.\begin{aligned} \cos D \cos A &= - \sin \odot \\ \cos D \sin A &= \cos \odot \cos \varepsilon \\ \sin D &= \cos \odot \sin \varepsilon \end{aligned}\right\} \quad (677)$$

If we substitute these values in the formulæ for $\alpha' - \alpha$ and $\delta' - \delta$, after developing $\sin (A - \alpha)$ and $\cos (A - \alpha)$, we obtain

$$\left.\begin{aligned} \alpha' - \alpha &= -k \sec\delta\,(\cos\odot \cos\varepsilon \cos\alpha + \sin\odot \sin\alpha) \\ \delta' - \delta &= -k \cos\odot\,(\sin\varepsilon \cos\delta - \cos\varepsilon \sin\delta \sin\alpha) \\ &\quad - k \sin\odot \sin\delta \cos\alpha \end{aligned}\right\} \quad (678)$$

If we retain the terms of the second order, (omitting, however, those which do not involve $\odot$, or the non-periodic terms), we find that the aberration in right ascension obtains the additional terms

$$-\tfrac{1}{4}k^2 \sin 1''(1+\cos^2\varepsilon)\cos 2\odot \sin 2\alpha \sec^2\delta$$
$$+\tfrac{1}{2}k^2 \sin 1'' \cos\varepsilon \sin 2\odot \cos 2\alpha \sec^2\delta$$

and the aberration in declination the terms

$$+\tfrac{1}{8}k^2 \sin 1''[\sin^2\varepsilon - (1+\cos^2\varepsilon)\cos 2\odot \cos 2\alpha] \tan\delta$$
$$-\tfrac{1}{4}k^2 \sin 1'' \cos\varepsilon \sin 2\odot \sin 2\alpha \tan\delta$$

Substituting the value of k in these terms, together with $\varepsilon = 23° 27' 30''$ (for 1850), and omitting insensible quantities, the corrections of the formulæ (678) will be

$$\left.\begin{array}{ll} \text{in } (\alpha'-\alpha), & -0''.000931 \sin 2(\odot-\alpha)\sec^2\delta \\ \text{in } (\delta'-\delta), & -0''.000466 \cos 2(\odot-\alpha)\tan\delta \end{array}\right\} \quad (678^*)$$

Example 1.—The mean longitude and latitude of *Spica* for January 10, 1860, are

$$\lambda = 201° 53' 22''.33 \qquad \beta = -2° 2' 36''.29$$

and the sun's longitude is

$$\odot = 289° 30'$$

Hence, we find, by (675), the aberration in longitude and latitude,

$$\lambda' - \lambda = -0''.85 \qquad \beta' - \beta = +0''.73$$

The corresponding mean right ascension and declination are

$$\alpha = 13^h 17^m 49^s.62 \qquad \delta = -10° 25' 44''.9$$

whence, by (678), taking $\varepsilon = 23° 27'.4$, we find the aberration in right ascension and declination,

$$\alpha' - \alpha = -0''.53 = -0^s.035 \qquad \delta' - \delta = +0''.99$$

Example 2.—The mean place of *Polaris* for 1820.0 was

$$\alpha = 0^h 57^m 1^s.505 = 14° 15' 22''.57$$
$$\delta = 88° 20' 54''.27$$

and for this date,

$$\odot = 280° 0' \qquad \varepsilon = 23° 27'.8$$

with which the aberration in right ascension and declination is found, by (678), to be

$$\alpha' - \alpha = +\,62''.51 = +\,4^s.167 \qquad\qquad \delta' - \delta = +\,20''.27$$

The additional terms of (678*) are in this case $-\,0''.158 = -\,0^s.011$ and $+\,0''.016$, and the more correct values are, therefore,

$$\alpha' - \alpha = +\,4^s.156 \qquad\qquad \delta' - \delta = +\,20''.29$$

390. *Gauss's Tables for computing the aberration in right ascension and declination.*—If we determine a and A by the conditions

$$a \sin(\odot + A) = k \sin \odot$$
$$a \cos(\odot + A) = k \cos \odot \cos \varepsilon$$

the formulæ (678) may be expressed as follows:

$$\begin{aligned} \alpha' - \alpha &= -\,a \sec \delta \cos(\odot + A - \alpha) \\ \delta' - \delta &= -\,a \sin \delta \sin(\odot + A - \alpha) - k \cos \odot \cos \delta \sin \varepsilon \\ &= -\,a \sin \delta \sin(\odot + A - \alpha) - \tfrac{1}{2} k \sin \varepsilon \cos(\odot + \delta) \\ &\qquad - \tfrac{1}{2} k \sin \varepsilon \cos(\odot - \delta) \end{aligned}$$

The first of the tables proposed by GAUSS* gives A and $\log a$ with the argument sun's longitude, and with these quantities we readily compute the aberration in right ascension and the first part of the aberration in declination. The second and third parts of the aberration in declination are taken directly from the second table with the arguments $\odot + \delta$ and $\odot - \delta$. The tables have been recomputed by NICOLAI with the constant $k = 20''.4451$, and are given in WARNSTORFF's edition of SCHUMACHER's *Hülfstafeln.*

The value of ε for 1850 is employed in computing these tables. The rate of change of ε is so slow that the tables will answer for the whole of the present century, unless more than usual precision is desired.

391. In the preceding investigation of the aberration formulæ we have, for greater simplicity, assumed the earth's orbit to be a circle and its motion in the orbit uniform. Let us now inquire what correction these formulæ will require when the true elliptical motion is employed.

* *Monatliche Correspondenz*, XVII. p. 312.

If u is the true anomaly of the earth in the orbit, reckoned from the perihelion, at the time t from the perihelion passage, r the radius vector, a the mean distance of the earth from the sun, or the semi-major axis of the ellipse, we have

$$r = \frac{a(1-e^2)}{1+e\cos u}$$

The true direction of the earth's motion at any time is not, as in the circular orbit, at right angles to the direction of the sun, but in that of the tangent to the curve. If we denote the angle which the tangent makes with the radius vector by $90° - i$, we have, by the theory of curves,

$$\cot(90° - i) = \frac{1}{r} \cdot \frac{dr}{du}$$

whence, by the above equation of the ellipse,

$$\tan i = \frac{e \sin u}{1 + e \cos u}$$

and the true *direction* of the earth's motion will be taken into account in our formulæ (675), if for $\odot$ we substitute $\odot - i$.

If v_1 denotes the true velocity of the earth in its orbit at the time t, we have

$$v_1 = r \sec i \frac{du}{dt}$$

and if f is the area described by the radius vector in the time t, F the whole area of the ellipse described in the period T, we have, by KEPLER'S first law,

$$\frac{f}{t} = \frac{F}{T}$$

or

$$\frac{df}{dt} = \frac{F}{T}$$

We also have, by the theory of the ellipse,

$$F = \pi a^2 \sqrt{(1-e^2)}$$

$$\frac{df}{dt} = \frac{r^2}{2} \cdot \frac{du}{dt}$$

and hence

$$\frac{du}{dt} = \frac{2\pi a^2 \sqrt{(1-e^2)}}{Tr^2}$$

which, substituted in the above value of v_1, together with the value of r, gives

$$v_1 = \frac{a}{\sqrt{(1-e^2)}} \cdot \frac{2\pi}{T} \cdot (1 + e \cos u) \sec i$$

The *mean* value of this velocity is that value which it would have if the small periodic terms depending on u and i were omitted (Art. 361); thus, denoting the mean velocity by v, we have

$$v = \frac{a}{\sqrt{(1-e^2)}} \cdot \frac{2\pi}{T} \tag{679}$$

$$v_1 = v(1 + e \cos u) \sec i \tag{680}$$

If, then, V is the velocity of light, and we put

$$k_1 = \frac{v_1}{V \sin 1''} = k(1 + e \cos u) \sec i$$

we can at once adapt our equations (675) to the case of the elliptical orbit, by introducing k_1 for k and $\odot - i$ for $\odot$, so that we have

$$\begin{aligned} \lambda' - \lambda &= -k(1 + e \cos u) \cos(\odot - \lambda - i) \sec i \sec \beta \\ \beta' - \beta &= -k(1 + e \cos u) \sin(\odot - \lambda - i) \sec i \sin \beta \end{aligned}$$

Developing the sine and cosine of $(\odot - \lambda) - i$, we have

$$\begin{aligned} \cos(\odot - \lambda - i) \sec i &= \cos(\odot - \lambda) + \sin(\odot - \lambda) \tan i \\ \sin(\odot - \lambda - i) \sec i &= \sin(\odot - \lambda) - \cos(\odot - \lambda) \tan i \end{aligned}$$

and substituting the value of $\tan i$, we find

$$\begin{aligned} \lambda' - \lambda &= -k \cos(\odot - \lambda) \sec \beta - ke \cos(\odot - u - \lambda) \sec \beta \\ \beta' - \beta &= -k \sin(\odot - \lambda) \sin \beta - ke \sin(\odot - u - \lambda) \sin \beta \end{aligned}$$

The longitude of the sun's perigee is

$$\Gamma = \odot - u$$

by the introduction of which we have, finally,

$$\left.\begin{aligned} \lambda' - \lambda &= -k \cos(\odot - \lambda) \sec \beta - ke \cos(\Gamma - \lambda) \sec \beta \\ \beta' - \beta &= -k \sin(\odot - \lambda) \sin \beta - ke \sin(\Gamma - \lambda) \sin \beta \end{aligned}\right\} \tag{681}$$

These formulæ differ from (675) only by the second terms, which therefore are the corrections for the eccentricity of the

orbit. But we observe that these terms involve only quantities which *for a fixed star* are very nearly constant, so that for the same star they will have, sensibly, the same values for very long periods: the corrections themselves being exceedingly small, since $e = 0.01677$, and hence $ke = 0''.3429$. They may, therefore, be regarded either as constant corrections, or as corrections having only a slow secular change; and in either case they will be combined with the *mean* place of the star, and may be altogether disregarded in the correction for the annual aberration.* The formulæ (675), derived from the circular orbit, will therefore be considered as complete (for the fixed stars), and, consequently, also (678), which are derived from the same hypothesis.

392. *The sun's aberration.*—Since β is less than $1''$, there is no sensible aberration in latitude. The aberration in longitude must be found by the complete formula (681), for in the case of the sun λ is variable. Hence, writing $\odot$ for λ, the aberration of the sun is found by the formula

$$\odot' - \odot = -20''.4451 - 0''.3429 \cos(\Gamma - \odot) \qquad (682)$$

in which for this century we may employ $\Gamma = 280°$ without an error of $0''.01$.

We could derive, from this, formulæ for the sun's aberration in right ascension and declination; but the practical method is to treat the sun as a planet, and to employ the planetary aberration which is given in a subsequent article.

393. *To find the diurnal aberration in right ascension and declination.*—Let

v' = the velocity of a point of the terrestrial equator, arising from the rotation of the earth,

$$k' = \frac{v'}{V \sin 1''} = k \cdot \frac{v'}{v} \qquad (683)$$

The diurnal aberration in the places of stars, as observed from a point on the equator, may be investigated in the same manner as the annual aberration, by substituting the equator for the ecliptic, and, consequently, right ascensions and declinations for

* Bessel, *Tabulæ Regiomontanæ*, XIX.

titudes and longitudes. The nadir of the point of observation is then to be substituted in the place of the sun:* so that if we put

Θ = the right ascension of the zenith, or the sidereal time,

the formulæ (675) are rendered immediately applicable to the present case by putting $180° + \Theta$, α, δ, and k' for $\odot$, λ, β, and k; whence we have, *for a point on the terrestrial equator*,

$$\alpha' - \alpha = k' \cos(\Theta - \alpha) \sec \delta$$
$$\delta' - \delta = k' \sin(\Theta - \alpha) \sin \delta$$

Since every point on the surface of the earth moves in a plane parallel to the equator, and this plane is to be regarded as coincident with the plane of the celestial equator, the same formulæ are applicable to every point, provided we introduce into the expression of k' the actual velocity of the point. This velocity varies directly with the circumference of the parallel of latitude, or with its radius; and this radius for the latitude φ is $\rho \cos \varphi'$, φ' being the geocentric latitude and ρ the radius of the earth for that latitude. Hence we have only to put $v'\rho \cos \varphi'$ for v', or $k'\rho \cos \varphi'$ for k', and we obtain for the diurnal aberration in right ascension and declination, for any point of the earth's surface, the formulæ

$$\left.\begin{aligned} \alpha' - \alpha &= k'\rho \cos \varphi' \cos(\Theta - \alpha) \sec \delta \\ \delta' - \delta &= k'\rho \cos \varphi' \sin(\Theta - \alpha) \sin \delta \end{aligned}\right\} \quad (684)$$

It only remains to determine k'. For this purpose, we have, by (679),

$$v = \frac{a}{\sqrt{(1 - e^2)}} \cdot \frac{2\pi}{T}$$

which, if T is the length of the sidereal year in sidereal days, gives the value of v for one sidereal day. The motion of a point on the earth's equator in one sidereal day is equal to the circumference of the equator: so that, if a' is the equatorial radius, we have the value of v' referred to the same unit as v, by the formula

$$v' = 2\pi a'$$

* For the observer is moving directly *from* the west point of his horizon, which is 90° of right ascension in advance of the nadir; and the point from which the earth in its annual revolution is moving is 90° of longitude in advance of the sun.

whence

$$\frac{v'}{v} = \frac{Ta'\sqrt{1-e^2}}{a}$$

But if we put

$$p = \text{the sun's mean horizontal parallax,}$$

we have

$$\sin p = \frac{a'}{a}$$

and hence we find

$$k' = kT \sin p \sqrt{1-e^2}$$

or, taking STRUVE'S value of $k = 20''.4451$, BESSEL'S value of $T = 366^d.25637$, ENCKE'S value of $p = 8''.57116$, and the eccentricity $e = 0.01677$,

$$k' = 0''.31112$$

This quantity is so small that we may in (684) employ $\cos\varphi$ for $\rho\cos\varphi'$ without sensible error; and hence the diurnal aberration may be found by the formulæ

$$\left.\begin{aligned} \alpha' - \alpha &= 0''.311 \cos\varphi \cos(\Theta - \alpha) \sec\delta \\ \delta' - \delta &= 0''.311 \cos\varphi \sin(\Theta - \alpha) \sin\delta \end{aligned}\right\} \quad (685)$$

The quantity $\Theta - \alpha$ is the hour angle of the star; whence it follows that the diurnal aberration in right ascension for a star on the meridian is $+ 0''.311 \cos\varphi \sec\delta = + 0^s.0207 \cos\varphi \sec\delta$; and the diurnal aberration in declination is then zero.

394. The illustration given in Art. 388 applies also to the diurnal aberration. In one sidereal day each star appears to describe a small ellipse whose major axis is $\sin k' \cos\varphi$, and minor axis $\sin k' \cos\varphi \sin\delta$, the radius of the sphere being unity. For an observer at the pole, where $\cos\varphi = 0$, this ellipse becomes a point, and the diurnal aberration disappears.

395. *The velocity of light.*—The constant k was determined by STRUVE by a comparison of the apparent places of stars at different seasons of the year, and not from the known velocity of light. We can, therefore, determine the velocity of light from this constant. We have, from the preceding articles,

$$V = \frac{v'}{k' \sin 1''} = \frac{v'}{T \sin p \sin k \sqrt{(1-e^2)}}$$

in which, if we take v' = the velocity per second of a point of the earth's equator resulting from the diurnal rotation, V will be the velocity of light per second. If, then, we take $v' = \frac{2\pi a'}{n}$ we have the following formula for determining the velocity of light from the aberration constant:

$$V = \frac{2\pi a'}{nT\sin p\sin k\sqrt{(1-e^2)}} \tag{686}$$

This will give the velocity in one sidereal or one mean second, according as we take $n = 86400$ or $n = 86164$, the number of seconds of either kind in a sidereal day. With Bessel's value of the equatorial radius, Art. 80, and the values of T, p, k, and e, above employed, we find

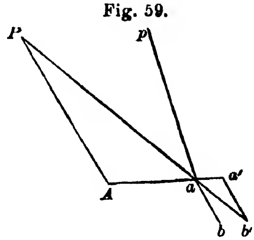
in one sid. second, $V = 191058$ miles $= 307473000$ metres;
in one mean second, $V = 191581$ miles $= 308314000$ metres.

The time required by light to traverse the mean distance of the earth from the sun is

$$\frac{a}{V} = \frac{nT\sin k\sqrt{1-e^2}}{2\pi} = 497^s.78 = 8^m\ 17^s.78 \text{ mean time.}$$

396. *Planetary aberration.*—When the observed body is a planet, and, therefore, in motion relatively to the earth, the aberration above considered is not the complete aberration; but we must further take into account the time required by light to come from the planet to the earth; for in this time the planet will have sensibly changed its place. Let us suppose that the ray of light which reaches the telescope at the time t left the planet at the time T; let P (Fig. 59) be the planet's place in space at the time T, and p its place at the time t; A the place of the object-end of the telescope at the time T, a its place at the time t, ab the direction of the axis of the telescope at the time t, $a'b'$ the position of the axis at the time t' when the light reaches the eye-end of the telescope. Then it is evident that

Fig. 59.



1st. The straight line AP gives the true direction of the planet at the time T;

2d. The straight line ap gives the true direction at the time t;

3d. The straight line ba or $b'a'$ gives the apparent direction at the time t or t' (the difference between which may be regarded as infinitely small);

4th. The straight line $b'a$ gives the apparent direction for the time t, freed from the aberration of the fixed stars.

Now, the points P, a, b' lie in a straight line, and the portions Pa, ab' will be proportional to the intervals of time $t - T$, $t' - t$, if the velocity of light is uniform. In consequence of the immense velocity of light, the interval $t' - T$ will always be so small that during this interval we may suppose the motion of the earth to be uniform and rectilinear; consequently, that A, a, a' also lie in a right line, and the portions Aa, aa' are also proportional to the intervals $t - T$, $t' - t$. Hence it follows that the lines AP and $b'a'$ are parallel, and, therefore, that the first place is identical with the third; that is, that *the true place at the time T is identical with the apparent place at the time t.*

The time $t - T$ will be the product of the distance Pa into $497^s.78$, which is the time in which light describes the mean distance of the earth from the sun (Art. 395), this mean distance being taken as the unit. In this computation we may take for the distance either Pa or PA or pa, without sensible difference in the resulting value of $t - T$.

From these principles there arise three methods by which a planet's (or a comet's) apparent place may be found from the true place for a given time t:

I. From the given time t we subtract the time required by light to come from the planet to the earth. We thus obtain the reduced time T for which the true place is identical with the apparent place for t.

II. The true place and the distance being known for the time t, we compute the reduction $t - T$. Thus, by means of the diurnal motion of the planet (in longitude and latitude, or in right ascension and declination) we can reduce the true place from the time t to the time T; and the true place thus found is the apparent place at the time t.

III. The true place of the planet at the time T as seen from the point in which the earth is situated at the time t (or the direction aP) is computed, to which is applied the

aberration of the fixed stars, and the result is the apparent place at the time t.*

397. If α and δ are the true right ascension and declination of a planet or comet at a time t, α' and δ' the apparent values for the same time, r' its distance from the earth, the mean distance of the earth from the sun being unity, $\Delta\alpha$, $\Delta\delta$, the motion of the planet in right ascension and declination in one mean hour, we have, according to the method II. of the preceding article,

$$\left.\begin{aligned} \alpha' - \alpha &= - r'k''\Delta\alpha \\ \delta' - \delta &= - r'k''\Delta\delta \end{aligned}\right\} \qquad (687)$$

in which

$$k'' = \frac{497.78}{3600} \qquad\qquad \log k'' = 9.14073$$

These formulæ may also be used for computing the sun's aberration in right ascension and declination, if we take for r' the radius vector of the earth.

HELIOCENTRIC OR ANNUAL PARALLAX OF THE FIXED STARS.

398. The heliocentric parallax of a star is the difference between its true places seen from the earth and from the sun. If the orbit of the earth were a circle with a radius equal to the mean distance from the sun, the maximum difference between the heliocentric and geocentric places of any star would occur when the radius vector of the earth was at right angles to the line drawn from the earth to the star. This maximum corresponds, then, to the horizontal geocentric parallax; and its effect upon the apparent places of stars might be investigated by the methods followed in Chapter IV.; but we prefer to employ here the method just exhibited in the investigation of the aberration. on account of the analogy in the resulting formulæ.

We shall call the maximum angle subtended by the *mean distance* of the earth from the sun, at the distance of the star, the *constant* of *annual parallax of the star*, or, simply, its *annual parallax*. If then we put

---

* See GAUSS, *Theoria Motus Corporum Cœlestium.* Art. 71, from which the above article is chiefly extracted. Also, for the application of method III., see the same work, Art. 118, *et seq.*

$p =$ the annual parallax,
$a =$ the mean distance of the earth from the sun,
$\Delta =$ the distance of the star from the sun,

we have

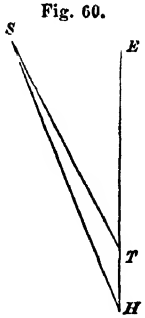
$$\sin p = \frac{a}{\Delta}$$

or, if we take $a = 1$, according to the usual practice, we have, for so small a quantity,

$$p = \frac{1}{\Delta \sin 1''} \tag{688}$$

399. *To find the heliocentric parallax of a star in longitude and latitude at a given time, the annual parallax being given.*—Let T (Fig. 60) be the place of the earth in its orbit, H that of the sun. Conceive a plane to be passed through the line HT and a star S; the intersection of this plane with the plane of the ecliptic is the line HT, which, produced to the celestial sphere, meets it in a point E of the ecliptic whose longitude is the earth's heliocentric longitude, or $180° + \odot$ (putting $\odot$ for the geocentric longitude of the sun at the given time). If then we also put

Fig. 60.



$r =$ the distance of the earth from the sun at the given time,
$\vartheta =$ the angle SHE,
$\vartheta' =$ " " STE,

the triangle SHT gives

$$\sin(\vartheta' - \vartheta) = \frac{r}{\Delta} \sin \vartheta'$$

or

$$\vartheta' - \vartheta = pr \sin \vartheta' \tag{689}$$

This formula corresponds to the formula (670) for the aberration reckoned in a direction *from* a point (E) of the ecliptic, only in the present case this point is in longitude $180° + \odot$, while in the case of the aberration it was in longitude $90° + \odot$. The formulæ for the aberration may therefore be immediately applied to the parallax if we put pr for k, and $180° + \odot$ for $90° + \odot$, or $90° + \odot$ for $\odot$. We thus find, by (675),

$$\left.\begin{aligned} \lambda' - \lambda &= -pr \sin(\lambda - \odot) \sec \beta \\ \beta' - \beta &= -pr \cos(\lambda - \odot) \sin \beta \end{aligned}\right\} \tag{690}$$

400. *To find the heliocentric parallax of a star in right ascension and declination, the annual parallax being given.*—By (678), putting pr for k, and $90° + \odot$ for $\odot$, we have, at once,

$$\left.\begin{aligned}\alpha' - \alpha &= -pr \sec\delta\,(\cos\odot\sin\alpha - \sin\odot\cos\varepsilon\cos\alpha)\\ \delta' - \delta &= -pr\sin\odot\,(\cos\varepsilon\sin\delta\sin\alpha - \sin\varepsilon\cos\delta)\\ &\quad - pr\cos\odot\sin\delta\cos\alpha\end{aligned}\right\}\quad(691)$$

401. It can be shown from (690) that, neglecting the small variations produced by the ellipticity of the earth's orbit, the effect of the annual parallax, considered alone, is to cause the star to appear to describe a small ellipse about its mean place in one sidereal year; an effect entirely analogous to that of the annual aberration, Art. 388. But the maximum and minimum of parallax occur when the earth is 90° from the points at which the maximum and minimum of aberration occur: so that the major axes of the parallax and aberration ellipses are at right angles to each other. The combined effect of both aberration and parallax is still to cause the star to describe an ellipse, the major axis of which is equal to the hypothenuse of a right triangle, of which the two legs are respectively equal to the major axes of the two ellipses. For this combined effect is expressed by the following formulæ (taking $r = 1$ for a circular orbit):

$$\begin{aligned}(\lambda' - \lambda) &= -[k\cos(\odot - \lambda) - p\sin(\odot - \lambda)]\sec\beta\\ (\beta' - \beta) &= -[k\sin(\odot - \lambda) + p\cos(\odot - \lambda)]\sin\beta\end{aligned}$$

which, if we assume c and γ to be determined by the conditions

$$\begin{aligned}c\sin\gamma &= k\sin\lambda - p\cos\lambda\\ c\cos\gamma &= k\cos\lambda + p\sin\lambda\end{aligned}$$

or

$$\begin{aligned}c\sin(\lambda - \gamma) &= p\\ c\cos(\lambda - \gamma) &= k\end{aligned}$$

become

$$\begin{aligned}(\lambda' - \lambda) &= -c\cos(\odot - \gamma)\sec\beta\\ (\beta' - \beta) &= -c\sin(\odot - \gamma)\sin\beta\end{aligned}$$

in which we have $c = \sqrt{(k^2 + p^2)}$. This form for the total effect is entirely analogous to that for the aberration alone.

MEAN AND APPARENT PLACES OF STARS.

402. The formulæ above given enable us to derive the apparent from the mean place, or the mean from the apparent place;

but in their present form their computation is exceedingly troublesome. We owe to BESSEL a very simple arrangement by which their application is facilitated.

In all catalogues of stars the mean places only can be given, and these only for a certain epoch. For each star there is given also the annual precession in right ascension and declination: so that the mean place for any time after or before the epoch of the catalogue is readily obtained, as in the example of Art. 374. But, since the annual precession is variable, there is generally added its *secular variation*, which is the variation of the precession in one hundred years. Finally, there is given the star's proper motion.

If the epoch of the catalogue is t_0, and the mean place is required for the time t. and we denote by

p, the precession for the epoch t_0,
Δp, its secular variation,
μ, the proper motion,

then, since in computing the whole precession for the interval $t - t_0$ we must employ the annual precession for the middle of the interval, the reduction of the mean place to the time t will be

$$\left[p + \frac{\Delta p}{200}(t - t_0) + \mu\right](t - t_0)$$

This form applies both to the right ascension and the declination.*

In this way the mean place is brought up to the *beginning* of any given year. If then we wish the apparent place for a time τ from the beginning of the year, τ being expressed in fractional parts of the year, we have first to obtain the mean place for the given date by adding the precession and proper motion for the interval τ, and then the apparent place, by further adding the nutation and aberration. Hence, denoting the mean right ascension and declination at the beginning of the year by α and δ, the apparent right ascension and declination for the given time τ by

* The annual proper motions being also variable (Art. 379), it would seem that their values given for the epoch of the catalogue could not be carried forward to another time without correction. But, to avoid the necessity for this separate correction, it may be included in the secular variation of the precession, as is done by ARGELANDER in his catalogue, "DLX *Stellarum Fixarum Positiones Mediæ, ineunte anno* 1830."

α' and δ', the annual proper motions in right ascension and declination by μ and μ', we have, by (663), (668), and (678),

$$
\begin{array}{lll}
\alpha' = \alpha + \tau(m + n \sin\alpha \tan\delta) + \tau\mu & & \text{(Precession and proper motion.)} \\
\quad -(15''.8148 + 6''.8650 \sin\alpha\tan\delta)\sin\Omega & \text{for } 1800 & \\
\qquad\ \ 15\ .8321 \qquad 6\ .8682 & \phantom{\text{for }} 1900 & \\
\quad +(\ 0\ .1902 + 0\ .0825 \sin\alpha\tan\delta)\sin 2\Omega & & \\
\quad -(\ 0\ .1872 + 0\ .0813 \sin\alpha\tan\delta)\sin 2\text{☾} & & \\
\quad +(\ 0\ .0621 + 0\ .0270 \sin\alpha\tan\delta)\sin(\text{☾} - \Gamma') & & \\
\quad -(\ 1\ .1644 + 0\ .5055 \sin\alpha\tan\delta)\sin 2\odot & & \\
\quad +(\ 0\ .1173 + 0\ .0509 \sin\alpha\tan\delta)\sin(\odot - \Gamma) & & \\
\quad -(\ 0\ .0195 + 0\ .0085 \sin\alpha\tan\delta)\sin(\odot + \Gamma) & & \text{(Nutation.)} \\
\qquad\qquad - 9''.2231 \cos\alpha\tan\delta\cos\Omega & 1800 & \\
\qquad\qquad\quad 9\ .2240 & 1900 & \\
\qquad\qquad + 0\ .0897 \cos\alpha\tan\delta\cos 2\Omega & & \\
\qquad\qquad - 0\ .0886 \cos\alpha\tan\delta\cos 2\text{☾} & & \\
\qquad\qquad - 0\ .5510 \cos\alpha\tan\delta\cos 2\odot & & \\
\qquad\qquad\ \cdot\ 0\ .0093 \cos\alpha\tan\delta\cos(\odot + \Gamma) & & \\
\quad - 20''.4451 \cos\epsilon\cos\odot\cos\alpha\sec\delta & & \\
\quad - 20\ .4451 \sin\odot\sin\alpha\sec\delta & & \text{(Aberration.)}
\end{array}
$$

$$
\begin{array}{lll}
\delta' = \delta + \tau . n\cos\alpha + \tau\mu' & & \text{(Precession and proper motion.)} \\
\quad - 6''.8650 \cos\alpha\sin\Omega + 9''.2231 \sin\alpha\cos\Omega & \text{for } 1800 & \\
\qquad 6\ .8682 \qquad\qquad\qquad\quad 9\ .2240 & \phantom{\text{for }} 1900 & \\
\quad + 0\ .0825 \cos\alpha\sin 2\Omega - 0\ .0897 \sin\alpha\cos 2\Omega & & \\
\quad - 0\ .0813 \cos\alpha\sin 2\text{☾} + 0\ .0886 \sin\alpha\cos 2\text{☾} & & \\
\quad + 0\ .0270 \cos\alpha\sin(\text{☾} - \Gamma') & & \text{(Nutation.)} \\
\quad - 0\ .5055 \cos\alpha\sin 2\odot + 0\ .5510 \sin\alpha\cos 2\odot & & \\
\quad + 0\ .0509 \cos\alpha\sin(\odot - \Gamma) & & \\
\quad - 0\ .0085 \cos\alpha\sin(\odot + \Gamma) + 0\ .0093 \sin\alpha\cos(\odot + \Gamma) & & \\
\quad - 20''.4451 \cos\epsilon\cos\odot(\tan\epsilon\cos\delta - \sin\alpha\sin\delta) & & \\
\quad - 20\ .4451 \sin\odot\cos\alpha\sin\delta & & \text{(Aberration.)}
\end{array}
$$

Now, it is to be remarked that the two numerical coefficients of $\sin\Omega$, $\sin 2\Omega$, $\sin 2\text{☽}$, &c. in the formula for α' are in each case very nearly in the ratio of m to n;* and hence, if, according to the method of BESSEL, we put

$$
\begin{array}{ll}
6''.8650 = ni & 15''.8148 = mi + h \\
6\ .8682 & 15\ .8321 \\
0\ .0825 = ni' & 0\ .1902 = mi' + h' \\
0\ .0813 = ni'' & 0\ .1872 = mi'' + h'' \\
0\ .0270 = ni''' & 0\ .0621 = mi''' + h''' \\
0\ .5055 = ni^{\text{iv}} & 1\ .1644 = mi^{\text{iv}} + h^{\text{iv}} \\
0\ .0509 = ni^{\text{v}} & 0\ .1173 = mi^{\text{v}} + h^{\text{v}} \\
0\ .0085 = ni^{\text{vi}} & 0\ .0195 = mi^{\text{vi}} + h^{\text{vi}}
\end{array}
$$

* This relation is not accidental, but results from the general theory of nutation, which, the student will remember, is only the periodical part of the precession.

we shall have

$$
\begin{aligned}
\alpha' = \alpha &+ [\tau - i \sin \text{☊} + i' \sin 2\text{☊} - i'' \sin 2\text{☾} + i''' \sin (\text{☾} - \Gamma') \\
&\qquad - i^{\text{iv}} \sin 2\odot + i^{\text{v}} \sin (\odot - \Gamma) - i^{\text{vi}} \sin (\odot + \Gamma)]\ [m + n \sin \alpha \tan \delta] \\
&- [9''.2231 \cos \text{☊} - 0''.0897 \cos 2\text{☊} + 0''.0886 \cos 2\text{☾} \\
&\quad\ 9\ \ 2240 \\
&\qquad + 0''.5510 \cos 2\odot + 0''\,0093 \cos (\odot + \Gamma)] \cos \alpha \tan \delta \\
&- 20''.4451 \cos \varepsilon \cos \odot \cos \alpha \sec \delta \\
&- 20\ .4451 \sin \odot \sin \alpha \sec \delta \\
&+ \tau\mu \\
&- h \sin \text{☊} + h' \sin 2\text{☊} - h'' \sin 2\text{☾} + h''' \sin (\text{☾} - \Gamma') \\
&\qquad - h^{\text{iv}} \sin 2\odot + h^{\text{v}} \sin (\odot - \Gamma) - h^{\text{vi}} \sin (\odot + \Gamma)
\end{aligned}
$$

and

$$
\begin{aligned}
\delta' = \delta &+ [\tau - i \sin \text{☊} + i' \sin 2\text{☊} - i'' \sin 2\text{☾} + i''' \sin (\text{☾} - \Gamma') \\
&\qquad - i^{\text{iv}} \sin 2\odot + i^{\text{v}} \sin (\odot - \Gamma) - i^{\text{vi}} \sin (\odot + \Gamma)]\ n \cos \alpha \\
&+ [9''.2231 \cos \text{☊} - 0''.0897 \cos 2\text{☊} + 0''.0886 \cos 2\text{☾} \\
&\quad\ 9\ .2240 \\
&\qquad + 0''.5510 \cos 2\odot + 0''.0093 \cos (\odot + \Gamma)] \sin \alpha \\
&- 20''.4451 \cos \varepsilon \cos \odot (\tan \varepsilon \cos \delta - \sin \alpha \sin \delta) \\
&- 20\ .4451 \sin \odot \cos \alpha \sin \delta \\
&+ \tau\mu'
\end{aligned}
$$

Putting then, in accordance with BESSEL's original notation, as employed in the American Ephemeris for 1865 and subsequent years,

$$
\begin{aligned}
A &= \tau - i \sin \text{☊} + i' \sin 2\text{☊} - i'' \sin 2\text{☾} + i''' \sin (\text{☾} - \Gamma') \\
&\qquad - i^{\text{iv}} \sin 2\odot + i^{\text{v}} \sin (\odot - \Gamma) - i^{\text{vi}} \sin (\odot + \Gamma) \\
B &= -9''\,2231 \cos \text{☊} + 0''\,0897 \cos 2\text{☊} - 0''.0886 \cos 2\text{☾} \\
&\quad\ 9\ \ 2240 \\
&\qquad - 0''.5510 \cos 2\odot - 0''.0093 \cos (\odot + \Gamma) \\
C &= -20''.4451 \cos \varepsilon \cos \odot \\
D &= -20\ .4451 \sin \odot \\
E &= -h \sin \text{☊} + h' \sin 2\text{☊} - h'' \sin 2\text{☾} + h''' \sin (\text{☾} - \Gamma') \\
&\qquad - h^{\text{iv}} \sin 2\odot + h^{\text{v}} \sin (\odot - \Gamma) - h^{\text{vi}} \sin (\odot + \Gamma)
\end{aligned}
$$

which quantities are dependent on the time, and are wholly independent of the star's place; and also

$$
\begin{aligned}
a &= m + n \sin \alpha \tan \delta & a' &= n \cos \alpha \\
b &= \cos \alpha \tan \delta & b' &= -\sin \alpha \\
c &= \cos \alpha \sec \delta & c' &= \tan \varepsilon \cos \delta - \sin \alpha \sin \delta \\
d &= \sin \alpha \sec \delta & d' &= \cos \alpha \sin \delta
\end{aligned}
$$

which depend on the star's place, we have

$$\left.\begin{aligned} \alpha' &= \alpha + Aa + Bb + Cc + Dd + E + \tau\mu \\ \delta' &= \delta + Aa' + Bb' + Cc' + Dd' + \tau\mu' \end{aligned}\right\} \qquad (692)$$

The logarithms of A, B, C, D are given in the Ephemeris for every day of the year. The residual E never exceeds $0''.05$, and may usually be omitted. The logarithms of a, b, c, d, a', b', c', d' are usually given in the catalogues, but where not given are readily computed by the above formulæ. When the right ascension is expressed in time, the values of a, b, c, d, above given, are to be divided by 15.

403. If we substitute the values of m and n, namely,

$$\begin{array}{llll} \text{for } 1800, & m = 46''.0623 & & n = 20''.0607 \\ \phantom{\text{for }} 1900, & m = 46\ .0908 & & n = 20\ .0521 \end{array}$$

we find the following values of i, i', &c.:

	i	i'	i''	i'''	i^{iv}	i^{v}	i^{vi}
1800	0.34221	0.00411	0.00405	0.00185	0.02520	0.00254	0.00042
1900	0.34252				0.02521		

	h	h^{iv}	
1800	$+0''.052$	$+0''.004$	and h', h'', h''', h^{v}, h^{vi} inappreciable.
1900	$+0\ .045$	$+0''.003$	

The terms in i^{v} and i^{vi} in the expression of A may be combined in a single term; for, putting

$$\begin{aligned} j \cos J &= (i^{\text{v}} - i^{\text{vi}}) \cos \Gamma \\ j \sin J &= -(i^{\text{v}} + i^{\text{vi}}) \sin \Gamma \end{aligned}$$

we have

$$i^{\text{v}} \sin(\odot - \Gamma) - i^{\text{vi}} \sin(\odot + \Gamma) = j \sin(\odot + J)$$

and taking for 1800, $\Gamma = 279^\circ\ 30'\ 8''$; and for 1900, $\Gamma = 281^\circ\ 12'\ 42''$, we find

	j	J
1800	$+$ 0.00294	83° 10′
1900	$+$ 0.00293	81 ·55.

Hence the values of A, B, and E may be expressed as follows:

$$
\begin{aligned}
A &= \tau - \underset{0.34252}{0.34221} \sin ☊ - 0.02520 \sin 2\odot + \underset{0.00293}{0.00294} \sin (\odot + \underset{81\ \ 55}{83^\circ\ 10'}) && \underset{\text{“}\ \ 1900}{\text{for } 1800} \\
&\quad + 0.00411 \sin 2☊ - 0.00405 \sin 2☾ + 0.00135 \sin (☾ - \Gamma') \\
B &= - \underset{9\ .2240}{9''.2231} \cos ☊ - 0''.5510 \cos 2\odot - 0''.0093 \cos (\odot + \underset{281\ \ 13}{279^\circ\ 30'}) && \underset{\text{“}\ \ 1900}{\text{“}\ \ 1800} \\
&\quad + 0\ .0897 \cos 2☊ - 0\ .0886 \cos 2☾ \\
E &= - \underset{0\ .045}{0''.052} \sin ☊ - \underset{0\ .003}{0''.004} \sin 2\odot && \underset{\text{“}\ \ 1900}{\text{“}\ \ 1800}
\end{aligned}
$$

These values agree (within quantities practically inappreciable) with those given by Dr. PETERS (*Numerus Constans Nutationis*, pp. 75, 76). It is necessary to remark that in the British Association Catalogue and the British Nautical Almanac, the preceding values of C and D are denoted by A and B, and *vice versa*.* See p. 94.

404. When the catalogue does not give the logs of a, b, c, &c., another form of reduction, also proposed by BESSEL, may sometimes be preferred. By putting

$$
\begin{aligned}
f &= mA + E & i &= C \tan \varepsilon \\
g \cos G &= nA & h \cos H &= D \\
g \sin G &= B & h \sin H &= C
\end{aligned}
$$

we find

$$
\left.
\begin{aligned}
\alpha' &= \alpha + f + \tau\mu + g \sin (G + \alpha) \tan \delta + h \sin (H + \alpha) \sec \delta \\
\delta' &= \delta + i \cos \delta + \tau\mu' + g \cos (G + \alpha) + h \cos (H + \alpha) \sin \delta
\end{aligned}
\right\} \quad (692^*)
$$

The values of f, $\log g$, G, $\log h$, H, $\log i$, and $\log \tau$ are given in the Ephemeris for every day of the year.

405. A star's apparent place may be reduced to its mean place and referred to the mean equinox of any given date by reversing the signs of the reductions as above determined. By the *apparent* place of a star we here mean the *apparent geocentric* place. The *observed* place (that seen from the surface of the earth) differs

---

* This interchange of letters, most unnecessarily introduced by BAILY in the British Association Catalogue, produces considerable inconvenience, as in most of the European catalogues of stars BESSEL's notation is preserved, while in the English Almanac BAILY's notation is followed. In the American Ephemeris for 1865 and subsequent years the notation of BESSEL has been restored: an example which will doubtless be followed by the British Almanac at an early day.

from this by the diurnal aberration and the refraction; but the first of these corrections depends on the latitude of the observer and the star's hour angle, and the second upon the star's zenith distance: so that neither of them can be brought into the computation of a star's position until the place of observation and the local time are given.

406. *The fictitious year.*—In the preceding investigations, we have used the expression "beginning of the year," without giving it a definite signification. For the purpose of introducing uniformity and accuracy in the reduction of stars' places, BESSEL proposed a *fictitious year*, to begin at the instant when the sun's mean longitude is 280°. This instant does not correspond to the beginning of the tropical year on the meridian of Greenwich; that is, the (mean) sun is not at this instant on the meridian of Greenwich, but on a meridian whose distance from that of Greenwich can always be determined by allowing for the sun's mean motion. This meridian at which the fictitious year begins will vary in different years; but, since the sun's mean right ascension is equal to his mean longitude (Art. 41), the sidereal time at this meridian when the fictitious year begins is always $18^h\ 40^m$ (= 280°). By the employment of this epoch, therefore, the reckoning of sidereal time from the beginning of the year is simplified, and, accordingly, it is now generally adopted as the epoch of the catalogues of stars. In the value of log A, which involves the fraction of a year (τ), the same origin of time must be used; and this is attended to in the computation of the Ephemerides, which now give not only the logarithms of A, B, C, and D, but also the value of τ (or its logarithm) reckoned from the beginning of the fictitious year and reduced to decimal parts of the mean tropical year.

For all the purposes of reduction of modern observations, the computer need not enter further into this subject, and may depend upon the Ephemerides.* But, as the subject is inti-

* The reduction of observations made between 1750 and 1850 will be most conveniently performed by the aid of the *Tabulæ Regiomontanæ* of BESSEL. The constants used by BESSEL differ materially from those now adopted in the American and British Almanacs. Professor HUBBARD has given a very simple table by which the values of log A, log B, log C, and log D as given in the *Tab. Reg.* may be reduced to those which follow from the use of PETERS's constants, in the *Astronomical Journal*, Vol. IV. p. 142. The special and general tables for the reduction of stars' places,

mately connected with that of time in general, I shall prosecute it a little further.

407. *The sun's mean motion, and the length of the year.*—According to BESSEL,* the sun's mean longitude at mean noon at Paris in 1800, January 0, is

$$279° \; 54' \; 1''.36$$

and the sun's sidereal motion in 365.25 mean days is

$$360° - 22''.617656$$

(By January 0 is denoted the noon of December 31 in the common mode of expressing the date; and, consequently, Jan. 1, 2, &c. denote 1 day, 2 days, &c. from the epoch, while in the common mode they mean the beginning of the 1st day, 2d day, &c.)

The sidereal motion is referred to a *fixed* point of the ecliptic; but the mean longitude is referred to the *moving* vernal equinox. Hence the change of the mean longitude in any time is the sidereal motion in that time *plus* the general precession; and therefore, adopting here BESSEL's precession constant,† in order to follow his computations,

$$\begin{array}{lcl}
\text{Sid motion in } 365^d.25 & = & 360° - 22''.617656 \\
\text{General precession} & = & \quad\;\; + 50\;.223500 + 0''.0002443610t \\
\hline
\text{Mean motion in } 365^d.25 & = & 360° + 27\;.605844 + 0\;.0002443610t
\end{array}$$

and, dividing by 365.25,

$$\text{Mean daily motion} = 59' \; 8''.3302 + 0''.0000006902t$$

where t is the number of years after 1800. The secular change of the mean motion, expressed by the second term, brings with it a secular change of the length of the tropical year. This year

---

given in the *Washington Astronomical Observations*, Vol. III., Appendix C, are also adapted to the new constants.

For the reduction of observations from 1850 to 1880, the *Tab. Reg.* have been continued by WOLFERS and ZECH (*Tabulæ Reductionum Observationum Astronomicarum Annis* 1860 *usque ad* 1880 *respondentes, auctore* J. PH. WOLFERS: *Additæ sunt, Tabulæ Regiomontanæ annis* 1850 *usque ad* 1860 *respondentes* ab ILL. ZECH *continuatæ*. Berlin, 1850) In the continuation by ZECH, which extends from 1850 to 1860, all the constants are the same as those used by BESSEL; in the continuation by WOLFERS, from 1860 to 1880, BESSEL's precession constant is retained, but PETERS's nutation constant is adopted.

* *Astron. Nach.*, No. 133. † Ibid.

is the time in which the sun changes his mean longitude exactly 360°, and is, therefore, found by dividing 360 by the mean daily motion: thus, if we put

$$Y = \text{the length of the tropical year in mean solar days,}$$

we find

$$Y = 365^d.242220027 - 0^d.00000006886\,t$$

where the value of the second term for $t = 100$ is $0^s.595$, which is the diminution of the length of the tropical year in a century.

The length of the sidereal year is invariable, and is readily found by adding to 365.25 the time required by the sun to move through 22″.617656 at the rate of his sidereal motion; or, putting

$$Y' = \text{the length of the sidereal year,}$$

by the proportion

$$360° - 22''.617656 : 360° = 365^d.25 : Y'$$

which gives

$$Y' = 365.256374416 \text{ mean solar days,}$$
$$= 366.256374416 \text{ sidereal days}$$

408. *The epoch of the sun's mean longitude.*—This term denotes the instant at which the common year begins. The value of the longitude itself at this instant is frequently called "the epoch," and is denoted by E. Its value for January 0 of any year, $1800 + t$, is found by adding the motion in 365 days for each year not a leap year, and the motion in 366 days for each leap year. The motion in 365 days is found from the above value for 365.25 days by deducting one-fourth the mean daily motion, or 14′ 47″.083: so that, if f denotes the remainder after the division of t by 4, we have, for the epoch of $1800 + t$, Jan. 0, at Paris,

$$E = 279°\ 54'\ 1''.36 + 27''.605844\,t + 0''.0001221805\,t^2 - (14'\ 47''.083)f \qquad (693)$$

To extend this formula to years preceding 1800, we must put $f - 4$ in the place of f: so that the multiplier of $(-\ 14'\ 47''.083)$ will be, for example, -1, -2, -3, -4, -1, &c. for the years 1799, '98, '97, '96, '95, &c. But these rules for f will give the

mean longitude at the *beginning* of the leap years too great by the motion in one day (since the additional day is not added until the end of February); and therefore the epoch for these years is January 1 instead ot January 0. A general table containing the mean longitude at mean noon for every day of the year, computed from the mean longitude for Jan. 0 by the formula, will be applicable to leap years if in the months of January and February we increase the argument of the table by one day, as in Table VI. of the *Tab. Reg.*

409. *To find the beginning of the fictitious year.*—Denoting the mean time from the beginning of the fictitious year to Jan. 0 of any year by k, we have

$$k = -\frac{E - 280^\circ}{\text{mean daily motion}}$$

whence, taking the daily motion = 59′ 8″.3302, we find, in decimals parts of a mean day,

$$\begin{aligned} k = & -0.10107289 + 0.0077799535\,t \\ & \qquad\quad -\tfrac{1}{4}f \quad + 0.000000034433\,t^2 \end{aligned}$$

The quantity k is evidently equal to the east longitude from Paris of that terrestrial meridian on which the fictitious year begins (Art. 406).

410. In the *Tabulæ Regiomontanæ* the argument is the *reduced date* as it would be reckoned at the meridian in the east longitude k, the beginning of the fictitious year being always denoted by January 0. If then d is the west longitude from Paris of any place, the instant of mean noon at this place corresponds to the instant $k + d$ of the fictitious meridian, and therefore $k + d$ is the reduction to apply to the mean time at the place to obtain the argument with which to enter those tables.

But, if the sidereal time at the place d is given, it is most expedient to reckon the time at once in sidereal days from the beginning of the fictitious year. Accordingly, in the tables containing the values of log A, log B, &c. for the reduction of stars, the argument is the *sidereal date* at the fictitious meridian. To obtain this date, it is to be observed, first, that the tables are immediately available on the fictitious meridian for the sidereal time $18^h\,40^m$, without any reduction of the date. For any other

meridian, at the sidereal time $18^h\ 40^m$ the argument of the table will be the *reduced date;* but at any other sidereal time g the argument must be this reduced date increased by

$$\frac{g - 18^h 40^m}{24^h}$$

which must be always taken < 1 and positive; or by the quantity

$$g' = \frac{g + 5^h\ 20^m}{24^h}$$

omitting one whole day if $g + 5^h\ 20^m \gtreqless 24^h$. Now, in order that the local date may correspond with that supposed in the tables, the day must be supposed to begin at the instant when that point is on the meridian whose right ascension is $18^h\ 40^m$. Therefore, whenever the right ascension of the sun is as great as $18^h\ 40^m$, so that the point in question culminates *before* the sun, one day must be added to the common reckoning. Hence the formula for preparing the argument of the tables will be

Argument = Reduced date $+ g' + i$;
in which we must take $i = 0$ from the beginning of the year to the time when the sun's R. A. $= g$, and $i = +1$ after this time.

The values of g' are given on p. 16 of the *Tab. Reg.* for given values of g. The values of k are given in Table I.

The values of $\log A$, $\log B$, $\log C$, $\log D$ are also given in the Berlin Jahrbuch for the fictitious date; and the constants of precession, nutation, and aberration are the same as those employed by BESSEL in the *Tab. Reg.*

411. *Conversion of mean into sidereal time, and vice versa.*—In the explanation of this subject in Chapter II. we said nothing of the effect of nutation, which we will now consider. Let us go back to the definitions and state them more precisely.

1st. The first mean sun, which may be denoted by $\odot_1$, moves uniformly in the ecliptic, returning to the perigee with the true sun. The longitude of this fictitious sun referred to the *mean* equinox is called *the sun's mean longitude.*

2d. The second mean sun, which may be denoted by $\odot_2$, moves

uniformly in the equator, returning to the *mean* equinox with the first mean sun.

3d. The sidereal day begins with the transit of the *true* equinox; and the sidereal time is the hour angle of the *true* equinox.

Hence it follows that

$$\text{the mean R. A. of } \odot_2 = \text{the mean long. of } \odot_1 = \text{the sun's mean longitude;}$$

and since when $\odot_2$ is on the meridian, its R. A. reckoned from the *true* equinox is also the hour angle of the true equinox, it also follows that

$$\begin{aligned} V_0 &= \text{the sidereal time at mean noon.} \\ &= \text{true R. A. of } \odot_2 \\ &= \text{mean R. A. of } \odot_2 + \text{nutation of the equinox in R. A.} \\ &= \text{sun's mean longitude} + \text{nutation of the equinox in R. A.} \end{aligned}$$

The nutation of the equinox in R. A. is found from the first equation on p. 626 by putting $\alpha = 0$, $\delta = 0$, whence

$$\text{nutation of equinox in R. A.} = \Delta\lambda \cos \varepsilon$$

which is the quantity given in the Nautical Almanac as the "equation of the equinoxes in right ascension."

Since the nutation is contained in the value of V_0 given in the Almanac for each mean noon, no further attention to it is needed when that work is consulted; and the rules given in Chapter II. are therefore practically complete.

For the conversion of time between 1750 and 1850, the *Tab., Reg.* furnish the following facilities:—Table VI. gives the right ascension of the second mean sun, corrected for the solar nutation of the equinox, for every mean noon at the fictitious meridian k. Since the fictitious year always begins with the same mean longitude of the sun (or right ascension of $\odot_2$), the numbers of this table are general, and may be used for every year. The number taken from this table for any given date (which must be the *reduced date* above explained) are then corrected for the lunar nutation of the equinox in right ascension, which is given in Table IV. for all dates between 1750 and 1850. We thus obtain the *sidereal time at mean noon* ($= V_0$) at the fictitious meridian on the given day. Any given mean time at another meridian is then converted into the corresponding sidereal time.

or *vice versa*, according to the rules in Chapter II., employing the V_0 for the fictitious meridian precisely as it was there employed for the meridian of Greenwich.—The longitude of the place to be used here is $k + d$, d being the west longitude of the place from Paris, and k the east longitude of the fictitious meridian from Paris given in Table I.

REDUCTION OF THE APPARENT PLACE OF A PLANET OR COMET.

412. The *observed* place of a planet (or comet) being freed from the effect of refraction, diurnal aberration, and geocentric parallax, we have the apparent geocentric place, referred to the true equator and equinox of the time of observation, and affected by the planetary aberration. For the calculation of a planet's orbit from three or more observations at different times, it is necessary to refer its places at these times to the same common fixed planes, which is most readily effected by reducing all the places to the equinox of the beginning of the year in which the observations are made, or, when the observations extend beyond one year, to the beginning of any assumed year. To effect this, we must apply to each apparent geocentric place—1st. The aberration (687), with its sign reversed, in computing which the position of the observer on the surface of the earth may be considered by taking r' equal to the actual distance of the planet from the observer at the time of observation. This distance is found from the geocentric distance at the same time with the parallax, by the equation (137).

2d. The nutation for the date of the observation, with its sign reversed.

3d. The precession from the date of the observation to the assumed epoch, which will be subtracted or added according as the epoch precedes or follows the date.

But the nutation and precession are most conveniently computed together by the aid of the constants A and B used for the fixed stars. These constants being taken for the date, a, b, a', and b' are to be computed as in Art. 402, with the right ascension and declination of the planet; and then to the α and δ, already corrected for aberration, we apply the corrections $-(Aa + Bb)$ and $-(Aa' + Bb')$ respectively. The place thus obtained is the *true place of the planet referred to the mean equinox of the beginning of the year.* If the several observations are in different

years, they are then to be reduced to the same epoch by simply applying the annual precession, c being the annual precession in right ascension, and c' that in declination.

When the distance of the planet is not known, the aberration is taken into account by Method III. of Art. 396; but the details of this subject belong to the computation of orbits, which is reserved for Physical Astronomy.—See GAUSS, *Theor. Mot. Corp. Cœl.*, Art. 118 *et seq.*

CHAPTER XII.

DETERMINATION OF THE OBLIQUITY OF THE ECLIPTIC AND THE ABSOLUTE RIGHT ASCENSIONS AND DECLINATIONS OF STARS BY OBSERVATION.

413. THE most obvious method of finding the obliquity of the ecliptic is to measure the sun's apparent declination at either the northern or the southern solstice; for at these points, assuming the sun to be exactly in the ecliptic, the declination is equal to the obliquity. Indeed, without any reference to the sun's absolute declination, a rude approximate value of the obliquity is at once derived by taking one-half of the difference of the meridian altitudes of the sun on the 21st of June and the 21st of December. Upon this principle the ancients succeeded in measuring the obliquity by observing the greatest and least lengths of the meridian shadow of a gnomon.

414. In what follows, we suppose the sun's declination to be observed. This is obtained from the true meridian zenith distance (ζ) of the sun's centre, and the known latitude of the place of observation (φ), by the formula*

$$\delta = \varphi - \zeta$$

---

* The sign of ζ is to be changed when the sun is north of the zenith of the observer.

415. Now, the sun's declination is equal to the obliquity only when it has reached its maximum (northern or southern) limit, that is, precisely at the solstitial points. But, since the sun will, in general, not arrive at the solstice at the same time that it culminates at the particular meridian at which the observation is made, we cannot directly measure this maximum by meridian observations. But we can measure the declination at several successive transits near the solstice, and then by interpolation infer the maximum value. A simpler practical process (which we shall explain fully below) is to reduce each observation to the solstice; but this requires us to know (at least approximately) the *time* when the sun arrives at the solstice, and this, again, supposes a knowledge of the position of the equinoctial points, which are 90° distant from the solstitial points.

The position of the equinoctial points may be determined by observing the sun's declination on several successive days near the time of the equinoxes, and, by interpolation, finding the time when the declination is zero. At the same time, a comparison must be made between the times of transit of the sun and some star, adopted as a *fundamental* star: so that the distance of the star from the equinoctial point, or its right ascension, is fixed. We may then regard the star as a fixed point of comparison by which the instants when the sun arrives at any given points (as the solstices) may be determined. But, instead of finding the equinoctial points by a direct interpolation, it is preferable in this case also to refer each observation to the equinox, which, as will be seen below, requires an approximate knowledge of the obliquity of the ecliptic.

The determination of these two elements, the obliquity of the ecliptic and the position of the equinoctial points, is, therefore, effected by successive approximations: but, in the actual state of astronomy, the approximations are already so far carried out that the remaining error in either element can be treated as a differential which, by a judicious arrangement of the observations, produces only insensible errors of a higher order in the other element. I proceed to treat fully of the precise practical methods.

416. *Determination of the obliquity of the ecliptic.*—Let D be the sun's apparent declination derived from an observation near the solstice; A its apparent right ascension at the time of the obser-

vation, derived from the solar tables; ε the apparent obliquity of the ecliptic for the same time. If the sun were exactly in the ecliptic, we should have, by (34),

$$\sin A \tan \varepsilon = \tan D$$

but accuracy requires that the sun's latitude, β, should be taken into account. We have, by (29),

$$\tan D - \tan \varepsilon \sin A = \frac{\sin \beta}{\cos D \cos \varepsilon}$$

which, if we put

$$\tan D' = \tan \varepsilon \sin A \tag{694}$$

becomes

$$\tan D - \tan D' = \frac{\sin (D - D')}{\cos D \cos D'} = \frac{\sin \beta}{\cos D \cos \varepsilon}$$

whence, with sufficient accuracy, since β never exceeds 1″,

$$D - D' = \beta \sec \varepsilon \cos D \tag{695}$$

Hence, if the correction $\beta \sec \varepsilon \cos D$ is subtracted from the given declination D, we shall obtain the *reduced* declination D', from which, by (694), we can deduce ε. It is evident that D' is the declination of the point in which the ecliptic is intersected by the declination circle drawn through the sun's centre, and we may call the quantity $\beta \sec \varepsilon \cos D$ the *reduction to the ecliptic.* Near the solstices, however, this reduction does not sensibly differ from β, since $\cos \varepsilon$ and $\cos D$ are then very nearly equal. We shall, therefore, in the present problem, find the reduced declination by the formula $D' = D - \beta$; and then we have, by (694),

$$\tan \varepsilon = \tan D' \operatorname{cosec} A \tag{696}$$

Instead of computing ε from this equation directly, it is usual to employ its development in series by which the difference of ε and D' is obtained. For, since A near the northern solstice is nearly 90°, if we put

$$u = 90° - A$$

u will be a small angle whose cosine and secant will not differ much from unity, and the equation (696), expressed in the form

$\tan D' = \tan \varepsilon \cos u$, will be developed in the series [Pl. Trig., Art. 254]

$$D' - \varepsilon = q \sin 2\varepsilon + \tfrac{1}{2} q^2 \sin 4\varepsilon + \tfrac{1}{3} q^3 \sin 6\varepsilon + \&c.$$

in which

$$q = \frac{\cos u - 1}{\cos u + 1} = - \tan^2 \tfrac{1}{2} u$$

and the terms of the series are expressed in arc. Reducing to seconds, and putting

$$x = \text{the reduction to the solstice},$$

or

$$x = \frac{\tan^2 \frac{1}{2} u}{\sin 1''} \sin 2\varepsilon - \frac{\tan^4 \frac{1}{2} u}{2 \sin 1''} \sin 4\varepsilon + \&c \tag{697}$$

we have, *at the northern solstice*,

$$\varepsilon = D' + x = D - \beta + x \tag{698}$$

The reduction x can be tabulated, for any assumed value of ε, with the argument u. The changes of the tabular numbers depending on a change of the obliquity may also be given in the table: so that these numbers may be readily made to correspond to any assumed obliquity.

For the southern solstice, we take $u = 270° - A$, and the equation (696) will give $\tan D' = - \tan \varepsilon \cos u$, the development of which gives the algebraic sum $D' + \varepsilon$; but we can avoid the use of two formulæ by throwing this change of sign upon ε, regarding the obliquity obtained from the southern solstice as negative, during the computation. This simply changes the sign of the reduction x.

417. Let us now inquire what effect an error in the right ascensions taken from the tables, or in u, will produce in the computed value of ε. Differentiating the equation (696) with reference to ε and $A = \pm 90° - u$, we find

$$d\varepsilon = \tfrac{1}{2} \tan u \sin 2\varepsilon \, du$$

If we suppose the error in the tabular right ascension of the sun to be in any case as great as one second of time (the actual probable error, however, being much less), and, therefore, substitute in this equation $du = 15''$, $\varepsilon = 23° \, 27'.5$, we find

$$d\varepsilon = 5''.48 \tan u$$

For $u = 10°$, this gives $d\varepsilon = 0''.97$. The sun's motion being about 1° per day, we shall have $u < 10°$ for observations within ten days of the solstice, and the error in the computed obliquity less than 1″, even if the error in the right ascensions is as great as 15″. But this error will be wholly eliminated if observations equidistant from the solstice preceding and following it are combined; for then u, and consequently also $d\varepsilon$, will have equal numerical values with opposite signs, and the errors will destroy each other in the mean.

418. The mean of the values of the obliquity found from a number of observations, preceding and following the solstice and symmetrically disposed, will, therefore, be taken as the value of the obliquity at the time of the solstice, free from errors in the right ascension, and affected only by the unavoidable errors of observation and by any errors that may exist in the refraction and parallax or in the latitude of the place of observation. The error in the latitude is eliminated by taking the mean of the values of the obliquity found at the northern and the southern solstices. The error of the refraction tables will at the same time be partially eliminated; but not wholly, since these errors have probably different values at zenith distances differing so much as 47°; but a sensible error in the mean resulting from any probable error in the present value of the solar parallax is not to be feared.

Before taking the mean, however, it is proper to deduct from each value the nutation of the obliquity ($\Delta\varepsilon$, Art. 381), for the times of the two solstices respectively, whereby we obtain the *mean obliquity;* and then to reduce this to the same fixed epoch, as the beginning of the year, by allowing for the annual decrease. The value of this annual decrease adopted in (646) is $0''.4738$; but this value was deduced by PETERS from theory, while the value derived directly from observations at distant periods is, according to BESSEL, $0''.457$, and, according to PETERS, $0''.4645$

In combining a number of determinations made at the same place in different years, it is not indispensable that there should be observations at both solstices in every year, provided there are *in all* as many determinations at the northern as at the southern solstice.

419. EXAMPLE.—Find the obliquity of the ecliptic from the

following apparent declinations of the sun's centre, observed at the Washington Observatory by Professor COFFIN and Lieutenant PAGE, with the mural circle.

1846.	D	1846.	D
June 16	23° 21′ 56″.02	December 14	— 23° 14′ 17″.26
" 19	26 28 .19	" 15	17 33 .82
" 20	27 6 .79	" 16	20 22 .94
" 23	26 39 .92	" 18	24 32 .69
" 27	20 17 .84	" 21	27 20 .43
		" 22	27 19 .64
		" 23	26 49 .82
		" 29	14 1 .20

Taking $5^h\ 8^m\ 11^s.2$ as the longitude of Washington from Greenwich, we find, for apparent noon at Washington, the following values of the sun's right ascension and latitude from the Nautical Almanac:

1846.	A	β	1846.	A	β
June 16	$5^h 38^m 37^s.13$	+ 0″.18	December 14	$17^h 26^m 52^s.73$	+ 0″.35
" 19	5 51 5.77	— 0 .19	" 15	17 31 18.43	+ 0 .46
" 20	5 55 15.44	— 0 .32	" 16	17 35 44.38	+ 0 .57
" 23	6 7 44.44	— 0 .63	" 18	17 44 36.91	+ 0 .72
" 27	6 24 22.00	— 0 .72	" 21	17 57 56.69	+ 0 .70
			" 22	18 2 23.39	+ 0 .64
			" 23	18 6 50.09	+ 0 .50
			" 29	18 33 28.11	— 0 .19

Supposing no tables of the reduction at hand, let us first reduce the observations at the summer solstice by the original equation (696). Subtracting β from the observed values of D, we then have

	D'	log tan D'	log cosec A	log tan ϵ	ϵ
June 16	23° 21′ 55″.84	9.6355081	0.0018927	9.6374008	23° 27′ 23″.61
" 19	26 28 .38	.6370823	03278	4101	25 .22
" 20	27 7 .11	.6373056	00930	3986	23 .23
" 23	26 40 .55	.6371524	02478	4002	23 .51
" 27	20 18 .56	.6349452	24592	4044	24 .25

Apparent obliquity = 23 27 23 .96

For the sake of comparison, I add the results of the computation by the series (697), which, however, will be far less convenient than the above direct computation, unless a table of the reduction is used.

	u	D	Red. to solstice.	Red. for ⊙ lat.	ϵ
June 16	$+27^m\,22^s.87$	23° 21′ 56″.02	+ 5′ 27″.77	− 0″.18	23° 27′ 23″.61
" 19	+ 8 54.23	26 28 .19	0 56 .84	+ 0 .19	25 .22
" 20	+ 4 44.56	27 6 .79	0 16 .13	+ 0 .32	23 .24
" 23	− 7 44.44	26 39 .92	0 42 .96	+ 0 .63	23 .51
" 27	− 24 22.00	20 17 .84	7 5 .69	+ 0 72	24 .25

Apparent obliquity = 23 27 23 .96
Nutation* = + 8 .24
Reduction to Jan. 0. 1846 = 0″.4645 × 0.469 = + 0 .22
Mean obliquity 1846.0 = 23 27 32 .42

In the same manner, for the southern solstice we have:

	u	D	Red. to solstice.	Red. for ⊙ lat.	ϵ
Dec. 14	$+33^m\,7^s.27$	− 23° 14′ 17″.26	− 13′ 6″.48	− 0″.35	23° 27′ 24″.09
" 15	+ 28 41.57	17 33 82	9 50 .24	− 0. 46	24 .52
" 16	+ 24 15.62	20 22 .94	7 1 .98	− 0 .57	25 .49
" 18	+ 15 23.09	24 32 .69	2 49 .70	− 0. 72	23 .11
" 21	+ 2 3.31	27 20 .43	0 3 .03	− 0 .70	24 .16
" 22	− 2 23.39	27 19 .64	0 4 .09	− 0 .64	24 .37
" 23	− 6 50.09	26 49 .82	0 33 49	− 0 .50	23 .81
" 29	− 33 28.11	14 1 .20	13 23 .05	+ 0 .19	24 .06

Apparent obliquity = 23 27 24 .20
Nutation = + 8 .98
Reduction to Jan. 0. 1846 = 0″.4645 × 0.971 = + 0 .45
Mean obliquity 1846.0 = 23 27 33 .63

The results from the two solstices being combined in order to

---

* The nutation for 1846 is found by the formula (Art. 381)

$$\Delta\epsilon = 9''.2235 \cos \Omega - 0''.0897 \cos 2\Omega + 0''.0886 \cos 2☾ + 0.5509 \cos 2⊙ + 0.0093 \cos (⊙ + \Gamma)$$

For the northern solstice June 21, 9^h, I have taken ☊ = 214° 27′, ☾ = 69°, ⊙ = 90°, Γ = 280°; for the southern solstice, Dec. 21. 16^h, ☊ = 204° 45′, ☾ = 319°, ⊙ = 270°. Γ = 280°. To proceed with theoretical rigor, the nutation should be found for the time of each observation.

eliminate the error of the assumed latitude of Washington,* we have, finally,

Mean obliquity for 1846.0 from observation	$= 23° \ 27' \ 33''.03$
The same by PETERS's formula (646) with the annual decrease $0''.4645$	$=$ " " $32 \ .85$

420. The secular variation of the obliquity is found by comparing its values at very distant epochs. The observations of BRADLEY from 1753 to 1760 gave for 1757.295 the mean obliquity 23° 28′ 14″.055. The observations at the Dorpat Observatory gave for 1825.0 the mean obliquity 23° 27′ 42″.607. Hence

$$\text{Annual var.} = -\frac{31''.448}{67.705} = -\ 0''.4645$$

BESSEL found $-0''.457$ by comparing BRADLEY's observations with his own.

The secular variation is also found in Physical Astronomy, theoretically. The value thus obtained by PETERS in his *Numerus Constans Nutationis* is $-0''.4738$, as given in the formulæ (646).

421. *Determination of the equinoctial points, and the absolute right ascension and declination of the fixed stars.*—The declinations of the fixed stars are either directly measured by the fixed instruments of the observatory, or deduced immediately from their observed meridian zenith distances (corrected for refraction) by the formula $\delta = \varphi - \zeta$. The practical details, which depend on the instrument employed, will be given in Vol. II. Here we have only to observe that the immediate result of such a measurement is the *apparent* declination at the time of observation, which must then be reduced to the mean declination for some assumed epoch by the formulæ of the preceding chapter.

The position of the equinoctial points is determined as soon as we have found the right ascension of *one* fixed star; and this is done by deducing from observation the *difference* between the

* The latitude employed in deducing the declinations was 38° 53′ 39″.25. The latitude given by the culminations of *Polaris* is 38° 53′ 39″.52 (*Washington Astr. Obs.*, Vol. I., App. p. 113). If we adopt the latter value, the obliquity derived from the northern solstice will be increased by 0″.27, and that derived from the southern solstice will be diminished by the same quantity; and the difference then remaining between the two results will be only 0″.67.

sun's right ascension and that of the star at the time the sun is at the equinoctial points. For this purpose a bright star is selected, which can be observed in the daytime and at either equinox, and which is not far from the equator. On a day near the equinox the times of transit of the sun and the star are noted by the sidereal clock; and at the time of the sun's transit his declination is also measured. Let

$T =$ the clock time of the sun's transit,
$t =$ " " " star's "
$A, D, \beta =$ the sun's apparent right ascension, declination, and latitude at the time T,
$\alpha =$ the star's apparent right ascension at the time t,
$\varepsilon =$ the apparent obliquity of the ecliptic at the time T;

then, correcting the sun's declination by the formula (695), or,

$$D' = D - \beta \sec \varepsilon \cos D$$

we have, by (694),

$$\sin A = \tan D' \cot \varepsilon \tag{699}$$

Thus A becomes known, and hence, also, α by the formula

$$\alpha = A + (t - T) \tag{700}$$

in which $t - T$ is the true sidereal interval between the observations corrected for the clock rate.

The observation is to be repeated on a number of days preceding and following each equinox. The star's apparent right ascension is in each case to be freed from the effects of aberration, nutation, and precession (also proper motion and annual parallax, if known). Each observation thus furnishes a value of the star's mean right ascension at the epoch to which the reduction is made. In order to learn what combination of these values will best eliminate constant errors in the elements upon which A depends, let us examine the effects of these errors. We speak only of *constant* errors; the *accidental* errors of observation being reduced to their minimum effect by taking the mean of a large number of observations.

The correction which the assumed value of the obliquity requires being denoted by $d\varepsilon$, the corresponding correction of A is found, by differentiating (699), to be

$$dA = - d\varepsilon \frac{2 \tan A}{\sin 2\varepsilon}$$

The correction of the declination D' is composed of the corrections in the latitude φ, and the zenith distance ζ; since, by the formula $D = \varphi - \zeta$, we have

$$dD = d\varphi - d\zeta$$

But $d\zeta$ is itself composed of the corrections required in the refraction and the sun's parallax and the correction for any error peculiar to the zenith distance ζ, which affects the meridian instrument employed in the observation. Denoting the correction of the refraction by dr, that of the sun's parallax by $dp \sin \zeta$, that of the instrument for the zenith distance ζ by $f(\zeta)$, we have

$$dD = d\varphi - [dr - dp \sin \zeta + f(\zeta)]$$

The effect of this correction upon A is found, by differentiating (699) with reference to D' (regarding dD as equal to dD'), to be

$$dA = dD \frac{2 \tan A}{\sin 2D'}$$

If then α' denotes the corrected mean right ascension of the star, free from all constant errors, we have

$$\alpha' = \alpha + \left[d\varphi - dr + dp \sin \zeta - f(\zeta)\right] \frac{2 \tan A}{\sin 2D'} - d\varepsilon \frac{2 \tan A}{\sin 2\varepsilon}$$

This formula shows that nearly all the errors will be eliminated by taking the mean between two observations taken at the same zenith distance (or the same declination), the one near the vernal, the other near the autumnal equinox. For, the first observation being taken when the declination is D' and right ascension A, at the second one the same declination D' will give the right ascension $180° - A$, the tangent of which is the negative of that of A. The temperature being generally different at the two seasons of the year, we cannot assume that the error in the refraction tables will be the same at both observations unless we can also assume that the *law* of correction of the refraction for temperature is perfectly known. So, also, we must admit the possibility that such changes of temperature change the instrumental correction; but the corrections of the latitude and the parallax will remain the same. Hence, if α_1 is the mean right

ascension computed from the observation at the autumnal equinox, the corrected right ascension will be

$$\alpha' = \alpha_1 - \left[d\varphi - dr_1 + dp \sin \zeta - f_1(\zeta)\right] \frac{2 \tan A}{\sin 2D'} + d\varepsilon \frac{2 \tan A}{\sin 2\varepsilon}$$

in which dr_1 and $f_1(\zeta)$ denote the corrections for the same zenith distance as before, but for a different temperature. The mean value of α' obtained from the two observations is then

$$\alpha' = \tfrac{1}{2}(\alpha + \alpha_1) + \left[dr_1 - dr + f_1(\zeta) - f(\zeta)\right] \frac{\tan A}{\sin 2D'}$$

This mean is thus freed entirely from the effects of the errors of latitude and the assumed obliquity, and the remaining error is composed merely of the *difference* of the errors of refraction and of the instrument arising from differences of temperature. The difference of temperature at the vernal and autumnal equinoxes, though considerable, is not so great but that we may assume the quantity $dr_1 - dr$ to be evanescent in the present state of the refraction tables. To eliminate the effects of temperature upon the instrument, the only course is to make a special investigation of its errors at various temperatures.

It follows from this discussion that the absolute right ascension of a star can be accurately determined by means of observations at both equinoxes so arranged that for every observation near the vernal equinox at the right ascension A there will be a corresponding one at the autumnal equinox at the right ascension $180° - A$. This condition is satisfied nearly enough by regarding as corresponding observations those which are taken between the declinations $0°$ and $+ 2°$ *after* the vernal and *before* the autumnal equinox, between $0°$ and $- 2°$ *before* the vernal and *after* the autumnal; between $+ 2°$ and $+ 4°$; $- 2°$ and $- 4°$, &c. On account of the very complete elimination of errors, it is safe to extend the observations even as far as $+ 14°$ and $- 14°$.*

EXAMPLE.—The following observations of the sun and γ *Pegasi* on the meridian were taken at the Washington Observatory in the year 1846:†

* BESSEL: *Fundamenta Astronomiæ*, pp. 12, 14.

† The transits were taken with the "West Transit," the declinations with the Mural Circle. Both the first and second limbs of the sun were observed on the seven threads of the transit instrument, and the declination of both the north and the south limbs with the mural.

Feb. 23. $D =$	$9^\circ\ 46'\ 15''.85$	Oct. 17. $D = -$	$9^\circ\ 17'\ 53''.12$
" " $T =$	$22^h\ 26^m\ 28^s.11$	" " $T =$	$13^h\ 28^m\ 40^s.01$
" " $t =$	$0\ \ 5\ \ 18\ .99$	" 16. $t =$	$0\ \ 5\ \ 22\ .97$

The times of transit are corrected for the supposed error and rate of the clock.

For the dates of the two observations, the apparent obliquity of the ecliptic and the sun's latitude are as follows:

	Feb. 23.	Oct. 17
ε	$23^\circ\ 27'\ 26''.10$	$23^\circ\ 27'\ 24''.35$
β	$+\ 0\ .33$	$-\ 0\ .13$
whence		
$-\beta \sec \varepsilon \cos D$	$-\ 0\ .35$	$+\ 0\ .14$
D'	$9\ \ 46\ \ 16\ .20$	$-\ 9\ \ 17\ \ 52\ .98$
$\log \tan D'$	$n9.236063$	$n9\ 214105$
$\log \cot \varepsilon$	0.362585	0.362595
$\log \sin A$	$n9.598648$	$n9.576700$
A	$22^h\ 26^m\ 28^s.17$	$13^h\ 28^m\ 40^s.14$
$A - T$	$+\ \ 0.06$	$+\ \ 0.13$
$t + A - T = \alpha$	$0\ \ 5\ \ 19\ .05$	$0\ \ 5\ \ 23\ .10$
Reduction to 1850.0	$+\ \ 12\ .15$	$+\ \ 8\ .12$
Mean α for 1850.0 $=$	$0\ \ 5\ \ 31\ .20$	$0\ \ 5\ \ 31\ .22$

The reduction to 1850 is here used because it can be taken directly from the general tables for reducing the apparent places of stars to mean places, given in the volume of Washington Observations for 1847. Taking the mean of the two observations, we have, finally,

$$\text{Mean R. A. of } \gamma \text{ Pegasi for } 1850.0 = 0^h\ 5^m\ 31^s.21$$

422. When, by the combination of a great number of observations, the right ascension of a fundamental star is thus established, the right ascensions of all other stars follow from the differences of time between their several transits and that of the fundamental star. But, in the present state of the star catalogues, it will be preferable not to limit the object of these observations to determining a single star. The constant use of the same fundamental stars as "clock stars" (stars near the equator by which the clock correction and rate are found) gives to the *relative* right ascensions of these stars (as derived from all their observed transits during one or more years) a high degree of accuracy. Assuming,

therefore, that the *relative* right ascensions of the clock stars are correct, the object of our observations of the sun will be to determine the *common* correction of the absolute right ascensions of *all* these stars. Accordingly, if we deduce the sun's apparent right ascension directly from each observation by applying to the clock time of the transit of the sun's centre the clock correction obtained from the fundamental stars, and compare this with the apparent right ascension computed from the observed declination, we have the correction which the right ascensions of these stars require. All that has been said above respecting the grouping of the observations at the two equinoxes, of course, applies equally well to this process.

Thus, in the preceding example, taking the clock times of the sun's transits there given as the directly observed right ascensions (since they have actually been corrected for the clock error obtained from a number of fundamental stars), we shall have

	Feb. 23.	Oct. 17.
Observed R. A of ☉,	$22^h\,26^m\,28^s.11$	$13^h\,28^m\,40^s.01$
Computed " "	" " 28 .17	" " 40 .14
Correction of clock stars,	+ 0 .06	+ 0 .13

whence

Mean correction of the R. A. of the clock stars $= +\,0^s.10$

CHAPTER XIII.

DETERMINATION OF ASTRONOMICAL CONSTANTS BY OBSERVATION.

423. I SHALL not attempt to enter into all the details of the methods by which the various astronomical constants are determined from observations, but shall confine myself to a sketch of their general principles, which will serve as an introduction to the special papers to be found in astronomical memoirs and other sources.

THE CONSTANTS OF REFRACTION.

424. The general refraction formula (191) involves the two constants α and β, both of which may be found from theory by the formulæ (178) and (176). But, as the refraction formula was deduced from an hypothesis, it was not to be expected that the theoretical values of α and β would give refractions in entire accordance with observation. The discrepancies, however, are exceedingly small: so small, indeed, that the formula may be regarded as representing well enough the *law* of refraction, without resorting to any new hypothesis; and to perfect it we have only to give the constants slightly amended values, whereby the computed refractions are made to harmonize entirely with those deduced from observation. To deduce the corrections of α and β, we can employ the concise expression of the refraction (213), or

$$(1 - \alpha)\, r = \sin^2 z \sqrt{\frac{2}{\beta}} \cdot Q$$

The factor $1 - \alpha$ differs so little from unity that we may regard it as constant in determining the small correction of r, and, therefore, by differentiating, we have

$$(1 - \alpha)\, dr = \sin^2 z \sqrt{\frac{2}{\beta}} \left[\frac{dQ}{d\alpha} \cdot d\alpha + \left(\frac{dQ}{d\beta} - \frac{Q}{2\beta}\right) d\beta\right]$$

By (217) and (210) we have

$$\frac{dQ}{d\alpha} = \frac{dQ}{dx} \cdot \frac{dx}{d\alpha} = \frac{1 - x}{x} \cdot \frac{\beta}{\sin^2 z}\, Q'$$

where Q' is known by (218), and, therefore, the coefficient of $d\alpha$ can be computed. Also, since $\frac{dQ}{d\beta}$ is given by (220), and Q by (212), the coefficient of $d\beta$ is known. We should, however, find the correction of the constant α_0, or that which corresponds to the normal temperature and barometric pressure of the refraction table. By (205) we have

$$d\alpha = \frac{d\alpha_0}{1 + \varepsilon(\tau - \tau_0)} \cdot \frac{p}{p_0}$$

As for $d\beta$, it is evidently the same as $d\beta_0$.

But, since α_0 can require but a very small correction, great precision in the coefficient of $d\alpha_0$ is not necessary; and, if we neglect the second and higher powers of α_0, it is easily seen that this coefficient will be reduced to $\frac{r}{\alpha_0}$, r being the refraction computed for the actual state of the air by the tables. This amounts to assuming that r and dr vary directly in proportion to α_0 and $d\alpha_0$; an assumption which is very nearly correct, as can be seen from the approximate formula (159), in which we have very nearly $2k\delta_0 = \alpha_0$. We may also in our differential formula put unity in the place of the factor $1 - \alpha$; and hence if we put

$$A = \frac{r}{\alpha_0}$$

$$B = \sin^2 z \sqrt{\frac{2}{\beta}} \left(\frac{dQ}{d\beta} - \frac{Q}{2\beta} \right)$$

we shall have

$$dr = A\,d\alpha_0 + B\,d\beta_0 \tag{701}$$

It only remains to show how this differential formula is to be applied in deducing $d\alpha_0$ and $d\beta_0$. The observations best suited to our purpose are those of the zenith distance of a circumpolar star at its upper and lower culminations. Let

z', z_1' = the observed zenith distances above and below the pole respectively,

z, z_1 = the true zenith distances obtained by employing the tabular refraction,

δ, δ_1 = the declination of the star at the two culminations respectively,

φ = the assumed latitude of the place of observation.

The true zenith distances which would be obtained by a table of refractions founded on the corrected constants will be $z + dr$ and $z_1 + dr_1$; and, therefore, if $d\varphi$ denotes the correction of the assumed latitude, we shall have

$$90^\circ - (\varphi + d\varphi) = z + dr + 90^\circ - \delta$$
$$90^\circ - (\varphi + d\varphi) = z_1 + dr_1 - (90^\circ - \delta_1)$$

whence, by taking the mean,

$$90^\circ - \varphi - d\varphi = \tfrac{1}{2}(z + z_1) + \tfrac{1}{2}(\delta_1 - \delta) + \tfrac{1}{2}(dr + dr_1)$$

The quantity $\delta_1 - \delta$ is merely the very small change of the star's declination between the two culminations, arising from precession and nutation, which is accurately known. If we substitute the values of dr and dr_1 in terms of $d\alpha$ and $d\beta$, and then put

$$a = \tfrac{1}{2}(A + A_1) \qquad b = \tfrac{1}{2}(B + B_1)$$
$$n = \tfrac{1}{2}(z + z_1) + \tfrac{1}{2}(\delta_1 - \delta) + \varphi - 90^\circ$$

we have the equation of condition

$$d\varphi + a\,d\alpha_0 + b\,d\beta_0 + n = 0 \qquad (702)$$

By employing a number of stars which culminate at various zenith distances, we shall obtain a number of such equations, in which the coefficients a and b will have different values: so that the solution of all these equations by the method of least squares will determine the three unknown quantities $d\varphi$, $d\alpha_0$, and $d\beta_0$.

THE CONSTANT OF SOLAR PARALLAX.

425. *The constant of solar parallax is the sun's mean equatorial horizontal parallax*, or its horizontal parallax when its distance from the earth is equal to the semi-major axis of the earth's orbit. The constant of parallax of any planet is also its parallax when its distance from the earth is equal to the semi-major axis of the earth's orbit: so that the constant of solar parallax belongs to the whole solar system.

The *relative* dimensions of the orbits of the planets are known from the periodic times of their revolutions about the sun, since, by KEPLER's third law, the squares of their periodic times are proportional to the cubes of their mean distances from the sun.

that is, to the cubes of the semi-major axes of their orbits. The *ratios* of these distances are therefore known.

Again, the *form* and *position* of each orbit are known from Physical Astronomy;* and therefore the ratio of the planet's distance from the earth at any given time to the earth's mean distance is also known.

According to these principles, if the distance of any planet from the earth can be found at any time, the dimensions of all the orbits are also found: in other words, when we have found the parallax of one planet we have also found that of all the planets, as well as that of the sun.

426. *To find a planet's, or the sun's, parallax by meridian observations.*—Let the meridian zenith distance of the planet's centre be observed on the same day at two places nearly on the same meridian, but in very different latitudes. After correcting the observed quantities for refraction, let

ζ', ζ_1' = the apparent zenith distances at the north and south places of observation, respectively,
ζ, ζ_1 = the true (geocentric) zenith distances,
p, p_1 = the parallax for the zenith distances ζ and ζ_1,
π, π_1 = the equatorial horizontal parallax at the respective times of observation,
Δ, Δ_1 = the geocentric distances of the planet at these times.
δ, δ_1 = the geocentric declination of the planet at the same times,
π_0 = the sun's mean equatorial horizontal parallax,
Δ_0 = " " " distance from the earth,
R = the earth's equatorial radius.

Also for the places of observation let

φ, φ_1 = the astronomical latitudes,
φ', φ_1' = the reduced or geocentric latitudes,
ρ, ρ_1 = the radii of the terrestrial spheroid for these latitudes.

We have

$$\sin \pi = \frac{R}{\Delta} \qquad \sin \pi_1 = \frac{R}{\Delta_1} \qquad \sin \pi_0 = \frac{R}{\Delta_0}$$

* They are found from three complete observations of the right ascension and declination of each planet at three different times (GAUSS, *Theoria Motus Corporum Cœlestium*), and therefore from the observed *directions* of the planet, the absolute distance being unknown.

and therefore

$$\sin \pi = \frac{\Delta_0}{\Delta} \sin \pi_0 \qquad \sin \pi_1 = \frac{\Delta_0}{\Delta_1} \sin \pi_0$$

The quantities Δ and Δ_1 are to be found from the planetary tables, or directly from the Nautical Almanac, where they are expressed in terms of Δ_0 as the unit: so that their values there given are the values of the ratios $\frac{\Delta}{\Delta_0}$ and $\frac{\Delta}{\Delta_0}$. Hence we shall put $\Delta_0 = 1$ in the preceding formulæ, and also put the arcs for their sines (since the greatest planetary parallax is only 35″): so that we have

$$\pi = \frac{\pi_0}{\Delta} \qquad \pi_1 = \frac{\pi_0}{\Delta_1}$$

Then, by (114),

$$p = \rho \pi \sin [\zeta' - (\varphi - \varphi')] = \frac{\rho \pi_0}{\Delta} \sin [\zeta' - (\varphi - \varphi')]$$

$$p_1 = \rho_1 \pi_1 \sin [\zeta_1' - (\varphi_1 - \varphi_1')] = \frac{\rho_1 \pi_0}{\Delta_1} \sin [\zeta_1' - (\varphi_1 - \varphi_1')]$$

But we also have

$$\zeta = \varphi - \delta \qquad \zeta_1 = \varphi_1 - \delta_1$$

and hence

$$\zeta - \zeta_1 = (\zeta' - p) - (\zeta_1' - p_1) = \varphi - \varphi_1 - (\delta - \delta_1)$$

from which we obtain

$$p - p_1 = \zeta' - \zeta_1' - (\varphi - \varphi_1) + (\delta - \delta_1)$$

As the small difference $\delta - \delta_1$ will be accurately known, the observations being taken nearly on the same meridian, all the quantities in the second member of this equation may be regarded as known. Hence, putting

$$\left.\begin{aligned} n &= \zeta' - \zeta_1' - (\varphi - \varphi_1) + (\delta - \delta_1) \\ a &= \frac{\rho}{\Delta} \sin [\zeta' - (\varphi - \varphi')] - \frac{\rho_1}{\Delta_1} \sin [\zeta_1' - (\varphi_1 - \varphi_1')] \end{aligned}\right\} \quad (703)$$

we obtain the equation

$$a \pi_0 = n \qquad (704)$$

which determines π_0. If the zeniths of the two places of observation are on opposite sides of the star (which is the most favor-

able case), the zenith distance at the southern place must be taken with the negative sign in the above formulæ. The coefficient a then becomes an arithmetical sum, and it is evident that the greater the value of a, the greater will be the degree of accuracy in the determination of π_0.

But, in order to give this method all the precision necessary in finding so small a quantity as π_0, the quantity n must not depend upon the absolute zenith distances observed (which involve the errors of divided circles and the whole errors of the refraction table at these zenith distances), nor upon the quantity $\varphi - \varphi_1$ (which involves the errors in the latitudes of the places), but upon micrometric measures. For this purpose, the planet is compared with a star nearly in the same parallel of declination, and always *with the same star at both places of observation*, the comparison stars being previously selected and agreed upon by the observers. The star and planet should differ so little in declination that they will both pass through the field of the meridian telescope, the instrument remaining firmly clamped between the transits of the two objects; and then the *difference* of apparent declination of the planet and star will be directly measured with the micrometer. This difference is to be corrected for the difference of refraction at the zenith distances of the planet and star, which difference of refraction, being very small, can be computed with the greatest accuracy.* If then

D = the declination of the star,

$\Delta\delta$, $\Delta\delta_1$ = the observed differences of declination of the star and planet (corrected for refraction) at the two places of observation,

the observed apparent declination of the planet at the northern place is

$$D + \Delta\delta = \varphi - \zeta'$$

and at the southern place

$$D + \Delta\delta_1 = \varphi_1 - \zeta_1'$$

whence

$$\Delta\delta - \Delta\delta_1 = -(\zeta' - \zeta_1') + (\varphi - \varphi_1)$$

and the value of n in (703) becomes

$$n = \delta - \delta_1 - (\Delta\delta - \Delta\delta_1) \qquad (705$$

* Vol. II. *Correction of micrometer observations for refraction.*

where $\Delta\delta$ and $\Delta\delta_1$ are in each case the planet's declination *minus* the star's declination, and their signs are to be carefully observed. For computing the coefficient a, the apparent zenith distances will be obtained by the formulæ

$$\zeta' = \varphi - (D + \Delta\delta) \qquad \zeta_1' = \varphi_1 - (D + \Delta\delta_1)$$

so that we have

$$a = \frac{\rho}{\Delta} \sin [\varphi' - (D + \Delta\delta)] - \frac{\rho_1}{\Delta_1} \sin [\varphi_1' - (D + \Delta\delta_1)] \quad (706)$$

and then, as before,

$$a\pi_0 = n$$

A great number of such corresponding observations will be necessary in order to determine π_0 with accuracy; and all the equations of the form just given are to be combined by the method of least squares. Thus, from the equations

$$a\pi_0 = n, \qquad a'\pi_0 = n', \qquad a''\pi_0 = n'', \text{ \&c.}$$

we obtain the final equation

$$[aa]\,\pi_0 = [an] \qquad \text{or} \qquad \pi_0 = \frac{[an]}{[aa]}$$

in which $[aa] = aa + a'a' + a''a'' +$ &c., and $[an] = an + a'n' + a''n'' +$ &c.

427. *To find the solar parallax by extra-meridian observations of a planet.*—The preceding process will require but a slight modification. The difference of apparent declination of the planet and a neighboring star is measured at both stations with a micrometer attached to an equatorial telescope, and is to be corrected for refraction. The quantity n will then be found by (705). The coefficient a will now be the difference of the coefficients of parallax in declination, computed by the formulæ (143), according to which, if we put

$$\tan \gamma = \frac{\tan \varphi'}{\cos (\Theta - \alpha)} \qquad \tan \gamma_1 = \frac{\tan \varphi_1'}{\cos (\Theta_1 - \alpha_1)}$$

we shall have

$$a = \frac{\rho}{\Delta}.\frac{\sin\varphi'\sin(\gamma-\delta)}{\sin\gamma} - \frac{\rho_1}{\Delta_1}.\frac{\sin\varphi_1'\sin(\gamma_1-\delta_1)}{\sin\gamma_1} \tag{707}$$

in which Θ and Θ_1 are the local sidereal times of the observations, α and α_1 the right ascensions, δ and δ_1 the declinations of the planet at these times. The equation of condition from each pair of corresponding observations of the same star will then be, as before, $a\pi_0 = n$.

If several comparisons are made at either place on the same day, these must first be combined, and reduced, as it were, to a single comparison. Thus, if we put

$$c = \frac{\rho}{\Delta}.\frac{\sin\varphi'\sin(\gamma-\delta)}{\sin\gamma}$$

we have, for each comparison of the planet with the star,

$$\delta = D + \Delta\delta + c\pi_0$$

and if m such comparisons are made, their mean will be

$$\delta = D + \frac{1}{m}\,\Sigma(\Delta\delta) + \pi_0.\frac{\Sigma(c)}{m}$$

In like manner, at the second place, we shall have for m_1 observations the equation

$$\delta_1 = D + \frac{1}{m_1}\,\Sigma(\Delta\delta_1) + \pi_0\frac{\Sigma(c_1)}{m_1}$$

and, taking the difference of these equations, we shall put

$$n = \delta - \delta_1 - \left(\frac{\Sigma(\Delta\delta)}{m} - \frac{\Sigma(\Delta\delta_1)}{m_1}\right)$$

$$a = \frac{\Sigma(c)}{m} - \frac{\Sigma(c_1)}{m_1}$$

The equation of condition $a\pi_0 = n$ will then represent all the observations on the same day at the two places.

428. The equations of condition will involve smaller numbers and be more easily solved if the unknown quantity is, not the whole parallax, but the correction of some assumed value of the parallax not greatly in error. In this case we may correct each observed difference $\Delta\delta$ for parallax, employing the assumed value

of π_0; and, proceeding as before, we shall have the equation of condition $a\,\Delta\pi_0 = n$, in which $\Delta\pi_0$ is the required correction of π_0.

429. If but one limb of the planet is observed at one or both the stations, it will be necessary to introduce the correction for the semidiameter. As the semidiameter itself should then be regarded as an unknown quantity, to be found if possible from the observations, its complete expression, in terms of all the corrections which the observations may require, is to be employed. This will be found in Article 435.

430. The differences of right ascension of the planet and a neighboring star may also be employed in the same manner as the differences of declination, the places of observation being in that case in widely different longitudes. We have only to introduce into (707) the coefficients of the parallax in right ascension computed by the first equation of (143), and in the expression of n substitute right ascensions for declinations.

431. The only planets which are near enough to the earth for the successful application of this method are Mars and Venus.

Mars is nearest to the earth at the time of opposition, and for this time the British Nautical Almanac furnishes an Ephemeris of stars to be observed with the planet. All the oppositions, however, are not equally favorable. The mean distance of Mars from the sun being $= 1.524$, and the eccentricity of the orbit $= 0.0933$, while the mean distance of the earth $= 1$ and the eccentricity of its orbit $= 0.017$, it follows that for an opposition in which Mars is at its perihelion while the earth is at its aphelion, the distance of the two bodies will be 0.365; but for one in which Mars is at its aphelion and the earth at its perihelion, their distance will be 0.683. Thus the former case will be nearly twice as favorable as the latter.

Venus is nearest to the earth at the time of inferior conjunction, but at that time can very rarely be compared micrometrically with stars, as the observations would be made with the sun above the horizon. The most favorable position of this planet is at or near its stationary points, where the changes of the planet's place are small and may therefore be accurately computed, while the distance from the earth is still not too great.*

* GERLING, *Astron. Nach*, No. 599.

The United States Astronomical Expedition to Chili under Lieut. J. M. Gilliss, in the years 1849–52, was set on foot for the purpose of determining the solar parallax by the above method. That indefatigable and accurate observer collected a large mass of valuable material, a great part of which, however, could not be used in the manner originally intended, for want of *corresponding* observations at northern observatories. In the thorough discussion of this material by Dr. B. A. Gould* will be found a full exposition of the modifications which the method required in order to make use of all the observations.

The constant of solar parallax is also found by the transits of Venus over the sun's disc, Art. 356.

THE CONSTANT OF LUNAR PARALLAX.

432. *The constant of lunar parallax is the moon's mean equatorial horizontal parallax*, or the equatorial horizontal parallax corresponding to the moon's mean distance from the earth.†

To find the moon's parallax by meridian observations at two stations on the earth's surface.

The stations will be assumed to be in opposite hemispheres of the earth: so that at every observation the moon will culminate south of the zenith of the northern station, and north of the zenith of the southern station. They will also be assumed to be nearly on the same meridian. At each station, the apparent declinations of the moon's bright limb at the instants of transit are to be observed on the same day, and, consequently, since the meridians are not remote, at nearly the same time. In order to eliminate constant errors of the refraction tables and instrumental errors, the *difference* of the moon's declination and that of a star nearly in the same parallel is to be observed, and the same comparison stars should be used at both stations. The observed difference of declination is to be corrected for the difference of the refraction at the zenith distance of the moon and star, and then applied to the assumed declination of the star. We shall thus obtain the apparent declination of the moon's limb affected only by parallax. Let

* *U. S. Naval Expedition to Chili*, Vol. III.

† The constant adopted in the lunar tables is for the mean distance affected by the constant part of the perturbations of the radius vector.

δ, δ_1 = the apparent declinations of the limb observed at the north and south stations respectively,

D, D_1 = the geocentric declinations of the moon's centre at the respective times of observation,

φ, φ_1 = the geographical latitudes of the stations,

γ, γ_1 = the reductions of the latitudes for the earth's compression,

ρ, ρ_1 = the distances of the stations from the earth's centre, the equatorial radius being unity,

P, P_1 = the moon's horizontal parallax at the times of the observation, respectively;

then, the apparent zenith distance of the limb and the geocentric zenith distance of the centre of the moon being, for the northern station,

$$\zeta' = \varphi - \delta \qquad \text{and} \qquad \zeta = \varphi - D$$

we have, by (255),

$$\sin (D - \delta) = [\rho \sin (\zeta' - \gamma) \mp k] \sin P$$

where k is the constant ratio of the radii of the moon and the earth, for which the value 0.272956 may be assumed; and the upper or lower sign of k is to be used according as the upper or lower limb is observed.

At the southern station we have

$$\zeta_1' = \delta_1 - \varphi_1 \qquad \zeta_1 = D_1 - \varphi_1$$

and hence, taking the reduction γ_1 as a positive quantity,

$$\sin (D_1 - \delta_1) = - [\rho_1 \sin (\zeta_1' - \gamma_1) \pm k] \sin P_1$$

where the sign of k is reversed, since the same limb will be an upper limb at one station and a lower limb at the other. For brevity, put

$$m = \rho \sin (\zeta' - \gamma) \mp k$$
$$m_1 = \rho_1 \sin (\zeta_1 - \gamma_1) \pm k$$

then, from the equations

$$\sin (D - \delta) = m \sin P \qquad \sin (D_1 - \delta_1) = - m_1 \sin P_1$$

we derive,* neglecting powers of sin P above the third,

$$D - \delta = \frac{m \sin P}{\sin 1''} + \tfrac{1}{6} \cdot \frac{m^3 \sin^3 P}{\sin 1''}$$

$$D_1 - \delta_1 = - \frac{m_1 \sin P_1}{\sin 1''} - \tfrac{1}{6} \cdot \frac{m_1{}^3 \sin^3 P_1}{\sin 1''}$$

If now the times of the two observations reckoned at the same first meridian are T and T_1, and for the middle time $t = \frac{1}{2}(T + T_1)$ we deduce from the lunar tables the hourly increase of the moon's declination, or $\frac{dD}{dt}$, we shall have, with regard to second differences,

$$D_1 - D = (T_1 - T) \frac{dD}{dt}$$

Again, if we denote the moon's horizontal parallax at the time t by p, and compute from the tables its hourly increase for this time, or $\frac{dp}{dt}$, we shall have

$$\sin P = \sin p + \cos p \sin 1'' (T - t) \frac{dp}{dt}$$

$$\sin P_1 = \sin p + \cos p \sin 1'' (T_1 - t) \frac{dp}{dt}$$

Taking the difference of the above values of $D - \delta$ and $D_1 - \delta_1$, we obtain, therefore,

$$0 = \left[(T_1 - T) \frac{dD}{dt} - (\delta_1 - \delta)\right] \sin 1'' + (m^3 + m_1{}^3) \frac{\sin^3 p}{6}$$

$$+ \cos p \sin 1'' \frac{dp}{dt} \left[m(T - t) + m_1 (T_1 - t)\right]$$

$$+ (m + m_1) \sin p \tag{708}$$

The parallax is sufficiently well known for the accurate computation of the terms in $\sin^3 p$ and $\frac{dp}{dt}$: so that the only unknown quantity in this equation is the last term. In this term we have

$$m + m_1 = \rho \sin (\zeta' - \gamma) + \rho_1 \sin (\zeta_1' - \gamma_1) \tag{709}$$

* By the formula, [Pl. Trig. (413)],

$$x = \sin x + \tfrac{1}{6} \sin^3 x + \&c.$$

where the second member is to be reduced to seconds by dividing it by sin 1''.

which is independent of k, and thus free from any error in that quantity. Small errors in k will not appreciably affect the other terms of the equation.

Thus every pair of corresponding observations gives an equation of the form

$$0 = n + a \sin p \tag{710}$$

from which the parallax p at the mean time of each pair of observations could be derived. But, in order to combine all these equations, we must introduce in the place of the variable p the constant mean parallax, which is effected as follows. Let

π = the horizontal parallax taken from the lunar tables for the time t,
π_0 = the constant mean parallax of the tables,
p_0 = the true value of this constant.

The form of the moon's orbit is well known: so that for any given time the ratio of the radius vector to the semi-major axis, as employed in the tables, is to be regarded as correct; that is, the ratio

$$\mu = \frac{\sin \pi}{\sin \pi_0} \tag{711}$$

derived from the tables, is to be regarded as the ratio between the *true* parallax at the given time and the *true* constant: so that we have also

$$\mu = \frac{\sin p}{\sin p_0} \qquad \text{or} \qquad \sin p = \mu \sin p_0$$

and the equation (710) becomes

$$a \sin p_0 + \frac{n}{\mu} = 0 \tag{712}$$

The quantities a, n, and μ being computed for each pair of corresponding observations, we thus obtain a number of equations, all involving the same unknown constant $\sin p_0$, which are then to be solved by the method of least squares.

433. The quantities ρ and γ, which enter into the coefficient m, will be computed for an assumed value of the compression of the earth. But, in order to see the effect of the compression, we

may isolate the terms which involve it, as follows. Neglecting the fourth powers of the eccentricity e, we have, by (84) and (83),

$$\rho = 1 - \tfrac{1}{2} e^2 \sin^2 \varphi$$
$$\gamma = \frac{e^2 \sin 2\varphi}{2 \sin 1''}$$

But when we neglect the fourth powers of e, or the square of the compression c, we have, by (81),

$$c = \tfrac{1}{2} e^2$$

by which we obtain the somewhat simpler forms,

$$\rho = 1 - c \sin^2 \varphi$$
$$\gamma = \frac{c \sin 2\varphi}{\sin 1''}$$

These values substituted in m give, by neglecting the square of c,

$$\begin{aligned} m &= (1 - c \sin^2 \varphi) \sin\left(\zeta' - \frac{c \sin 2\varphi}{\sin 1''}\right) \mp k \\ &= (1 - c \sin^2 \varphi)(\sin \zeta' - c \sin 2\varphi \cos \zeta') \mp k \\ &= \sin \zeta' - c(\sin^2 \varphi \sin \zeta' + \sin 2\varphi \cos \zeta') \mp k \end{aligned}$$

and, similarly,

$$m_1 = \sin \zeta_1' - c(\sin^2 \varphi_1 \sin \zeta_1' + \sin 2\varphi_1 \cos \zeta_1') \pm k$$

The effect of the compression will be insensible in the terms involving $\sin^3 p$, in which we may take

$$m^3 = (\sin \zeta' \mp k)^3 \qquad\qquad m_1^3 = (\sin \zeta_1' \pm k)^3$$

and the same approximation is allowable in the term in $\frac{dp}{dt}$. If then we make these substitutions in (708), we obtain the following expanded equation:

$$\begin{aligned} 0 = {} & \left[(T_1 - T)\frac{dD}{dt} - (\delta_1 - \delta)\right] \sin 1'' + \left[(\sin \zeta' \mp k)^3 + (\sin \zeta_1' \pm k)^3\right] \frac{\sin^3 p}{6} \\ & + \cos p \frac{dp}{dt} \sin 1'' \left[(\sin \zeta' \mp k)(T - t) + (\sin \zeta_1' \pm k)(T_1 - t)\right] \\ & + \mu \sin p_0 \ (\sin \zeta' + \sin \zeta_1') \\ & - c \mu \sin p_0 \left[\sin^2 \phi \sin \zeta' + \sin 2\phi \cos \zeta' + \sin^2 \phi_1 \sin \zeta_1' + \sin 2\phi_1 \cos \zeta_1'\right] \end{aligned}$$

If this equation be divided by μ, it may be expressed under the form

$$0 = n + x\,(a - cb) \tag{713}$$

where the notation is as follows:

$$n = \frac{\sin 1''}{\mu}\left[(T_1 - T)\frac{dD}{dt} - (\delta_1 - \delta)\right]$$
$$+ \frac{\sin^3 p}{6\,\mu}\left[(\sin \zeta' \mp k)^3 + (\sin \zeta_1' \pm k)^3\right]$$
$$+ \frac{\sin 1''}{\mu}\left[(\sin \zeta' \mp k)(T - t) + (\sin \zeta_1' \pm k)(T_1 - t)\right]\frac{dp}{dt}\cos p$$
$$a = \sin \zeta' + \sin \zeta_1'$$
$$b = \sin^2 \varphi \sin \zeta' + \sin 2\varphi \cos \zeta' + \sin^2 \varphi_1 \sin \zeta_1' + \sin 2\varphi_1 \cos \zeta_1'$$
$$x = \sin p_0$$

It is here to be observed that we have taken γ_1 as a positive quantity even for the southern station: so that $\sin 2\,\varphi_1$ must be taken positively in computing b.

Let us now suppose we have obtained from a large number of such corresponding observations the equations

$$\begin{aligned} 0 &= n + x\,(a - cb) \\ 0 &= n' + x\,(a' - cb') \\ 0 &= n'' + x\,(a'' - cb'') \\ &\text{\&c.} \end{aligned}$$

Multiplying these respectively by a, a', a'', &c., and then forming their sum, we have

$$0 = [an] + [aa]\,x - [ab]\,cx$$

where $[an] = an + a'n' +$ &c., $[aa] = aa + a'a' +$ &c., &c. The last term is very small: so that an approximate value of x may be found by neglecting it, whence

$$(x) = -\frac{[an]}{[aa]}$$

which value may then be employed with sufficient accuracy in the term $[ab]cx$; we thus find the complete value

$$x = -\frac{[an]}{[aa]} + \frac{[an]}{[aa]}\frac{[ab]}{[aa]}\cdot c \tag{714}$$

This is essentially the method by which OLUFSEN* has discussed the observations made by LACAILLE in the years 1751, 1752, and 1753, at the Cape of Good Hope, and the corresponding observations made at Paris, Bologna, Berlin, and Greenwich. He found from all the observations the final equation

$$x = 0.01651233 + 0.02449201\, c$$

Consequently, if we take the most probable value of $c = \frac{1}{299.1528}$, there results

$$x = \sin p_0 = 0.01659420$$

The parallax given by the lunar tables of BURCKHARDT and DAMOISEAU is properly the sine of the parallax reduced to seconds. In order to compare this determination with the constants of these tables, we therefore take

$$p_0 = \frac{\sin p_0}{\sin 1''} = 3422''.8$$

The constant of BURCKHARDT'S tables is 3420''.5; that of DAMOISEAU'S, 3420''.9; that of HANSEN'S new tables, 3422''.06. This last value, which is derived from theory, agrees remarkably with that which is derived from direct observation; for the determination by HENDERSON from corresponding observations at Greenwich and the Cape of Good Hope is† 3421''.8, and the mean between this and OLUFSEN'S value is 3422''.3.

434. The correction of the moon's parallax may also be found from the observations of a solar eclipse at two places whose difference of longitude is great, as is shown in the chapter on eclipses, p. 541.

It is also possible to determine the moon's parallax by comparing the different zenith distances of the moon observed at one and the same place between her rising and setting, since the effect of so great a parallax is easily traced from its maximum when the moon is in the horizon to its minimum when at the least zenith distance. But this very obvious method, by which, in fact, HIPPARCHUS discovered the moon's parallax, depends too much upon the measurement of the *absolute* zenith distances to admit of any great degree of accuracy.

---

* *Astronomische Nachrichten*, No. 326. † Ibid, No. 338.

THE MEAN SEMIDIAMETERS OF THE PLANETS.

435. The apparent equatorial semidiameter of a planet when its distance from the earth is equal to the earth's mean distance from the sun is the constant from which its apparent semidiameter at any other distance can be found by the formula

$$s = \frac{s_0}{\varDelta} \tag{715}$$

in which s_0 is the mean semidiameter and $\varDelta$ the actual distance of the planet from the earth, the semi-major axis of the earth's orbit being unity. To find the value of s_0 from the values of s observed at different times, we have then only to take the mean of all its values found by the formula

$$s_0 = s\varDelta \tag{716}$$

taking $\varDelta$ from the tables of the planet for each observation.

But here it is to be remarked that, in micrometric measures of the apparent diameter of a planet, different values will be obtained by different observers or with different instruments. The spurious enlargement of the apparent disc arising from imperfect definition of the limb, or from the *irradiation* resulting from the vivid impression of light upon the eye, will vary with the telescope, and may also vary for the same telescope when eye pieces of different powers are employed. The irradiation may be assumed to consist of two parts, one of which is constant and the other proportional to the semidiameter. Those errors of the observer which are not *accidental* may also be supposed to consist of two parts, one constant and the other proportional to the semidiameter; the first arising from a faulty judgment of a contact of a micrometer thread with the limb of the planet, the second, from the variations in this judgment depending on the magnitude of the disc observed, and possibly also upon any peculiarity of his eye by which the irradiation is *for him* not the same quantity as for other observers. With the errors proportional to the semidiameter will be combined also any error in the supposed value of a revolution of the micrometer. The errors of the two kinds will, however, be all represented in the formula

$$s_0 = (s + x + sy)\varDelta \tag{717}$$

where x is the sum of all the constant corrections which the

observed value s requires, and sy is the sum of all those which are proportional to s. Now, let

$$\begin{aligned} s_1 &= \text{an assumed value of } s_0. \\ ds_1 &= \text{the unknown correction of this value} \\ s_0 &= s_1 + ds_1; \end{aligned}$$

then the above equation may be written

$$0 = s\varDelta - s_1 + x\varDelta + sy\varDelta - ds_1$$

But $sy\varDelta$ will be sensibly the same as s_0y. It will, therefore, be constant, and will combine with ds_1. We shall, therefore, put z for $sy\varDelta - ds_1$, and then, putting

$$n = s\varDelta - s_1$$

our equations of condition will be of the form

$$x\varDelta + z + n = 0 \tag{718}$$

from all of which x and z may be found by the method of least squares. But it will be impossible to separate the quantity ds_1 from z; we can only put

$$(s_0) = s_1 - z$$

whereas we have, for the true value,

$$s_0 = s_1 + ds_1 = s_1 - z + s_0y$$

or

$$s_0 = (s_0)\,(1 + y) \tag{719}$$

and then, if any independent means of finding y are discovered, the true value of s_0 can be computed.

THE ABERRATION CONSTANT AND THE ANNUAL PARALLAX OF FIXED STARS.

436. The constant of aberration is found by (669) when we know the velocity of light and the mean velocity of the earth in its orbit. The progressive motion of light was discovered by ROEMER, in the year 1675, from the discrepancies between the predicted and observed times of the eclipses of Jupiter's satellites. He found that when the planet was nearest to the earth the eclipses occurred about 8^m earlier than the predicted times, and when farthest from the earth about 8^m later than the predicted

times. The planet was nearer the earth in the first position than in the second by the diameter of the earth's orbit; and hence ROEMER was led to the true explanation of the discrepancy,—namely, that light was progressive and traversed a distance equal to the diameter of the earth's orbit in about 16^m. More recently, DELAMBRE, from a discussion of several thousand of the observed eclipses, found $8^m\ 13^s.2$ for the time in which light describes the mean distance of the earth from the sun. From this quantity, which is denoted by $\frac{a}{V}$, Art. 395, we obtain the aberration constant by the formula

$$k = \frac{a}{V} \cdot \frac{2\pi}{n\,T \sin 1'' \sqrt{1 - e^2}} \tag{720}$$

Hence, with the values $\frac{a}{V} = 493^s.2$, $T = 366.256$, $n = 86164$, $e = 0.01677$, we find $k = 20''.260$. DELAMBRE gives $20''.255$, which would result from the above formula if we omitted the factor $\sqrt{1 - e^2}$, as was done by DELAMBRE.

On account of the uncertainty of the observations of these eclipses (resulting from the gradual instead of the instantaneous extinction of the light reflected by the satellite), more confidence is placed in the value derived from direct observation of the apparent places of the fixed stars.

437. *To find the aberration constant by observations of fixed stars.*—Observations of the right ascension of a star near the pole are especially suitable for this purpose, because the effect of the aberration upon the right ascension is rendered the more evident by the large factor $\sec\delta$ with which in (678) the constant is multiplied. The apparent right ascension should be directly observed at different times during at least one year, in which time the aberration obtains all its values, from its greatest positive to its greatest negative value. If we suppose but two observations made at the two instants when the aberration reaches its maximum and its minimum, the earth at these times being in opposite points of its orbit, and if α' and α'' are the apparent right ascensions at these times (freed from the effects of the nutation and the precession in the interval between the observations), we shall have

$$k = \tfrac{1}{2}(\alpha' - \alpha'')\cos\delta$$

But, not to limit the observations to these two instants, let us take, for any time,

α = the assumed mean right ascension of the star + the nutation + proper motion,
α' = the observed right ascension,

and, further, let

$\Delta\alpha$ = the correction of the assumed mean right ascension,
Δk = the correction of the assumed aberration constant,

then, by (678), we have

$$\alpha' = \alpha + \Delta\alpha - (k + \Delta k)(\cos \odot \cos \varepsilon \cos \alpha + \sin \odot \sin \alpha) \sec \delta$$

or, putting

$$m \sin M = \sin \alpha$$
$$m \cos M = \cos \alpha \cos \varepsilon$$
$$\alpha' = \alpha + \Delta\alpha - (k + \Delta k)\, m \cos (\odot - M) \sec \delta \qquad (721)$$

Hence, collecting the known quantities, and putting

$$a = - m \cos (\odot - M) \sec \delta$$
$$n = \alpha + ak - \alpha'$$

we have the equation of condition

$$a \Delta k + \Delta\alpha + n = 0 \qquad (722)$$

Every observation throughout the year being employed to form such an equation, we can deduce from all the equations, by the method of least squares, the most probable values of Δk and $\Delta\alpha$. Those observations will have the greatest weight in determining Δk, which are near the positive and negative maxima of the aberration, where the coefficient a has its greatest numerical values. These maxima occur for $\cos (\odot - M) = -1$ and $\cos (\odot - M) = +1$; that is, for $\odot = 180° + M$ and $\odot = M$.

In this method it is assumed that the precession and nutation are so well known that the relative values of α are correct, or, in other words, that they are in error only by some quantity common to them all and denoted by $-\Delta\alpha$. Since the aberration completes its period in one year, the probable errors of the present values of the precession and the nutation constants will not become sensible in the investigation of the aberration if the

observations of each year are separately discussed. The period of the leading terms of the nutation being only nineteen years, if we extend the observations for aberration over a considerable portion of this period, it will be proper to introduce into our equations of condition a term involving the correction of the nutation constant, as will be seen hereafter.

438. The declinations may also be employed for determining the aberration. If we put

$$\begin{aligned} \delta &= \text{the assumed mean declination} + \text{the nutation,} \\ \Delta\delta &= \text{the correction of this value,} \\ \delta' &= \text{the observed value,} \end{aligned}$$

we have, by (678),

$$\delta' = \delta + \Delta\delta - (k + \Delta k)\,[(\sin\varepsilon\cos\delta - \cos\varepsilon\sin\delta\sin\alpha)\cos\odot + \sin\delta\cos\alpha\sin\odot]$$

or, putting

$$\begin{aligned} m'\sin M' &= \sin\delta\cos\alpha \\ m'\cos M' &= \cos\delta\sin\varepsilon - \sin\delta\cos\varepsilon\sin\alpha \end{aligned}$$

and then

$$\begin{aligned} a' &= -m'\cos(\odot - M') \\ n' &= \quad \delta + a'k - \delta' \end{aligned}$$

the equation of condition is

$$a'\Delta k + \Delta\delta + n' = 0 \tag{723}$$

439. If the pole star is employed, which has a sensible annual parallax, or any star whose parallax is even suspected, it will be proper to introduce into the equations of condition a term which represents its effect. We have, by (691), introducing the above auxiliaries,

$$\begin{aligned} \text{par. in R. A.} &= +pr\,m\sin(\odot - M)\sec\delta \\ \text{par. in dec.} &= +pr\,m'\sin(\odot - M') \end{aligned}$$

and hence the equation of condition from the right ascension will be

$$a\,\Delta k + bp + \Delta\alpha + n = 0 \tag{724}$$

and, from the declination,

$$a'\Delta k + b'p + \Delta\delta + n' = 0 \tag{725}$$

in which

$$b = rm \sin(\odot - M) \sec \delta$$
$$b' = rm' \sin(\odot - M')$$

The solution of the equations will now determine, not only Δk and either $\Delta\alpha$ or $\Delta\delta$, but also the parallax p.

440. It was by comparing the declinations deduced from the meridian zenith distances of stars, and more especially of the star γ *Draconis*, that BRADLEY discovered the aberration. The constant deduced from his observations by BUSCH is 20″.2116.

STRUVE's value of the constant was derived from the declinations of seven stars observed with a transit instrument in the prime vertical.*. The term representing the parallax was retained in the equations of condition, but merely to show the effect of parallax should it exist. This effect was in every case small, and, moreover, for the different stars had not always the same sign: so that he found the mean value of the constant from all the stars would not be changed as much as 0″.006 by any probable parallax. On account of the extraordinary precision of this determination of the aberration, I here quote the individual results and their probable errors from the *Astronomische Nachrichten*, Vol. XXI. p. 58.

	Aberration Constant	Probable Error.
v *Ursæ Maj.*	20″.4571	0″.0303
ι *Draconis*	20 .4792	0 .0224
δ *Cassiopeiæ*	20 .4559	0 .0462
o *Draconis*	20 .4039	0 .0229
b *Draconis*	20 .5036	0 .0322
P. XIX. 371	20 .3947	0 .0333
β *Cassiopeiæ*	20 .4227	0 .0352

whence, having regard to the probable errors, the mean was found 20″.4451 with the probable error 0″.0111 = $\frac{1}{90}$ of a second of arc.

Other modern determinations of the constant of aberration agree in giving a greater value than was found by DELAMBRE from the eclipses of Jupiter's satellites. Thus, LINDENAU found

* See Vol. II. Determination of the declinations of stars by their transits over the Prime Vertical, Artt. 138 *et seq.*

trom the right ascensions of the pole star $k = 20''.4486$, and the annual parallax of the star $= 0''.1444$; PETERS, from six hundred and three equations of condition, formed upon the right ascensions of the pole star, observed at Dorpat in the years 1822 to 1838, found $k = 20''.4255$, with the annual parallax $= 0''.1724$; LUNDAHL, from one hundred and two observed declinations of this star, found $k = 20''.5508$, and the parallax $= 0''.1473$; and PETERS, from two hundred and seventy-nine declinations of the same star, observed with the Repsold vertical circle of the Pulkova Observatory, found $k = 20''.503$, and the parallax $= 0''.067$*.

The parallax is so small a quantity that the discrepancies between these several values appear to be relatively great: nevertheless, we must consider them as surprisingly small when we remember that all these determinations rest upon observations of the absolute place of the star. *Differential* measures of the changes of a star's place with the micrometer are susceptible of greater refinement. Such a method I proceed to give in the next article.

441. *To find the relative parallax of two stars by micrometric measures of their apparent angular distance.*—It was first suggested by the elder HERSCHEL that if the absolute linear distances of two neighboring stars from our solar system were very unequal, their apparent angular distance from each other as seen from the earth would necessarily vary as the earth changed its position in its orbit. If one of the stars were so remote as to have no sensible parallax, changes in this apparent distance (provided they followed the known law of parallax) might be ascribed solely to the parallax of the nearer star; and in any case such changes might be ascribed to the relative parallax; that is, to the difference of the parallaxes of the two stars.

For the trial of this method BESSEL judiciously selected the star 61 *Cygni*, near which are two much smaller stars (at distances from it of about $8'$ and $12'$ respectively), and from a series of micrometric measures of its angular distance from each, extending through a period of more than a year, namely, from August 18, 1837, to October 2, 1838, obtained the first clearly demonstrated parallax of a fixed star.† A subsequent

* *Astron. Nach.*, Vol. XXII. p. 119. † Ibid. No. 366.

series extending from October 10, 1838, to March 23, 1840, fully confirmed the parallax, only slightly increasing its amount.* The first series gave the annual parallax 0″.3136: the final result from both series is 0″.37, with a probable error of ± 0″.01.

In the selection of this star, it was presumed, in accordance with the conception of HERSCHEL, that 61 *Cygni*, being between the fifth and sixth magnitudes, was much nearer than the comparison stars, which were both between the ninth and tenth magnitudes. A still stronger presumption in favor of its proximity was found in its great proper motion, which is among the greatest yet observed. Moreover, it is a double star, and the distance of the middle point of the line joining its two components, from each of the comparison stars, could be more accurately observed with the heliometer than the distance of two simple stars.†

The following is BESSEL'S method of reducing these observations.

Fig. 61.

Let A be the star (Fig. 61) whose parallax is sought, (if a double star, A will denote the middle point between its components); B the comparison star; P the pole of the equator. The observations will be reduced to some assumed epoch, as the beginning of one of the years over which the series extends. For this epoch let

s = the distance AB,
P = the position angle of the star B at $A = PAB$,
α, δ = the mean right ascension and declination of A,
p = the relative annual parallax of A and B.

If A' is the position of the star at the time of an observation, as affected by parallax, it is easily seen that the increase of the

---

* *Astron. Nach.*, No. 401.

† The observations were made with the great heliometer of the Königsberg Observatory. The distance of two simple stars is measured with this instrument by bringing the image of one star, formed by one half of the object glass, into *coincidence* with the image of the other star, formed by the other half of the object-glass. When one of the stars is double, the image of the simple star is brought to the middle point of the line joining the components of the double star. This point of bisection can be more accurately judged of by the eye than the coincidence of two superposed images, when the distance bisected is within certain limits. In the present case it was 16″.

distance AB or $A'B - AB$, which will be denoted by Δs, is given by the differential formula

$$\Delta s = - \Delta\alpha \cos\delta \,.\, \sin P - \Delta\delta \cos P$$

where $\Delta\alpha$ and $\Delta\delta$ are respectively the parallax in right ascension and declination, which are given by (691). Substituting these values, and then assuming the auxiliaries m and M, such that

$$\begin{aligned} m \cos M &= \sin\alpha \sin P + \cos\alpha \sin\delta \cos P \\ m \sin M &= (- \cos\alpha \sin P + \sin\alpha \sin\delta \cos P) \cos\varepsilon \\ &\qquad - \cos\delta \cos P \sin\varepsilon \end{aligned}$$

we have

$$\Delta s = prm \cos(\odot - M) \tag{726}$$

The effect of the proper motion of A upon the distance is found as follows. Let

χ = the angle which the great circle in which the star moves makes with the declination circle.

ρ = the annual proper motion on the great circle.

$\Delta'\alpha$, $\Delta'\delta$ = the given proper motion in right ascension and declination, reduced to the assumed epoch (Art. 379);

then, as in Art. 380, we find ρ and χ by the formulas

$$\left.\begin{aligned} \rho \sin\chi &= \Delta'\alpha \cos\delta \\ \rho \cos\chi &= \Delta'\delta \end{aligned}\right\} \tag{727}$$

Let τ be the time of any observation reckoned from the assumed epoch and expressed in fractional parts of a year. In the above diagram, if AA' now represents the proper motion on a great circle in the time τ, then $AA' = \tau\rho$; and, if the effect of the proper motion upon the distance is denoted by $\Delta's$, we have also $A'B = s + \Delta's$, $A'AB = P - \chi$, and the triangle $AA'B$ gives

$$\cos(s + \Delta's) = \cos(\tau\rho) \cos s + \sin(\tau\rho) \sin s \cos(P - \chi)$$

Developing this equation, and retaining only second powers of $\tau\rho$, we find

$$\Delta's = - \tau\rho \cos(P - \chi) + \frac{(\tau\rho)^2 \sin^2(P - \chi)}{2}$$

in which τ is the only variable. Taking then for the constants

$$\left.\begin{aligned} f &= -\rho\cos(P-\chi) \\ f' &= \frac{\rho^2\sin^2(P-\chi)}{2s} \end{aligned}\right\} \quad (728)$$

the computation of the correction for each observation is readily made by the formula

$$\Delta' s = f\tau + f'\tau\tau$$

The assumed proper motion may, however, be in error; and there may also be errors in the observed distances which are proportional to the time (such as any progressive change in the value of the micrometer screw, &c.). The correction for all such errors may be expressed by a single unknown correction y of the coefficient f, so that we shall take

$$\Delta' s = (f + y)\tau + f'\tau\tau \quad (729)$$

The corrections of micrometric measures for the effects of aberration and refraction* are treated of in Vol. II. Chapter X. We shall, therefore, suppose these corrections to have been applied, and shall take

s' = the observed distance at the time τ, corrected for differential aberration and refraction,

and then we shall have

$$s' = s + \Delta s + \Delta' s \quad (730)$$

This equation involves three unknown quantities, namely, the distance s, the parallax involved in Δs, and the correction y involved in $\Delta' s$. Let s_0 be an assumed value of s nearly equal to the mean of the values of s', and put

$$s = s_0 + x$$

The substitution of this in our equations of condition will introduce the small unknown quantity x in the place of the larger

* These effects are only differential, and so small that the errors in the total refraction and aberration may safely be assumed to have no sensible influence. It is also an advantage of this method of finding the parallax of a star, that it is free from the errors of the nutation and precession, which, being only changes in the position of the circles of reference, have no effect whatever upon the apparent distance of two stars.

one s, and will thus facilitate the computations. When all the substitutions are made in the expression of s', we obtain the following equation:

$$0 = s_0 - s' + f\tau + f'\tau\tau + x + \tau y + prm \cos(\odot - M)$$

To put this in the usual form, let us take

$$n = s_0 - s' + f\tau + f'\tau\tau$$
$$c = rm \cos(\odot - M)$$

then each observation gives the equation

$$x + \tau y + cp + n = 0 \tag{731}$$

and from all these equations we find, by the method of least squares, the most probable values of x, y, and p.

In the determination of so small a quantity as p, it is necessary to give to the micrometric measures the greatest possible precision. It is particularly important to find the effects of temperature upon the micrometer screw; for these effects, depending on the season, have a period of one year, like the parallax itself, and may in some cases so combine with it as completely to defeat the object of the observations. At the time BESSEL published his discussion of his observations on 61 *Cygni*, he had not completed his investigations of the effect of temperature upon the screw, and therefore introduced an indeterminate quantity k into his equations of condition, by which the effect upon the parallax might be subsequently taken into account when the correction for temperature was definitively ascertained. This was done as follows. He had assumed the correction of a measured distance for the temperature of the micrometer screw to be

$$\Delta''s = -0''.0003912\, s\,(t - 49°.2)$$

in which t is the temperature by Fahrenheit's scale, and s is expressed in revolutions of the screw. If the coefficient $0''.0003912$ should be changed by subsequent investigations to $0''.0003912 \times (1 + k)$, each observed distance would receive the correction $\Delta''s.k$, the quantity n in the equations of condition would become $n - \Delta''s.k$, and the equations would take the form

$$x + \tau y + cp - \Delta''s.k + n = 0 \tag{732}$$

The quantity k being left indeterminate, x, y, and p were found as functions of it. The value of p was thus found to be

$$= 0''.3483 - 0''.0533\, k, \text{ with the mean error } \pm 0''.0141$$

The final result of his investigation of the micrometer gives*

$$k = -0.4893 \text{ with the mean error } \pm 0.0903$$

and hence the corrected value of the parallax

$$= 0''.3744 \text{ with the mean error } \pm 0''.0149$$

If this result had been deduced by comparison with but one star, it could only be received as the *relative* parallax. BESSEL, however, employed two stars whose directions from 61 *Cygni* were nearly at right angles to each other, and found nearly the same parallax from both; whence it follows either that both these stars have the *same* sensible parallax, or, which is more probable, that both are so distant as to exhibit no sensible parallax. This conclusion would be confirmed if a comparison with other surrounding stars gave the same parallax, especially if these were of different magnitudes; for it would be in the highest degree improbable that all these stars were at the same distance from our solar system.

THE NUTATION CONSTANT.

442. *To find the constant of nutation from the observed right ascensions or declinations of a fixed star.*—In Art. 437 it was assumed that the observations by which the aberration constant was determined extended over only a year or two: so that the nutation affected all the observations by quantities which differed so little that any error in the total nutation would not sensibly affect the determination. When the observations are extended over a longer period, we may introduce into the equations of condition an additional term for the correction of the nutation. As before, let the mean right ascensions and declinations be reduced to their apparent values at the time of each observation by means

* According to PETERS in the *Astron. Nach. Ergänzungs-heft*, p. 55; derived from BESSEL's *Astronomische Untersuchungen*, Vol. I. p. 125.

of an assumed aberration and nutation, and denote these apparent values by α and δ, and put

$$\Delta\nu = \text{the correction of the nutation constant,}$$
$$\alpha',\ \delta' = \text{the observed right ascension and declination;}$$

then

$$\begin{aligned} \alpha' &= \alpha + \Delta\alpha + a\,\Delta k + bp + c\,\Delta\nu \\ \delta' &= \delta + \Delta\delta + a'\Delta k + b'p + c'\Delta\nu \end{aligned} \qquad (738)$$

in which, as before, $\Delta\alpha$ and $\Delta\delta$ are the corrections of the star's mean place, Δk the correction of the aberration constant, p the star's annual parallax, a and b, a' and b' are the coefficients found in Arts. 437, 438, and 439. It only remains to express c and c' in terms of known quantities.

In the physical theory, it is shown that the coefficients of those terms of the nutation formulæ (666) which depend upon $2\odot$, $\odot - \Gamma$, and $\odot + \Gamma$ involve not only the nutation constant (the coefficient of cos ☊), but also the precession constant; while all the other coefficients vary proportionally to the coefficient of cos ☊. If we put

$$\nu = \text{the assumed nutation constant,}$$
$$\nu' = \text{the true} \quad \text{"} \quad \text{"} \quad = \nu + \Delta\nu$$

and if we express the relation between ν and ν' by the equation

$$\nu' = \nu(1 + i)$$

and, in like manner, suppose the true precession constant to be

$$\psi = 50''.3798\,(1 + \zeta)$$

then, according to Peters,* the formulæ (666), adapted for any value of the constants, are for 1800,

$$\begin{aligned} \Delta\varepsilon = (1+i)\,[&9''.2231\cos\Omega - 0''.0897\cos 2\Omega + 0''.0886\cos 2\text{☾}] \\ &+ (1 - 2.162\,i + 3.162\,\zeta)\,[0''.5510\cos 2\odot + 0''.0093\cos(\odot + \Gamma')] \\ \Delta\lambda = (1+i)\,[&-17''.2405\sin\Omega + 0''.2073\sin 2\Omega - 0''.2041\sin 2\text{☾} \\ &+ 0''.0677\sin(\text{☾} - \Gamma')] \\ &+ (1 - 2.162\,i + 3.162\,\zeta)\,[-1''.2694\sin 2\odot + 0''.1279\sin(\odot - \Gamma) \\ &- 0''.0213\sin(\odot + \Gamma)] \end{aligned}$$

---

* *Numerus Constans Nutationis*, p. 46. We have omitted some terms which are inappreciable or of very short period. This omission will not affect the accuracy of the determination of the quantity ν.

The effect which any probable correction of the precession constant can have upon the very small terms of these formulæ is not only itself very small, but must entirely disappear when a great number of observations extending over a number of years are combined, since the principal terms which are affected by the precession—namely, those in $2\odot$—have a period of only six months. We can, therefore, here assume $\zeta = 0$. In the formulæ for the nutation in right ascension and declination (668), the terms in the first four lines will be multiplied by $1 + i$, and those in the last three lines by $1 - 2.162i$; so that, if we denote by β the sum of the corrections in R. A. contained in the first four lines, by γ the sum of the remaining corrections, and the coresponding corrections in dec. by β' and γ', we shall have

$$\text{Nutation in R. A.} = (1 + i)\beta + (1 - 2.162i)\gamma$$
$$\text{" " Dec.} = (1 + i)\beta' + (1 - 2.162i)\gamma'$$

or

$$\text{Nutation in R. A.} = \beta + \gamma + (\beta - 2.162\gamma)i$$
$$\text{" " Dec.} = \beta' + \gamma' + (\beta' - 2.162\gamma')i$$

in which $\beta + \gamma$ and $\beta' + \gamma'$ express the nutation computed according to the assumed constant. Hence we derive

$$c\,\Delta\nu = c\nu i = (\beta - 2.162\gamma)i$$
$$c'\Delta\nu = c'\nu i = (\beta' - 2.162\gamma')i$$

and, consequently,

$$\left.\begin{aligned} c &= \frac{\beta - 2.162\gamma}{\nu} \\ c' &= \frac{\beta' - 2.162\gamma'}{\nu} \end{aligned}\right\} \quad (734)$$

which will be readily computed for each observation if the lunar nutation (β, β') and the solar nutation (γ, γ') have been separately computed, as they usually are. All the equations of the form (733), whether constructed upon the right ascensions or the declinations, or both, will then be treated by the method of least squares, and the most probable values of $\Delta\alpha$, Δk, p, and $\Delta\nu$ will be found.

In this manner BUSCH,* from BRADLEY's observations of the declinations of twenty-three stars, made in the years 1727 to

* *Astron. Nach.*, No. 309.

1747, and embracing, therefore, a whole period of the nutation, found $k = 20''.2116$, $\nu = 9''.2320$. In this discussion the parallax of the stars was not taken into account.

Nearly the same value of the nutation constant follows from the more recent observations at the Pulkova Observatory. From the declinations of the pole star observed between 1822 and 1838, LUNDAHL found $\nu = 9''.2164$, and from the right ascensions of the same star PETERS found $9''.2361$. The value $9''.2231$, which PETERS has adopted in the *Numerus Constans Nutationis*, is the mean of the three values found by BUSCH, LUNDAHL, and himself, having regard to the weights of the several determinations as given by their probable errors.

THE PRECESSION CONSTANT.

443. If α_1, δ_1, and α_2, δ_2 are the mean right ascensions and declinations of the same star, deduced from observation at two distant epochs t_1 and t_2, by deducting from the observed values the aberration and nutation, the *annual variations* of the right ascension and declination for the mean epoch $\frac{1}{2}(t_1 + t_2)$ will be

$$a = \frac{\alpha_2 - \alpha_1}{t_2 - t_1} \qquad b = \frac{\delta_2 - \delta_1}{t_2 - t_1} \tag{735}$$

These annual variations include both the precession and the proper motion of the star; and, since both are proportional to the time, it will be impossible to distinguish the proper motion until the precession is obtained. If, however, we suppose that the proper motions of the different stars observe no law, or that they take place indiscriminately in all directions, it will follow that the mean value of the precession, deduced from such annual variations of a very large number of stars, will be free from the effect of the proper motions. The latter are, in fact, so various in direction, although, as will hereafter be shown, not entirely without law, that this mode of proceeding must lead at least to an approximation not very far from the truth. Accordingly, from the a and b, found as above for each star, we derive the m and n of Art. 374, by the equations*

---

* Both m and n may be found from the right ascensions alone by forming equations of the form

$$m + n \sin \alpha_0 \tan \delta_0 = a$$

from a number of stars and solving them by the method of least squares.

$$\left.\begin{array}{l} m + n \sin \alpha_0 \tan \delta_0 = a \\ n \cos \alpha_0 \qquad\quad = b \end{array}\right\} \qquad (736)$$

in which α_0 and δ_0 are taken for the mean epoch $\frac{1}{2}(t_1 + t_2)$. And from the m and n thus found we have, by (661),

$$\left.\begin{array}{l} \dfrac{d\psi}{dt} \cos \varepsilon_1 - \dfrac{d\vartheta}{dt} = m \\ \\ \dfrac{d\psi}{dt} \sin \varepsilon_1 = n \end{array}\right\} \qquad (737)$$

in which $\dfrac{d\psi}{dt}$ is the annual luni-solar precession (or the precession constant), and $\dfrac{d\vartheta}{dt}$ the annual planetary precession. But $\dfrac{d\vartheta}{dt}$ is very accurately obtained theoretically by substituting the known masses of the planets in the general formula deduced from the theory of gravitation: so that a value of the precession $\dfrac{d\psi}{dt}$ may be derived both from m and from n. In these formulæ, the value of ε_1 is to be employed as given by (646) for the epoch $t = \frac{1}{2}(t_1 + t_2)$.

Having thus obtained a preliminary value of the precession, the quantities $m + n \sin \alpha_0 \tan \delta_0$ and $n \cos \alpha_0$, computed from it for each star, can be compared with the a and b found by (735), and the differences which exceed the probable errors of observation may be regarded as resulting from the proper motion of the star. Those stars which are found to have a very large proper motion are then to be excluded from the investigation; and from the remaining ones a more accurate value of the precession will be obtained.

In this way, BESSEL, from 2300 stars whose places were determined by BRADLEY for 1755 and by PIAZZI for 1800, found the precession constant for the year 1750 to be 50″.37572, and for 1800, 50″.36354.* In this investigation those stars were excluded which in the preliminary computation exhibited annual proper motions exceeding 0″.3.

See also Article 445.

---

* *Fundamenta Astronomiæ*, p. 297, where the value 50″.340499 is found; and *Astron. Nach.*, No. 92, where the value is increased to 50″.37572.

THE MOTION OF THE SUN IN SPACE.

444. With a knowledge of the precession we are enabled to distinguish proper motions in a large number of stars. Upon comparing these proper motions, Sir W. HERSCHEL was the first to observe that they were not without law, that they did not occur indiscriminately in all directions, but that, in general, the stars were apparently moving *towards* the same point of the sphere, or *from* the diametrically opposite point. The latter point he located near the star *λ Herculis*. This common apparent motion he ascribed to a real motion of our solar system, a conclusion which has since been fully confirmed.

Nevertheless, there are many stars whose proper motions are exceptions to this law: these must be regarded as motions compounded of the real motions of the stars themselves and that of our sun. These real motions must, doubtless, also be connected by some law which the future progress of astronomy may develop;* but thus far they present themselves in so many directions that (like the whole proper motion in relation to the precession) they may be provisionally treated as accidental in relation to the common motion. Hence, for the purpose of determining the common point from which the stars appear to be moving, and towards which our sun is really moving, we may employ *all* the observed proper motions, upon the presumption that the real motions of the stars, having the characteristics of accidental errors of observation and combining with them, will be eliminated in the combination. Nevertheless, in order that the errors of observation may not have too great an influence, it will be advisable to employ only those proper motions which are large in comparison with their probable errors.

The direction in which a star appears to move in consequence of the sun's motion lies in the great circle drawn through the star and the point towards which the sun is moving. Let this point be here designated as the point *O*. If the great circle in which each star is observed to move were drawn upon an artificial globe,

* The law which we naturally expect to find is that of a revolution of all the stars of our system around their common centre of gravity. MÄDLER, conceiving that our knowledge of the proper motions is already sufficient for the purpose, has attempted to assign the position of this centre. He has fixed upon *Alcyone*, the principal star of the *Pleiades*, as the central sun. *Astron. Nach.*, No. 566. *Die Eigenbewegungen der Fixsterne in ihrer Beziehung zum Gesammtsystem*, von J. H. MÄDLER, Dorpat, 1856.

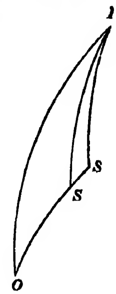
all these circles would intersect in the same point O, if the observations were perfect and the stars had no real motion of their own. But, the latter conditions failing, the intersections which would actually occur would form a group of points whose mathematical *centre of gravity* would, according to the theory of probabilities, be the point from which, or towards which, the common motions were directed. Thus, an approximate first solution might be obtained by a purely graphic process.

Let us then assume that an approximate solution has been found, and put

$A, D =$ the assumed approximate right ascension and declination of the point O.

It is then required to find a more exact solution by determining the corrections ΔA and ΔD which A and D require. Let P (Fig. 62) be the pole of the equator, and S a star whose apparent motion resulting from the sun's motion is in the great circle OSS'. The angle $PSS' = \chi$, which this great circle makes with the declination circle (reckoned in the usual manner from the north towards the east), is the supplement of the angle PSO. Hence, if α and δ are the right ascension and declination of the star, and λ the arc SO joining the star and the point O, we have, in the triangle POS,

Fig. 62.



$$\left.\begin{aligned} \sin\lambda \sin\chi &= \sin(\alpha - A)\cos D \\ \sin\lambda\cos\chi &= \cos(\alpha - A)\cos D \sin\delta - \sin D \cos\delta \end{aligned}\right\} \quad (738)$$

by which λ and χ are found for each star.

The angle χ thus computed will be equal to the observed angle which the path of the star makes with the declination circle only when A and D are correctly assumed. Let χ' be the *observed* angle, or that which results from the equations

$$\left.\begin{aligned} \rho \sin\chi' &= \Delta\alpha \cos\delta \\ \rho\cos\chi' &= \Delta\delta \end{aligned}\right\} \quad (739)$$

in which $\Delta\alpha$ and $\Delta\delta$ are the observed proper motions in right ascension and declination, and ρ the proper motion in the great circle. Then, when χ' differs from χ, the difference $\chi' - \chi$ is to be regarded as a function of the corrections ΔA and ΔD which the assumed values of A and D require. The variations of the

angle χ produced by the variations of A and D will be found from the triangle POS by the first differential formulæ (47); whence

$$\Delta\chi . \sin\lambda = (\chi' - \chi)\sin\lambda = \left(\frac{\cos(\alpha - A)\cos\delta\sin D - \sin\delta\cos D}{\sin\lambda}\right)\Delta A\cos D$$

$$+ \frac{\sin(\alpha - A)\cos\delta}{\sin\lambda}\Delta D \qquad (740)$$

Hence, we have only to compute for each star the values of χ and $\sin\lambda$ by (738), and of χ' by (739), and then, putting

$$n = (\chi - \chi')\sin\lambda$$

$$a = \frac{\cos(\alpha - A)\cos\delta\sin D - \sin\delta\cos D}{\sin\lambda}$$

$$b = \frac{\sin(\alpha - A)\cos\delta}{\sin\lambda}$$

we form the equation of condition,

$$a . \Delta A\cos D + b . \Delta D + n = 0$$

in which $\Delta A\cos D$ and ΔD are the unknown quantities. From all the equations thus formed the most probable values of ΔA and ΔD will be found by the method of least squares. The quantity $(\chi - \chi')\sin\lambda$ is the distance between the great circle in which the star really moves and that drawn from the star to the point O, measured at this point.

In this manner the position of the point O has been very closely determined. The earlier determinations founded on a comparatively small number of well established proper motions are those of

	W. HERSCHEL,	$A = 245° 53'$	$D = + 49° 38'$
and	GAUSS,	$A = 259\ 10$	$D = + 30\ 50$

Of the more recent determinations, the first in the order of time is that of ARGELANDER.* He employed 390 stars, the proper motions of which he found by comparing their positions as determined by himself for 1830† with those determined by BESSEL from BRADLEY'S observations for 1755.‡ He divided these stars into

* *Astron. Nach.*, No. 363. † DLX *Stell. Fix. Positiones Mediæ ineunte anno* 1830.

‡ *Fundamenta Astronomiæ.*

three classes according to their proper motions, and found, for the epoch 1792.5,

From	Whose annual proper motion was	$A=$	$D=$
23 stars	greater than 1″.0	256° 25′.1	+ 38° 37′.2
50 "	between 0″.5 and 1 .0	255 9 .7	+ 38 34 .3
319 "	" 0 .2 " 0 .5	261 10 .7	+ 30 58 .1

and, combining these results with regard to their respective weights,

$$A = 259° \; 51'.8 \qquad D = + 32° \; 29'.1$$

As supplementary to this computation, LUNDAHL compared 147 of BRADLEY'S stars not contained in ARGELANDER'S catalogue with POND'S catalogue of 1112 stars for 1830, and found*

$$A = 252° \; 24'.4 \qquad D = + 14° \; 26'.1$$

which ARGELANDER combined with his former results and found, for 1800,

$$A = 257° \; 54' \qquad D = + 28° \; 49'$$

OTTO STRUVE, employing 400 stars, mostly identical, however, with ARGELANDER'S and LUNDAHL'S stars, and determining their proper motions from the Dorpat observations compared with BRADLEY'S, found, for 1790,

$$A = 261° \; 21'.8 \qquad D = 37° \; 36'.0$$

GALLOWAY, from the southern stars observed by JOHNSON at St. Helena and HENDERSON at the Cape of Good Hope (for 1830), and by LACAILLE at the Cape of Good Hope (for 1750), found

$$A = 260° \; 1' \qquad D = + 34° \; 23'$$

Finally, MÄDLER, recomputing the proper motions of a large number of stars, with the aid of the best modern observations, has found, for 1800,†

From	Whose proper motion is	$A=$	$D=$
227 stars	greater than 0″.25	262° 38′.8	+ 39° 25′.2
663 "	between 0″.1 and 0 .25	261 14 .4	+ 37 53 .6
1273 "	" 0 .04 " 0 .01	261 32. 2	+ 42 21 .9

* *Astron. Nach.*, No. 398.

† *Die Eigenbewegungen der Fixsterne*, p. 227

and by combination, having regard to the number of stars in each class,

$$A = 261^\circ\ 38'\ 8 \qquad D = +\ 39^\circ\ 53'.9$$

445. It would at first sight seem that the existence of any law in the proper motions of the stars would vitiate the value of the precession constant found by BESSEL according to the method of Art. 443. Accordingly, OTTO STRUVE has attempted to determine both the precession constant and the motion of the solar system from equations of condition involving both. In order to accomplish this it was necessary to introduce into the equations the magnitude as well as the direction of the proper motions. But since the apparent angular motion of a star, so far as it depends upon the motion of our sun, is a function of the star's distance from us, it became necessary also to make an hypothesis as to the relative distances of the stars of different orders of magnitude. Thus, the new value of the precession constant given by him, and which we have (provisionally) adopted on page 606, is also exposed to the objection that it rests upon an hypothesis.

Astronomers have, therefore, been led to re-examine the grounds upon which BESSEL's determination rests. It is to be observed that the method which he employed would give a result entirely free from the effects of the sun's motion, if the stars employed were uniformly distributed over the sphere, and if the average distance of these stars in all directions from the sun were the same. MÄDLER, in the work above quoted, has shown that for 2139 stars distributed with tolerable uniformity, BESSEL's constant gives proper motions in right ascension the mean of which is only $-\ 0''.0003$. If now this quantity were applied to BESSEL's value of m and the proper motions again computed, their mean would come out exactly zero. Hence he concludes that these stars fully confirm BESSEL's constant, since the correction $-\ 0''.0003$ is insignificant. It appears, however, that, in drawing this inference without reservation, he has left out of view the second conclusion above stated, that the average distance of the stars on all sides of us should be the same. For, if the sun's motion produces greater apparent motions in stars near to us than in those more remote, a want of uniformity in the distances, notwithstanding the equal distribution of the stars, would produce a greater *amount* of proper motion in one hemisphere

than in the other; and the aggregate of all the proper motions, having regard to their signs, would not be zero.

Since it is probable that the average distance of stars of the *same magnitude* is the same on all sides of us (although there are not a few individual exceptions of small stars with large proper motions and large stars with small ones), a more satisfactory determination of the precession constant may result from future investigations in which not only all the stars employed shall be uniformly distributed, but those of each order of apparent magnitude shall be so distributed. It will be impossible to secure this condition if the larger stars are retained; for their distribution is too unequal. By confining the investigation to the small stars, there will also be obtained the additional advantage that the amount of the proper motions themselves will probably be very small, and thus have very little influence upon the precession constant, even if they are not wholly eliminated. The formation of accurate catalogues of the small stars is therefore essential to the future progress of astronomy in this direction.

END OF VOL. I.